Sci & Educ (2011) 20:831–862
DOI 10.1007/s11191-010-9315-2

# New Avenues for History in Mathematics Education: Mathematical Competencies and Anchoring

**Uffe Thomas Jankvist · Tinne Hoff Kjeldsen**

Published online: 21 November 2010
© Springer Science+Business Media B.V. 2010

**Abstract** The paper addresses the apparent lack of impact of 'history in mathematics education' in mathematics education research in general, and proposes new avenues for research. We identify two general scenarios of integrating history in mathematics education that each gives rise to different problems. The first scenario occurs when history is used as a 'tool' for the learning and teaching of mathematics, the second when history of mathematics as a 'goal' is pursued as an integral part of mathematics education. We introduce a multiple-perspective approach to history, and suggest that research on history in mathematics education follows one of two different avenues in dealing with these scenarios. The first is to focus on students' development of mathematical competencies when history is used a tool for the learning of curriculum-dictated mathematical in-issues. A framework for this is described. Secondly, when using history as a goal it is argued that an anchoring of the meta-issues in the related in-issues is essential, and a framework for this is given. Both frameworks are illustrated through empirical examples.

## 1 Introduction

History of mathematics in the teaching and learning of mathematics is an area which has attracted an increasing amount of interest within didactics of mathematics in the past four decades or so. This interest has resulted in several publications, newsletters, conferences

U. T. Jankvist (✉)
Department of Mathematics and Computer Science, Southern University of Denmark, Odense, Denmark
e-mail: utj@imada.sdu.dk

T. H. Kjeldsen
IMFUFA, Department of Science, Systems and Models, Roskilde University, Roskilde, Denmark
e-mail: thk@ruc.dk

more or less dedicated to this topic as well as working groups at more general conferences on mathematics education.[1]

Despite this increasing interest, the actual *impact* of the field of history in mathematics education, in the field of mathematics education research (MER) in general as well as in education practices, appears very little or close to none at all. Of course, history of mathematics is part of some mathematics programs across the world, but not many.[2] And of course some empirical studies have been carried out by researchers within the HPM (International Study Group on the Relations between the History and Pedagogy of Mathematics), but on the overall this amount is very small[3]—in particular, compared to the amount of empirical research conducted within the PME (International Group for the Psychology of Mathematics Education). This lack of impact is evident from the fact that the general mathematics education conferences, such as ICME (International Congress on Mathematical Education) and CERME (Congress of the European Society for Research in Mathematics Education), rarely (or never) have had any plenary lectures on the use of history in mathematics education, and that the previously published proceedings from the centennial celebration of ICMI (International Commission on Mathematical Instruction) in Rome (Menghini et al. 2009) does not touch upon the use of history in mathematics education, despite HPM being one of the two oldest affiliated study groups of ICMI (Jankvist 2010b).

In this paper we shall first touch on some possible reasons for this lack of impact. Then we identify two scenarios occurring when integrating history in mathematics education, and propose new avenues for research which we believe could have some impact and help put history in mathematics education on the agenda of MER. The treatment of these avenues requires the development of theoretical frameworks which we address next, followed by some illustrative, empirical examples. The paper ends with a concluding discussion of the questions raised and problems pointed to along the way, suggestions for dealing with these, and the possible impact of the proposed new avenues.

[1] Examples are: *The International Study Group on the Relations between the History and Pedagogy of Mathematics* (HPM), which together with the *International Group for the Psychology of Mathematics Education* (PME), became one of the first affiliated study groups of the *International Commission on Mathematical Instruction* (ICMI) in 1976 (Fasanelli and Fauvel 2007). HPM holds a satellite conference to ICME (which itself also has a topic study groups on history) every fourth year, alternating with the French initiative, the now quadrennial *European Summer University on the History and Epistemology in Mathematics Education* (ESU), which began in 1993 (Barbin et al. 2007). The years in between HPM and ESU, the *Congress of the European Society for Research in Mathematics Education* (CERME) has a working group dedicated to *History in Mathematics Education*, which began in Lyon 2009 and takes place again at CERME 7 in Poland, 2011. On top of this the *HPM Americas* section holds meetings every year. All these conferences and meetings produce proceedings thus adding to the amount of papers on history in mathematics education. The increasing interest can also be seen from papers published in more general journals on mathematics education such as *Educational Studies in Mathematics* (ESM) and *For the Learning of Mathematics* (FLM) of which special issues on history in mathematics education have appeared in 2007 and 1991, respectively. And also entire books on the subject are available: Fauvel (1990), Swetz et al. (1995), Jahnke et al. (1996), Calinger (1996), Katz (2000), Shell-Gellasch and Jardine (2005), and most importantly Fauvel and van Maanen (2000)—the most comprehensive sample on the topic to date.

