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The role of contextual, conceptual and procedural knowledge in activating mathematical competencies (PISA)

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Abstract This paper analyses the difficulties which Spanish student teachers have in solving the PISA 2003 released items. It studies the role played by the type and organisation of mathematical knowledge in the activation of competencies identified by PISA with particular attention to the function of contextual knowledge. The results of the research lead us to conclude that the assessment of the participant's mathematical competencies must include an assessment of the extent to which they have school mathematical knowledge (contextual, conceptual and procedural) that can be productively applied to problem situations. In this way, the school knowledge variable becomes a variable associated with the PISA competence variable.

Keywords Conceptual · Procedural · Contextual knowledge · Mathematical literacy · PISA tasks

1 Introduction

The *Programme for International Student Assessment* (PISA) provides valuable information on the skills of students, at the end of their compulsory schooling (15 years), in using mathematics in everyday-life situations (OECD 2004). The publication of the PISA 2003 results caused a considerable stir in the Spanish educational community and even in the media because of the far from brilliant results of Spanish students. The result was major debates concerned with improving these results in future assessments. In fact, the very effectiveness of the compulsory schooling system in Spain was brought into question.

Creemers and Kyriakides (2006) critically analyse the weaknesses of research in educational effectiveness. For example, they cite the research by Scriven (1994) that sees

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teacher subject knowledge as a factor affecting teacher effectiveness. They also cite the research of Darling-Hammond (2000) that finds this knowledge rarely correlates significantly with student performance.

Creemers and Kyriakides (2006) propose a dynamic multilevel model of effectiveness (educational system, school, classroom). It is based on the assumption that the relation of some factors of effectiveness with student performance does not have to be linear (it may be curvilinear) and that each factor is measured in relation to five dimensions (common for all the factors). At the classroom level, this dynamic model defines eight factors of effectiveness that describe the instructional role of the teacher and are consistently related with student performance. Some of them (e.g. orientation and teaching modelling) stress teaching through comprehension and promote the achievement of new educational goals such as the development of thinking skills of a higher order, and particularly problem-solving skills, beyond teaching based on the transmission of knowledge.

Ponte and Chapman (2006) discuss studies that focus on mathematics teachers' knowledge and practices in terms of issues, results and perspectives. Leikin and Levav-Waynberg (2007) reveal the complex relationship between different types of teacher knowledge to explain the gap between theory-based recommendations and school practice. The teacher effectiveness research has shown that effective teachers are expected to help pupils to use strategies and develop their own strategies which can help them solve different types of problems (Kyriakides et al. 2002).

Assuming the hypothesis that to teach any skill the first requirement is to possess the skill, the following question arises: Do the future Spanish teachers have this skill? We carried out a research project aimed at discovering the mathematical skills of students training to be elementary school teachers when confronted with the PISA tasks. The study (Sáenz 2007) provided an X-ray of their performance. It showed up many weak points in their mathematical knowledge, which is an essential attribute of their professional knowledge.

This present paper aims to study the results in more depth and analyse them against the background of the traditional curriculum of school mathematics, which is focused on conceptual and procedural contents. We now move from a merely evaluative interest to a more comprehensive curricular interest. How do the future teachers at compulsory school level organise their mathematical knowledge? How do they use this knowledge to activate the mathematical competence needed to solve a particular mathematical task?

In this way, we can trace and investigate the links between the traditional curriculum, which teachers find familiar, and a competence-based curriculum such as that forming the basis for PISA, which is a novelty for them. We believe that the functional model for school mathematics established by PISA identifies ideas and relations that prove extremely useful as a theoretical framework for this research.

2 The research framework

2.1 The PISA functional model

The mathematical literacy or competence notion in PISA (OECD 2003) deals with the capacity of students (at the end of their compulsory schooling) to analyse, reason and communicate efficiently as they pose, formulate, solve and interpret mathematical problems in a variety of situations. Rico (2006) explains that PISA is not linked to a curriculum in the traditional sense of a study programme, but it does form part of a precise and coherent

curricular structure. He defines a functional model of mathematical learning based on three elements:

- *Contextualised tasks*: These create situations in which the problems are located. They represent an approach of the PISA model towards the didactical phenomenology of mathematical structures (Freudenthal 1983).
- *Conceptual tools*: This is the mathematical content required to solve the problems, organised according to conceptual structures and procedures.
- *A cognitive subject*: The subject should activate certain cognitive competencies or processes to connect the real world in which problems are generated with the mathematics needed to solve them.

These components and their relations are represented in Fig. 1.

Concepts and procedures, which are the backbone of the traditional mathematical curriculum, are not the main focus of the PISA curricular framework. However, they are necessary if we are to ensure the functionality of the mathematical knowledge, which is the aim of this curricular framework. But in what way are they necessary? How does the type of school mathematics and the degree to which it is mastered influence the activation of cognitive competencies when the subject is faced with a contextualised task with a specific demand? The question is relevant because, although the aim of PISA is to assess students' competencies, educational practice acts directly on the contents of school mathematics and the system of tasks it incorporates into the teaching/learning process. To put it another way, in the functional model of learning, cognitive competence is the dependent variable and the mathematical tasks and contents are the independent variables inasmuch as they are controlled by the teacher.

In light of the above, and given that the relation between tasks and competencies (relation 1 in Fig. 1) is the direct object of study in PISA, our research is focused on the other two relations: mathematical tasks/content (relation 2) and mathematical content/competencies (relation 3).

We agree with Bransford, Brown and Cocking (2001) when they state that organised knowledge enables us to solve original problems and remember more relevant information than if we have only memorised isolated mathematical facts or procedures. For Rittle-Johnson and Koedinger (2005), properly structured knowledge requires people to integrate their contextual, conceptual and procedural knowledge within a domain. In our study, we

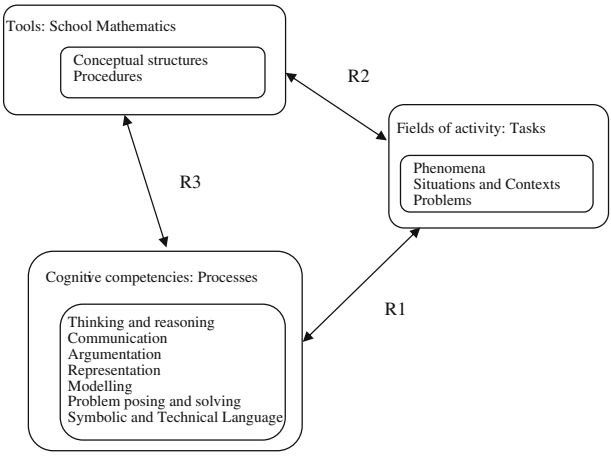


Fig. 1 Functional model of mathematical learning (Rico 2006)

propose that information on these three types of knowledge will offer keys for understanding the process of activating the competencies of our subjects.

2.2 Types of knowledge

Contextual knowledge is related to everyday-life problems in the real world and enters the school situation through the presentation of the problem in a context with its own story. It does not appear when the problems are presented using only mathematical symbols with numbers, operators and variables. Rittle-Johnson and Koedinger (2005) show that there is no agreement within the scientific community as to whether presenting arithmetical problems in a story context makes them easier or more difficult to solve. Linguistic difficulties appear to be one of the most important factors explaining the poor performance of students in some mathematical tasks (Movshovitz-Hadar, Zaslavsky & Inbar 1987; Radatz 1980). When analysing the conduct of our subjects, it is worth bearing in mind that the PISA functional model stipulates contextualised tasks and thus always poses problems by describing a situation of a personal, employment or scientific nature. In addition, it seems natural to think that linguistic difficulties can significantly affect the PISA competencies such as argumentation and communication, and of course, the deployment of the remaining competencies (thinking and reasoning, problem solving, etc.) demands a command of the natural language.

According to Groth and Bergner (2006), conceptual knowledge may be thought of as a connected web of knowledge; a cognitive network in which relations between nodes are as important as the discrete pieces of information constituting these nodes. For Hiebert and Lefevre (1986), conceptual knowledge is needed to understand problems and to generate new strategies or adapt known strategies to solve original problems. Although there is broad agreement on the importance of conceptual knowledge, it is less clear how this knowledge can be elicited and made to work in an integrated fashion with contextual and procedural knowledge when solving problems.

Procedural knowledge specifically nurtures the skills needed for solving problems. Procedures are a type of knowledge which implies step-by-step actions and simultaneous learning of all the components or steps of what may be a complex procedure.

Hiebert and Lefevre (1986) describe two types of procedural knowledge: knowledge of the formal language or system of symbolic representation of mathematical ideas and knowledge of the rules or algorithms used to solve mathematical tasks. They point out that whereas procedural knowledge may or may not be learned meaningfully, conceptual knowledge must be learned meaningfully. In this study, I understand procedural knowledge to be the second kind described by these authors, i.e. the knowledge of rules and algorithms used to solve mathematical tasks.

For various authors (Nesher 1986; Rittle-Johnson and Koedinger 2005), conceptual understanding and procedural skills are related in that the mastery of a mathematical object is not a single-phase process, but rather is developed iteratively between both types of knowledge. Conceptual understanding develops the system of cognitive control needed to execute the algorithm correctly, and conversely, procedural knowledge contributes to the understanding of the mathematical object.

2.3 The competencies

Competencies are the cognitive processes which should be activated to connect the real world in which the problem arises with mathematics and to solve the problem posed. They

represent the third dimension of the functional PISA model. To assess the level of overall competence (mathematical literacy) of the students, PISA is based on seven specific competencies, which are described below together with some indicators (OECD 2003):

1. *Thinking and reasoning*. This involves posing questions characteristic of mathematics ("How many… are there?" "How do we find…?"), knowing the kinds of answers that mathematics offers to such questions, distinguishing between different kinds of statements (definitions, theorems, conjectures, hypotheses, examples, conditioned assertions) and understanding and handling the extent and limits of given mathematical concepts.
2. *Argumentation*. This involves knowing what mathematical proofs are and how they differ from other kinds of mathematical reasoning, following and assessing chains of mathematical arguments of different types, developing intuitive processes and creating and expressing mathematical arguments.
3. *Communication*. This involves expressing oneself, in a variety of ways, on matters with a mathematical content, in oral as well as in written form, and understanding others' written or oral statements about such matters.
4. *Modelling*. This involves structuring the situation to be modelled; translating "reality" into mathematical structures; working with a mathematical model; validating the model; reflecting, analysing and offering a critique of a model and its results; communicating effectively about the model and its results (including the limitations of such results) and monitoring and controlling the modelling process.
5. *Problem posing and solving*. This involves posing, formulating and defining different kinds of mathematical problems and solving different kinds of problems in a variety of ways.
6. *Representation*. This involves decoding and encoding, translating, interpreting and distinguishing between different forms of representation of mathematical objects and situations and the interrelationships between the various representations; choosing and switching between different forms of representation, according to situation and purpose.
7. *Using symbolic, formal and technical language and operations*. This involves decoding and interpreting symbolic and formal language and understanding its relationship to natural language; translating from natural language to symbolic/formal language; handling statements and expressions containing symbols and formulae; using variables, solving equations and undertaking calculations.