[2] For examples see Fauvel and van Maanen (2000).

[3] For a survey of empirical studies on the use of history in mathematics, see Jankvist (2009c).

## 2 Two Different Purposes for Using History

In the literature there are different classifications of the 'whys' for using history in mathematics education (e.g. Fauvel 1991; Tzanakis and Arcavi 2000; Gulikers and Blom 2001; Fried 2001 and Furinghetti 2004). However, generally speaking it may be argued that mainly two different purposes for using history have been given: *history as a goal* and *history as a tool* (Jankvist 2009a).

Regarding the first, it is *not* a matter of teaching history per se. The word *goal* must be understood in the sense that it is considered a goal to show the students something about the historical development of mathematics. For example to show them that mathematics exists and evolves in time and space; that it is a discipline which has undergone an evolution over millennia; that mathematics has developed due to human activities; that the evolution of mathematics is due to many different cultures throughout history, and that these cultures have had an influence on the shaping of mathematics as well as the other way round. Of course, a use of history as a goal may have a positive side effect of assisting the learning of mathematics, but this is not the primary purpose for using history here.

That is, however, the case when history is used as a *tool*. In this sense, history is used as an assisting means, or an aid, in the teaching and learning of mathematics—it becomes a vehicle for teaching and learning mathematical ideas, concepts, theories, methods, algorithms, ways of argumentation and proof, etc. When restricting ourselves to the use of history as a tool, we may (sub) categorize such uses into at least three different types: history as a motivational and/or affective tool; history as a cognitive tool (e.g. the idea of epistemological obstacles); and the role of history in what may be referred to as the evolutionary arguments (the recapitulation argument or historical parallelism).[4]

Another way of distinguishing the use of history as a goal from that of a tool is to say that, where 'history as a goal' concerns the learning of some meta-perspective issues—or *meta-issues*—of mathematics and its history, 'history as a tool' focuses primarily on the internal or inner issues—*in-issues*—of mathematics itself. Of course, from a point of view of history of mathematics the realms of meta-issues and in-issues are not completely separable, because, as we explain in Sect. 4, we think of mathematics as a cultural and historical product of knowledge produced by human intellectual activities. This means, for example, that the meta-issue discussion of what mathematics is, is determined by its in-issues and at the same time the in-issues of mathematics are determined by what mathematics is. Nevertheless, from the perspective of doing research on history in mathematics education, the distinction between meta-issues and in-issues is a very useful analytical construct, in particular regarding the *intentions* of using history, as we shall see in the sections to come.

## 3 Identifying Two Scenarios Occuring When Integrating History

From the distribution of foci in the small number of empirical studies existing so far it is clear that researchers applying history in mathematics teaching and learning mainly use history as a tool—and in particular history as a cognitive tool (Jankvist 2009c).

From a non-historian's point of view one could rightfully ask why the history of mathematics should be the path to follow when wanting to teach students mathematical in-issues. Surely history has much to offer in relation to the teaching and learning of

[4] See Jankvist (2009a) for an elaboration of this.