Described in this way, the competencies are not task variables but subject variables, and for this reason, it is not possible a priori to establish which processes a student will activate to answer a proposed question. In general, it will be possible to link a task with various cognitive processes given that the subjects who resolve them may do so in different ways. For this reason, the strategy followed in the design of the PISA tasks involves including a new task variable which measures the complexity of the task. This variable considers three levels of task complexity with respect to the competencies required to solve it. OECD (2003) describes the three levels with their respective indicators as follows:

1. *Reproduction*: Familiar contexts, knowledge which has already been used, the application of standard algorithms, performing simple operations, using elemental formulae.
2. *Connection*: Less familiar contexts, interpreting and explaining, handling and relating different systems of representation, selecting and using strategies for solving non-standard problems.

3. *Reflection*: Tasks which require understanding, reflection, creativity, relating knowledge to resolve complex problems, generalisation and justification of the results obtained.

3 The goals of the study

Our aim is to identify the difficulties which the subject has in activating mathematical competencies depending on his or her mastery of school mathematics and the characteristics of the tasks proposed by PISA. To this end, our method is to study the type and level of organisation of school knowledge shown by the subjects when they solve these tasks. Thus, the goals of the study are:

1. To categorise the PISA tasks according to the types of knowledge (contextual, conceptual and procedural) required to solve them, i.e. to characterise relation 2 in Fig. 1.
2. To analyse the degree of empirical concordance between the classification of tasks according to the levels of complexity of the competencies (reproduction, connection and reflection) and their classification according to the types of knowledge required, i.e. to characterise relation 3 in Fig. 1.
3. To study the role played by the type and organisation of mathematical knowledge in the activation of competencies identified by PISA with particular attention to the function of contextual knowledge, i.e. to characterise in more depth relation 3 in Fig. 1.

4 The research methodology

4.1 Sample

The participants were 140 pre-service elementary teachers from the Autonomous University of Madrid, chosen from a sampling based on groups/classes.

They were first-year students and had not yet taken any course in teaching mathematics. Their most advanced mathematics studies were secondary-level mathematics. They had no prior experience of the PISA tests.

We chose first-year student teachers because the research was designed as an initial diagnostic evaluation to form the basis of a teaching training that allows good educational practice in compulsory education in accordance with the learning needs derived from the PISA approach.

4.2 Assessment instrument

The participants took a mathematics test composed of the 39 items released by the OECD from those used in PISA 2003; items 18, 24, 28, 29, 30, 31 and 32 were eliminated from the test, as their data was not available (INECSE 2005, available from http://www.oecd.org/). Three booklets were issued, so that each subject had to deal with a third of the released mathematics items, following an analogous procedure to that carried out in PISA 2003 with 15-year-old students (OECD 2004). The items were distributed in booklets in such a way that all the booklets had the same level of complexity, i.e. the proportion of reproduction, connection and reflection items was equivalent in each of the booklets. The booklets were distributed at random and in equal numbers to the subjects. Once the questions had been

answered, an analysis of variance was undertaken. No significant differences were found (at the 0.05 level) in the performance of the three groups.

The Appendix includes an example of the items, including those which proved most difficult for the student teachers, which are analysed in the section dealing with the results.

4.3 Data collection and analysis: research phases

1. Analysis by a group of experts in mathematical education (three secondary- and university-level mathematics teachers with teaching and research experience) of each released item according to the type of knowledge the solution demands. For this purpose, they used the Rittle-Johnson and Koedinger (2005) definition of contextual, conceptual and procedural knowledge. The degree of agreement between the three experts was verified applying Kendall's coefficient of concordance. This enabled the hypothesis of zero concordance to be rejected (at a 0.05 level of significance) and the conclusion that there was significant agreement between the valuations of the experts.
2. Test carried out by the student teachers. The difficulty of the items was classified according to the percentage of correct answers to each one. The following two checks were made using these data to test the theoretical classification established in phase 1: (a) the extent to which the data related to difficulty reflect the classification of the tasks according to type of knowledge required and (b) the degree to which there is concordance between the classification of the tasks according to the type of knowledge and according to the level of complexity of the competencies established by PISA 2003 (reproduction, connection and reflection).
3. Analysis of the responses of the subjects, identification of the errors committed and their classification according to the kind of deficiencies in knowledge shown with respect to the demands made by the tasks. Error identification was chosen because it has been shown that this is a good method of explaining the mathematical knowledge of the subjects by default (Movshovitz-Hadar et al. 1987; Radatz 1980). The classification of the errors committed according to the kind of deficiencies in knowledge shown with respect to the demands made by the tasks was carried out independently by each expert. The degree of agreement between the three experts was verified applying Kendall's coefficient of concordance. This enabled the hypothesis of zero concordance to be rejected (at a 0.05 level of significance) and the conclusion that there was significant agreement between the valuations of the experts.
4. Establishing the relationship between the type and organisation of mathematical knowledge of the subjects and the activation of their mathematical competencies according to the demands of the task. To establish this relationship, the errors committed by the subjects were analysed again, but this time from the perspective of difficulties in activating the competencies, using as a guide the indicators associated with the competencies proposed by PISA (see Section 2.3).

5 Results

5.1 Results of phases 1 and 2 of the research

Table 1 presents a classification from more to less difficulty of the items, according to the percentage of correct answers given by the student teachers. Column 3 indicates the level of

Table 1 Classification of the PISA items by three variables

Item no.	Percent correct	Level of complexity	Type of knowledge
2	12.1	Connections	Ct, C, P
8	18.9	Connections	Ct, C, P
7	22.2	Connections	Ct, C, P
10	35.8	Reflection	Ct, C, P
17	39.6	Reproduction	C, P
38	40.7	Reflection	Ct, C, P
36	44.4	Connections	Ct, C
22	52.8	Connections	C, P
23	56.6	Connections	Ct, C, P
27	61.1	Connections	P
9	62.3	Connections	Ct, P
26	66.7	Reproduction	P
21	66.7	Reflection	C
20	66.7	Reflection	Ct, C
33	67.9	Reproduction	Ct, P
5	69.8	Connections	C, P
25	72.2	Reproduction	P
1	75	Reproduction	Ct, P
13	75.8	Reflection	Ct, P
6	77.4	Reproduction	P
16	77.8	Reproduction	C, P
34	78.8	Connections	P
15	78.8	Connections	P
19	79.2	Connections	P
4	79.2	Reproduction	P
37	81.5	Reproduction	P
39	87.9	Reproduction	P
11	87.9	Reproduction	P
3	88.9	Reproduction	P
35	90.9	Connections	P
12	90.9	Reproduction	P
14	93.9	Reproduction	P

theoretical complexity of the item (reproduction, connections and reflection) established in PISA (OECD 2004). Column 4 specifies the type of knowledge needed to solve the problem, determined by the group of experts: contextual (Ct), conceptual (C) and procedural (P).

The profile of the performance of student teachers responds to the level of complexity of the items defined by PISA: seven of the eight most difficult items are connection or reflection items, and conversely, seven of the eight easiest items are reproduction items. Table 2 shows the differences in the percentages of correct answers across the three levels

Table 2 Percentages of correct answers across the three levels of complexity

Level of complexity	Percent correct
Reproduction	78
Connections	53
Reflection	51

of complexity. An analysis of variance of one factor (level of complexity) was executed with repeated measures. The differences are statistically significant (at a level of 0.05) in the percentage of correct answers to the reproduction tasks compared to those for connections and reflection: more than 50% of the variance in the percentage of correct answers is explained by the level of complexity of the tasks (η^2=0.57).

With regard to the difficulty of the tasks in relation to the type of knowledge required for solving them, we can see that the 11 easiest tasks only demand procedural knowledge when their content is analysed theoretically, whilst the four most difficult tasks require all three types of knowledge. This is a valuable initial piece of information for the verification of the schema characterising the PISA tasks according to the type of knowledge they require.

There is also a high degree of concordance in classifying the tasks according to the type of knowledge and the level of complexity of the competencies established in PISA 2003: the reproduction tasks basically demand procedural knowledge, and the need to integrate the three types of knowledge only arises in the connections or reflection tasks, as can be seen in Table 3 (the numbers in the cells refer to the number of items in each category).

5.2 Results of phases 3 and 4 of the research

The easy tasks give very little information in terms of establishing relationships between activated competencies and the type of knowledge used because the few errors committed can be reduced to technical errors in simple algorithms (procedural knowledge). In terms of competencies, these errors correspond to difficulties with competence 7 ("using symbolic, formal and technical language and operations"; see Section 2.3).

Analysis of the errors in the most difficult tasks reveals patterns which allow us to relate the main type of knowledge and the subject's competencies in more depth. We have already seen (Table 1) that the solution of the most difficult tasks demands that the subjects use all three types of knowledge, but when we analyse them from the point of view of errors committed, there is usually one kind of knowledge which is of basic importance. Analogously, when we analyse the errors according to the indicators of the various competencies, the activation of one of the competencies is revealed as essential. Table 4 is a result of cross-analysing the mistakes from both perspectives in the most difficult tasks. The number in each cell refers to the number of the item.

Next, we include a summary of the analysis of the eight most difficult tasks for student teachers to illustrate how this relationship was constructed (the tasks appear in the Appendix). The handling of the school knowledge variable is based on the identification carried out by experts of the type of knowledge required for each task (see Table 1). We have analysed the competence variable according to the set of indicators for each competence (see Section 2.3). The most common wrong answers given by the subjects are included between quotation marks.

Table 3 Number of items in each category

	P	C	Ct	P and Ct	P and C	C and Ct	P, C and Ct
Reproduction	10			2	2		
Connections	5			1	2	1	4
Reflection		1		1		1	2

Table 4 Classification of the most difficult items according to the knowledge and competencies mobilised (the number in each cell refers to the number of the ítem)

		School knowledge		
		Contextual	Conceptual	Procedural
Competencies	Thinking and reasoning		17, 8	
	Argumentation	36		
	Communication	7		
	Modelling	38	38	38
	Problem posing and solving	10	10	10
	Representation			22
	Using symbolic language, …			2

5.2.1 Difficulties with contextual knowledge

The following are the most common manifestations of these difficulties:

1. Using natural language in a generic way without the necessary mathematical detail or rigour. For example, in task 7, basing the argument on the idea that "the increase in the actual number of robberies is not a large increase".
2. It is more difficult to express mathematical arguments when only using natural language than when using numbers or formulae. For example, in task 7, the argument is much clearer when expressed as "the increase is, approximately, 10/500, which is small" than when the argument tries to say the same thing with words, but ends up saying something else (the subjects have difficulty using terms such as proportion or proportional).
3. Difficulties related to the processing of information on charts. These difficulties arise in the case of charts which are unfamiliar in the school environment, such as the bar chart in task 7 where the vertical axis is truncated at the bottom. Many students' reasoning is based on visual aspects such as the appearance of the chart. They do not argue in terms of the percentage increase in robberies or in terms of trends, which would be correct.
4. Using commonplace ideas suggested by the context of the problem without any relevance to the mathematical reasoning required. For example, in task 7, to state that "the presenter is right because robberies and crime have been increasing a great deal lately", without any reference to the chart. Or in task 36, to explain that "all the surveys fail and that the best forecast of votes is the ballot box" or to argue that "those who make the phone calls show greater interest".
5. Not all the properties, characteristics or attributes forming a mathematical concept or object are taken into account in the mathematical argument. Thus, in task 36, a correct answer involves verifying each of the characteristics of an election survey carried out correctly with statistical rigour: random choice of sample, adequate sample size, the right to vote and the closeness of the elections. In fact, 41% of the subjects chose newspaper 4 because "many people are questioned and the date is close to election day". They do not take into account that the survey is by telephone.

From the point of view of competencies, these difficulties have to do with difficulties in *creating and expressing mathematical arguments* and *expressing oneself, in a variety of ways, on matters with a mathematical content*. These indicators point to the competencies of argumentation and communication (see Section 2.3).

5.2.2 Difficulties with conceptual knowledge

Deficient conceptual understanding is manifested in various ways:

1. The routine mastery of a procedure or algorithm may conceal deficiencies in basic conceptual knowledge. Thus, in task 17, the main error made by the subjects consists in applying the algorithm of the average without bearing in mind the weighting, so that they calculated an average of "70 marks: (60+80)/2". However, deep understanding of the concept of average as a measure of the central position means bearing in mind all the distribution values.
2. In item 17, some subjects conclude that the average is 28, calculating (60+80)/5, without taking into account that the average has to be a value belonging to the range of the variable.
3. The visual aspects, the appearance of the diagram or figure, prevail over the formal attributes or characteristics of a mathematical concept. Thus, in task 8, certain answers (for example, A with C and B with D) appear to have been coupled together more because of the shape of the parterre rather than as a response to the formal concept of a figure's perimeter.
4. There are difficulties linked to cultural factors. As PISA is an international assessment, the formulation of the tasks may pose problems of comprehension in some countries. Thus, some Spanish subjects say that "they cannot answer task 8 because they do not understand the question". The fact is that the typical Spanish garden is surrounded by high wire fences and not by a simple wooden dividing fence. In addition, the word *parterre* used in the original is a Gallicism which is not very common in Spanish.

From the perspective of competencies, these difficulties are related to obstacles to *understanding and handling the extent and limits of given mathematical concepts*, which is an indicator of the competence of thinking and reasoning (see Section 2.3).

5.2.3 Difficulties with procedural knowledge

Common difficulties and mistakes appear in the handling of algorithms, formulae and graphs which reveal insufficient procedural knowledge. Specifically:

1. Difficulty in handling units of measurement, above all when they are not conventional. For example, task 2 requires the handling of a formula in an analogous way to typical school exercises. However, those taking the test do not bear in mind that, in the formula $n/p=140$, n is a velocity in steps per minute and p is the length (of step) in metres, so that they find it particularly difficult to move from steps per minute to metres per minute. They find it difficult to understand that steps per minute is a speed, although the step is not a conventional unit of length.
2. The mistake of not verifying the solution derived from the application of a procedure or algorithm. Thus, also in task 2, some subjects calculated that "Bernardo walked at 8,960 km/h", without questioning this result at all.
3. Difficulty with the combinatorial schema. In task 22, the mistakes made by students derive from the lack of a strategy of systematically counting all the possible groupings of the elements of a set. For example, they do not use the technique of the tree diagram. And 35% of the subjects say that there are 12 combinations, using justifications such as $3\times4=12$.

From the perspective of competencies, these difficulties suggest barriers to activating capacities such as *handling statements and expressions containing symbols and formulae and using variables, solving equations and undertaking calculations* (capacities which are

indicators of the competence of "using symbolic, formal and technical language and operations"). They are also obstacles to activating the capacity for *decoding, interpreting and distinguishing between different forms of representation of mathematical objects and situations, and the interrelationships between the various representations* (indicator of the competence for representation).

5.2.4 Deficiencies in integrating knowledge

There are two tasks, 10 and 38, in which an analysis of errors shows not so much a lack or misuse of school concepts or procedures as the incapacity for structuring knowledge in processes of mathematisation involving a certain originality or novelty. Thus, the high number of subjects who do not answer task 38 is very significant (25%), when compared with task 37, which is set in the same problem situation, but was answered by 100% of the subjects. Those who did try to answer it had many difficulties with the handling of the equation because they are used to substituting values in the variables but they are not used to manipulating the coefficients of the equation in searching for a new mathematical model; in this case, one which optimizes the *Ca* car (in task 37), they do so very well: 81.5% of correct answers compared to 40.7% correct answers for item 38 (see Table 1). This difficulty of *working with a mathematical model* points directly to the modelling competence (see Section 2.3).