mathematical in-issues, we of all acknowledge this, but at the same time we are not blind to the fact that the argument for history as a cognitive tool has been 'weak' in regard to impacting discourses in mathematics education research in general. One reason for this is that many other topics, methods, and approaches than just a use of history present themselves when it comes to teaching and learning mathematical concepts, theories, etc., for example mathematical modeling.[5] Another reason is that up to now, the only 'indispensable' argument for using history as a tool has been the evolutionary one (Jankvist 2009a). Based on Sfard's (2008) theory of thinking as communicating Kjeldsen and Blomhøj (Accepted) have argued that history may have a profound role to play in the teaching and learning of mathematics as an obvious (if not indispensable) strategy from the point of view of learning proper meta-discursive rules i.e. the meta rules of the mathematical discourse. A third reason is the problem, as pointed out by Fried (2001), that when using history as a cognitive tool there is a grave danger of the treatment of history becoming anachronical and 'Whig'[6]:

> […] in choosing and presenting historical material, mathematics educators must accommodate curricular needs, which are generally directed towards learning the kind of mathematics necessary for science and industry, that is, modern mathematics; they must filter history to suit predetermined needs, making history a tool to be used rather than a subject to be studied. However well intentioned, then, mathematics educators are forced into a Whiggist position – a position that has less to do with historical accuracy than with historical approach. (Fried 2008, p. 4)

According to Fried this might result in a dilemma for the teacher who wants to use history in mathematics education of either: (1) being "true to the commitments of mathematics education and be forced to adopt a Whiggist brand of history", or (2) being "true to the commitments of the history of mathematics and be forced to spend time on mathematical and philosophical ideas which may not be relevant to the general mathematics curriculum" (Fried 2008, p. 4). The mathematics curricula dictated by the mathematics necessary for science and industry, of which Fried talks, and degree level mathematics are curricula and syllabi which are most often 'concept driven'. That is to say, it is the mathematical concepts, methods, and theories that set the agenda for the mathematical activities that the students are to engage in. We shall illustrate below, how the dilemma pointed out by Fried can be resolved.

In the following we distinguish between two different scenarios:

1. one where history is not part of the existing curriculum, and therefore in some way or another must play the role of a tool in teaching the students topics (in-issues) within this curriculum, and
2. one where a curriculum includes elements of the history of mathematics with the intention of preserving the honesty to history and historical meta-issues as an intrinsic part, which is to say that the integration of history in a way relates to 'history as a goal'.

Where the first scenario introduces the problem of adopting a Whiggist brand of history, the second in fact also introduces a problem. In such existing curricula as embraced by scenario 2, there is a danger of the history becoming anecdotical, which is to say that the treatment of the historical issues becomes superficial and detached from the related

[5] The same argument can be put forward against a use of history as a motivating tool, plenty of other topics, perspectives, etc. may motivate students to study and learn mathematics.

[6] This refers to a reading of the past in which one tries to find the present. See also for example Schubring (2008), Kjeldsen (2010a), or Kjeldsen and Blomhøj (Accepted).

mathematical in-issues (Smestad 2002; Smestad 2003; Jankvist 2008a). The problem here then becomes how to bring about and to some extent 'ensure' that the students' treatments of and reflections about history in the selected cases are *anchored* in the mathematical content of these. By 'anchoring' we are referring to something that substantiates discussions and reflections about meta-issues on a basis of knowledge and understanding of the related in-issues, e.g. by revealing insights about the meta-issues that could not have been accessed or uncovered without knowing about the in-issues, or by providing in-issue evidence for meta-issue claims or viewpoints (Jankvist forthcoming).[7] From the point of view of using history as a tool, in-issues may to some extent refer to mathematical issues relevant to modern mathematics. However, when using history as a goal it is important to notice that the meta-issues may also be anchored in mathematical in-issues of the historical texts, which may differ from the modern ones. The important thing is that the meta-issue reflections rest on relevant mathematical content, which may be made up by historical as well as modern in-issues depending on the historical case in question.