Task 10 poses an original problem whose solution demands activation of the problem-solving competence rather than specific conceptual or procedural knowledge. This item highlights the difficulties associated with this complex mathematical competence. Most of those answering the task used poor, rigid and irrelevant strategies, such as unsystematic trial and error or "guesstimates". Very few used two parallel temporal axes to identify the periods at which times coincided. And in general, they did not check the process or results of the solution, so that they did not verify whether the solution they found fit the problem data. A typical wrong answer was "Sydney 7 A.M., Berlin 11 P.M."

Those responsible for the PISA study recognise the difficulty involved in assessing a competence which is as complex as mathematisation through a simple written pen-and-paper test under limited time constraints. Indeed, the complete process of mathematical activation, which starts with a real problem, moves on to the mathematical solution and then back to reality to verify the appropriateness of this solution, demands considerable resources in terms of time, information search and collaboration. Because of this, PISA opts for preparing a set of items which assess different aspects of the process of mathematisation. This may explain the difficulty of documenting and verifying some competencies such as modelling or problem solving which deal directly with the activity of mathematisation itself.

6 Conclusions

The classification of the PISA tasks according to the types of school knowledge required for solving them, following the approach of Rittle-Johnson and Koedinger (2005), has been shown to be a powerful tool for understanding the relationships which underlie the PISA functional model (see Fig. 1). This theoretical classification corresponds to the classification according to the degree of difficulty of the tasks, measured as a percentage of correct answers, and to the classification according to the level of complexity of the activated competencies, measured in terms of reproduction, connection and reflection (see Table 1).

A twofold analysis of errors made by those answering the tests has also turned out to be a very useful mechanism for understanding the functional PISA model. The errors can be classified into those related to insufficiencies and limitations in school knowledge and difficulties in activating competencies. This analysis has shown the importance of contextual knowledge as a link between traditional and school mathematics (which is expressed as concepts and procedures) and the activation of competencies required to solve the PISA tasks (aimed at expressing the functionality of knowledge). In fact, contextual knowledge appears in our research as a very important tool for solving the most difficult tasks: those which demand something more than a simple reproduction of algorithms or formulae.

The importance of this finding is that it puts the focus of attention on a connecting link between the variable controlled by the teacher (school mathematics) and the variable which cannot be handled directly, but which is the object of mathematical training (the activation of competencies). For example, the capacity to reject implausible solutions to a problem, which is an indicator that cuts across various PISA competencies, appears to be linked to procedural knowledge through contextual knowledge (see the analysis of task 2 above).

To take a second example, PISA proposes many tasks requiring the capacity for understanding and processing information given in the form of graphs. Those attempting to solve the tasks give evidence of many difficulties in using visual representations of the problems. As a result, they are influenced by contextual and superficial characteristics and do not comprehend the mathematical concepts involved (such as the proportion in task 7 or the perimeter in task 8). The difficulties in obtaining spatial information, already highlighted by Radatz (1980), here emerge linked to insufficiencies in contextual knowledge. In task 8, there are also difficulties derived from cultural factors, another manifestation of the significance of contextual knowledge.

Unlike contextual knowledge, the other types of knowledge (conceptual and procedural) are the object of a long research tradition. Pesek and Kirshner (2000) warn that the emphasis on procedural learning with a superficial conceptual base could interfere with the development of a sophisticated conceptual understanding. We have found support for this idea in the errors committed by the subjects. For example, the wrong answers appearing in task 17 as a result of the hurried application of the algorithm of the mean ("add and divide") rather than calculating with weighted measures. We consider that giving excessive importance to the algorithm over the underlying concept is particularly serious among those who are to be future teachers.

We have also found empirical support for the two types of procedural knowledge described by Hiebert and Lefevre (1986). In item 2, the main mistake arises from the lack of expertise in manipulating expressions containing formulas (first type of procedural knowledge according to the authors mentioned). In item 22, the basic mistake is related to weakness in the symbolic representation of a mathematical idea (second type of procedural knowledge); in this case, the tree diagrams representing a combinatorial problem.

Based on the idea that the complexity of a procedure depends on the number of steps which constitute it, Rittle-Johnson and Koedinger (2005) state that the use of a procedural scaffold (which consists in providing a step or action from the set of steps which the solution of the problem demands) facilitates the solution of the problem. This form of help could prevent the working memory of participants from collapsing and free cognitive resources for identifying relevant information and successfully completing other steps or components in the task. We have an example in task 2 of a procedure of certain complexity where the difficulty resides in a specific step of the procedure: practically 100% of the subjects complete the first calculation ($140 \times 0.8 = 112$), but there is a substantial drop in the

number of subjects who complete the second correct calculation (112×0.8) to obtain the speed in metres per minute. Here, the actual data involved within the context of the task influence their misuse. As a result, the students make a mistake when they do not multiply again by 0.8 to change from steps per minute to metres per minute.

We consider that classifying the PISA tasks according to the type of knowledge required is consistent with the proposal made by Rico (2006) of establishing the meaning of a school mathematical concept through the representation–phenomenon–definition triad (itself based on Frege's triad: sign, sense and reference). Rico states that we have to understand that there is a plurality of senses for the same mathematical concept because they can be represented by different systems of signs and there are various phenomena which refer and give different senses to the same concept. Being aware of the different senses for the same concept makes it easier to understand and thus use the concept in problem fields which may be far removed in everyday life.

Normally, the student teachers linked each concept with only one sense or phenomenon (the standard problem, which tends to appear in textbooks) and were not aware of the richness, breadth and relevance of the idea that mathematics has generated for applying to different situations in the real world. We found that the lack of conceptual profundity in understanding mathematical objects means that the only tools they used to tackle the mathematical tasks were elemental algorithms and stereotypical rules with a minimum of underlying conceptual knowledge. When the tasks were simple (reproduction, in PISA terms), they carried them out well, but as they become more complex (connection or reflection), problems arose because of the lack of the cognitive networks characterising deep conceptual understanding. In all, the results of the study suggest serious difficulties for future teachers when it comes to leading a process of teaching/learning with their students in line with the PISA proposals.

In our research, we have noted the difficulty in activating the competencies of argumentation and communication. We found that, among the nine most difficult tasks, there are three (7, 36 and 23) which directly require these competencies. It is worth noting here a problem of interpretation in the analysis of those tasks which demand the creation or expression of mathematical arguments in writing. A distinction has to be made between weak mathematical argumentation and what is an insufficiency or poverty in the natural language, which has to do with contextual knowledge. Thus, we see that, in task 7, the students who basically use mathematical expression argue and communicate better than those who write the argument in natural language without the support of any mathematical formulae.

To sum up, the results of the research allow us to state that assessment of the mathematical competencies of participants (in PISA terms) includes assessment of the extent to which they have school mathematical knowledge (contextual, conceptual and procedural) which they can productively apply to problem situations of any kind (personal, working, public, scientific, etc.). In this sense, the variable of the type of school knowledge is a variable associated with the cognitive competence variable. A fortunate conclusion resulting from this fact is that designing teaching for the development of students' competencies is not radically different from working on concepts and procedures through the curriculum. However, it does mean that we should reflect on the methodology of teaching the contents to be included in the integration of contextual knowledge. This knowledge is an essential link between the traditional curriculum and the curriculum which underlies the PISA approach.

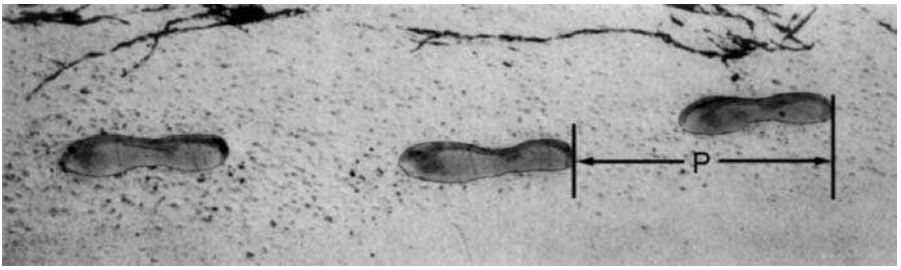
We consider that our study may provide useful information for designing the training of teachers at compulsory education level. We should start with organising their mathematical

knowledge to make it more functional; at the same time, they should be given the didactic tools needed to allow the development of functional mathematical thought in their students.

This study is only a starting point. We are aware that more research is needed, using more sophisticated statistical procedures, if we are to generalise the results we have found and state them in terms of strong causal statements. The classroom performance of these teachers during their career would then have to be monitored to study the relations between their knowledge, instructional conduct and the performance of their students.

Appendix

WALKING



The picture shows the footprints of a man walking. The pacelength P is the distance between the rear of two consecutive footprints.

For men, the formula, $\frac{n}{P} = 140$, gives an approximate relationship between n and P where,

n = number of steps per minute, and

P = pacelength in metres.

Question 1: WALKING

If the formula applies to Heiko's walking and Heiko takes 70 steps per minute, what is Heiko's pacelength? Show your work.

Question 2: WALKING

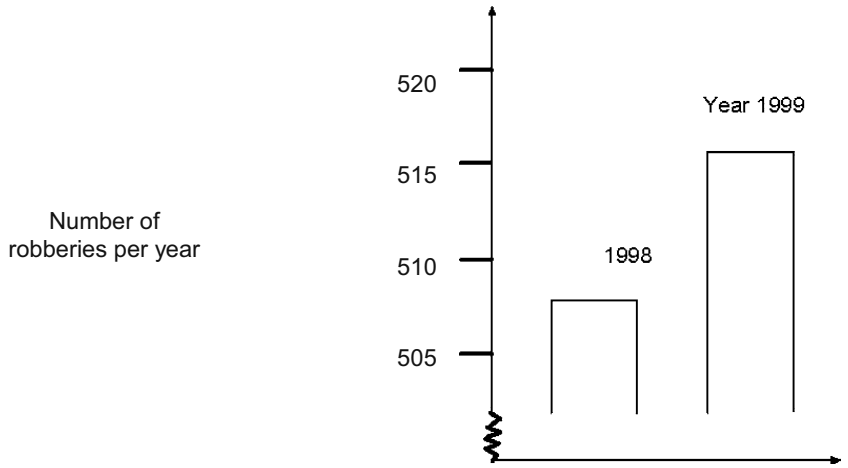
Bernard knows his pacelength is 0.80 metres. The formula applies to Bernard's walking.
Calculate Bernard's walking speed in metres per minute and in kilometres per hour. Show your working out.

ROBBERIES

Question 7: ROBBERIES

A TV reporter showed this graph and said:

"The graph shows that there is a huge increase in the number of robberies from 1998 to 1999."