In the following sections of this paper we shall propose ways of dealing with the problems related to the two scenarios above—ways, or *avenues*, that, if studied and investigated further, could help increase the impact of 'history in mathematics education' in mathematics education research in general. The way of dealing with the problem of the first scenario is to adopt a competence-based view of mathematics education, which was developed in the Danish so-called KOM-project[8] and amounted in a final report (Niss and Jensen 2002). In dealing with the problem of the second scenario, elements from the KOM-report, in particular the three so-called 'types of overview and judgment', are also adopted along with an introduction of so-called 'general topics and issues' of the history and historiography of mathematics (Jankvist 2010a). Common for both avenues is the reliance on a multiple-perspective approach to the history of the practice of mathematics (Kjeldsen forthcoming a). We explain this multiple-perspective approach first and then we describe the other theoretical constructs, and illustrate the use of them through actual empirical examples. The empirical examples we use to illustrate the first scenario are project work carried out by undergraduates studying for a degree in mathematics, whereas the empirical examples used to illustrate the second scenario are taken from an experimental study carried out in upper secondary school. However, we think our theoretical ideas regarding the first scenario can also be put into practice in general mathematics education.[9] This implies that we propose that a scholarly approach to history can develop students' mathematical competencies in e.g. secondary school. We are aware, though, that there are other ways to use past mathematics in mathematics education, for a discussion of this we refer to Kjeldsen (forthcoming b).

## 4 A Multiple-Perspective Approach to History of Mathematics

History of mathematics deals with how mathematics has developed over time, of course, but, as the discussion of a Whiggist brand of the history of mathematics indicates, historical consciousness can take quite different forms. We think of mathematics as a cultural and historical product of knowledge, created through human activities. Hence, we take

[7] The construct of anchoring has also been addressed and discussed as part of a plenary talk at the 6th ESU in Vienna. Interested readers are thus referred to the forthcoming proceedings from this conference.

[8] KOM is a Danish abbreviation for *competencies and mathematics learning*.

[9] See footnote 19.

concrete practices of mathematics as point of departure for understanding historical developments of mathematics. Generally speaking, as pointed out by Jensen (2003, p. 171), a strength of such a micro-history is that one's analyses find a rooting in the particular from where relations and connections can be unfolded. Regarding history of mathematics, such a micro-history also has the advantage of providing one with a way to disentangle one's historical analyses from the universal character of mathematics, and thereby avoiding the trap of Whig history.[10]

Taking the practice of mathematics as a starting point for history of mathematics, in order to understand how mathematics has been practiced and understood, how mathematical knowledge has been produced throughout time at different places, in different cultures, and in different intellectual contexts as well as how such understandings and productions have changed, localizes history of mathematics in space and/or intellectual milieus. Such an approach involves studying concrete episodes of the history of mathematics to gain insights into matters such as: why mathematicians asked the questions they did; why they treated the problems they dealt with in the way they did; what kinds of proofs or arguments they gave, and how these were perceived and received within the mathematical community(ies); why they introduced certain mathematical objects, definitions, areas of research etc.; and how all of the above influenced further developments in mathematics as well as changes in perceptions of mathematics.[11]

The ability to reflect upon one's own and other's mathematical practices from different angles is a premise for developing historical awareness in a qualified sense, i.e. acquiring insights into, on the one hand, how mathematicians have thought and acted in different intellectual contexts at different times, and on the other hand, how this have changed over time and through cultures. This point can be considered through a multiple perspective account where the history of mathematics is studied from different points of observation.[12] The perspectives can be of different kinds and the mathematics can be analyzed from different angles, e.g. from the nature of mathematics, from the point of view of applications, from proof techniques, from sub-disciplines of mathematics, from social influences, and so forth. This understanding of and approach to history can be implemented in mathematics education at a smaller scale, where the different angles are chosen with respect to the educational purpose(s) of using history in particular teaching and learning situations.

In the empirical examples belonging to the two scenarios which we present and unfold below, the students worked with different perspectives, and analyzed their historical sources from different angles. The methodological framework of epistemic objects, techniques and configurations[13] can be used in both kinds of scenarios to analyze the mathematics of historical sources, thereby having the students work with historical texts as historians, asking and answering historical questions about the mathematics. The term *epistemic objects* refers to the mathematical object(s) under investigation, i.e. the

[10] Recent history of mathematics has taken similar approaches, see e.g. Epple (2000), Corry (2004), Kjeldsen et al. (2004), Kjeldsen (2009a).

[11] See also Epple (2004, p. 133).