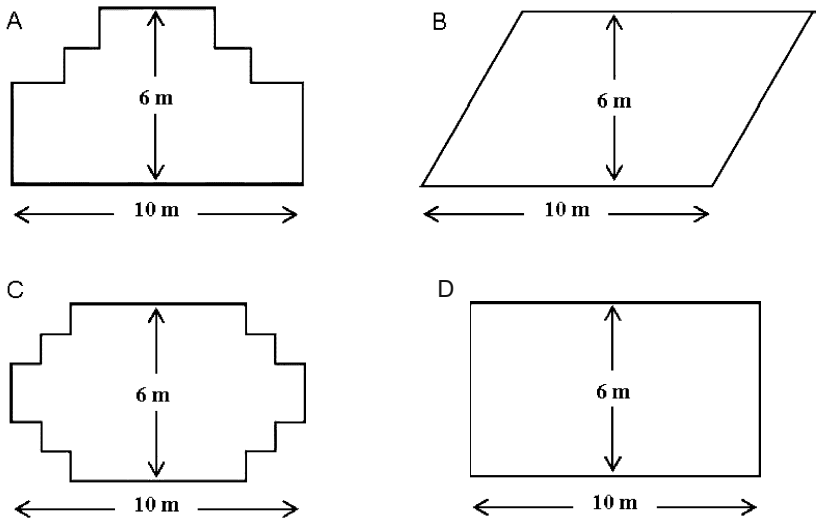


Do you consider the reporter's statement to be a reasonable interpretation of the graph? Give an explanation to support your answer.

CARPENTER

Question 8: CARPENTER

A carpenter has 32 metres of timber and wants to make a border around a garden bed. He is considering the following designs for the garden bed.



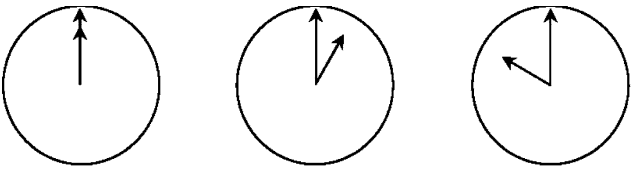
Circle either "Yes" or "No" for each design to indicate whether he garden bed can be made with 32 metres of timber.

Garden bed design	Using this design, can the garden bed be made with 32 metres of timber?
Design A	Yes / No
Design B	Yes / No
Design C	Yes / No
Design D	Yes / No

INTERNET RELAY CHAT

Mark (from Sydney, Australia) and Hans (from Berlin, Germany) often communicate with each other using “chat” on the Internet. They have to log on to the Internet at the same time to be able to chat.

To find a suitable time to chat, Mark looked up a chart of world times and found the following:



Question 9: INTERNET RELAY CHAT

At 7:00 PM in Sydney, what time is it in Berlin?

Answer: .....

Question 10: INTERNET RELAY CHAT

Mark and Hans are not able to chat between 9:00 AM and 4:30 PM their local time, as they have to go to school. Also, from 11:00 PM till 7:00 AM their local time they won’t be able to chat because they will be sleeping.

When would be a good time for Mark and Hans to chat? Write the local times in the table.

Place	Time
Sydney	
Berlin	

SCIENCE TESTS

Question 17: SCIENCE TESTS

In Mei Lin's school, her science teacher gives tests that are marked out of 100. Mei Lin has an average of 60 marks on her first four Science tests. On the fifth test she got 80 marks.

What is the average of Mei Lin's marks in Science after all five tests?

Average: .....

CHOICES

Question 22: CHOICES

In a pizza restaurant, you can get a basic pizza with two toppings: cheese and tomato. You can also make up your own pizza with **extra** toppings. You can choose from four different extra toppings: olives, ham, mushrooms and salami.

Ross wants to order a pizza with two different **extra** toppings.

How many different combinations can Ross choose from?

Answer: ..... combinations.

SUPPORT FOR THE PRESIDENT

Question 36: SUPPORT FOR THE PRESIDENT

In Zedland, opinion polls were conducted to find out the level of support for the President in the forthcoming election. Four newspaper publishers did separate nationwide polls. The results for the four newspaper polls are shown below:

Newspaper 1: 36.5% (poll conducted on January 6, with a sample of 500 randomly selected citizens with voting rights)

Newspaper 2: 41.0% (poll conducted on January 20, with a sample of 500 randomly selected citizens with voting rights)

Newspaper 3: 39.0% (poll conducted on January 20, with a sample of 1000 randomly selected citizens with voting rights)

Newspaper 4: 44.5% (poll conducted on January 20, with 1000 readers phoning in to vote).

Which newspaper's result is likely to be the best for predicting the level of support for the President if the election is held on January 25? Give two reasons to support your answer.

THE BEST CAR

A car magazine uses a rating system to evaluate new cars, and gives the award of "The Car of the Year" to the car with the highest total score. Five new cars are being evaluated, and their ratings are shown in the table.

Car	Safety Features (S)	Fuel Efficiency (F)	External Appearance (E)	Internal Fittings (T)
Ca	3	1	2	3
M2	2	2	2	2
Sp	3	1	3	2
N1	1	3	3	3
KK	3	2	3	2

The ratings are interpreted as follows:

3 points = Excellent
2 points = Good
1 point = Fair

Question 37: THE BEST CAR

To calculate the total score for a car, the car magazine uses the following rule, which is a weighted sum of the individual score points:

Total Score = (3 x S) + F + E + T

Calculate the total score for Car "Ca". Write your answer in the space below.

Total score for "Ca": .....

Question 38: THE BEST CAR

The manufacturer of car "Ca" thought the rule for the total score was unfair.

Write down a rule for calculating the total score so that Car "Ca" will be the winner.

Your rule should include all four of the variables, and you should write down your rule by filling in positive numbers in the four spaces in the equation below.

Total score = ……× S + ……× F + ……× E + ……× T.

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