[12] The term 'multiple perspective' approach to history is borrowed from the Danish historian Bernard E. Jensen (Jensen 2003, p. 16). For its use in history in mathematics education, see also Kjeldsen (2010a), Kjeldsen (forthcoming a), Kjeldsen and Blomhøj (Accepted).

[13] These concepts have been introduced into the historiography of mathematics by Epple (2004) inspired by a methodological framework originally developed by Rheinberger (1997) for the study of experimental science.

object(s) about which new knowledge is searched for. *Epistemic techniques* refer to the techniques, or the methods, that are used to gain insights into the objects. These techniques can be already well-established mathematical concepts, theories, methods, etc., or they can be developed in the process of the investigation of the objects. *Epistemic configuration* refers to the intellectual resources available at the specific historical episode being studied. It is important to notice that there is a dynamical process inherent in the notions of epistemic objects and techniques, in the sense that what in some mathematical investigations functioned as objects might be turned into techniques later or in other investigations (e.g. Kjeldsen 2009b; Epple 2004). The notions of epistemic objects and techniques are a useful working tool for micro-historical approaches to the history of mathematics, because they are "constructed to distinguish between how problem-generating and answer-generating elements of particular research episodes function, interact, and change in the course of the work of a specific mathematician or group of mathematicians" (Kjeldsen 2009b), and thereby are attached to concrete mathematical practices and intellectual contexts.

## 5 Theoretical Constructs for Dealing with Scenario 1

Besides the multiple-perspective approach to history as presented above, the theoretical construct for dealing with scenario 1 is, as mentioned, the competence-based view of mathematics education as it has been developed in the KOM-project (Niss and Jensen 2002). At present time the report of the KOM-project, the KOM-report, has only appeared in Danish, and the following description and quotes are therefore our own translations into English, which however are made consistently with the brief English presentation of the project in Niss (2003).

### 5.1 A Competence-based View of Mathematics Education

Niss and Jensen (2002) defines *mathematical competence* as "a well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge". Depending on the mathematical challenge in question different types of mathematical competencies are called for. All in all, eight different and relatively distinct competencies are identified and together these span the overall mathematical competence. In order to distinguish these eight competencies from the overall and unified competence, each of these is referred to as a *competency* (see below). Although the eight competencies are different, it is important to understand that they are mutually related, that there are overlaps, and that they normally cannot be put into play in isolation of one another.

The eight specific competencies are ordered into two groups:

I. the ability to ask and answer questions in and with mathematics, and
II. the ability to deal with mathematical language and tools.

From an overall point of view, the ability to cope with and in mathematics can be said to consist of exactly these two capacities.

The four specific competencies which make up the ability to ask and answer questions in and with mathematics (I) are:

1. mathematical thinking competency,
2. problem solving competency,

3. modeling competency,
4. reasoning competency.

The *mathematical thinking competency* comprises an awareness of the types of questions that are typical for mathematics and an insight into the types of answers that can be expected (Niss and Jensen 2002, p. 47), for example to be able to identify necessary and sufficient conditions for the specific properties of an object. Also the acts of abstracting and generalizing are part of this competency, as well as the act of distinguishing between different mathematical statements which can be made, e.g. conditional statements, definitions, theorems, phenomenological statements about single cases, and conjectures. The competency also includes being able to understand the role of quantifiers in such statements.

The *problem solving competency*, or problem tackling competency, involves the ability to be able to detect, formulate, delimitate, and specify different kinds of mathematical problems, pure and applied both, as well as being able to solve mathematical problems in their already formulated form, whether posed by oneself or by others (Niss and Jensen 2002, p. 49). The important thing to notice about this competency is that the word 'problem' is relative to the person who is trying to solve it; what is a routine task for one person may be a problem for another and the other way round.

The *modeling competency* consists, on the one hand, of being able to analyze the foundations and properties of existing models and being able to assess their range and validity and, on the other hand, it involves being able to perform active modeling in given contexts, i.e. mathematizing and applying it to extra-mathematical situations (Niss and Jensen 2002, p. 52). The boundary between dealing with applied mathematical problems, as in the previous competency, and active mathematical model building is fluid. The more we take into consideration specific features of the elements comprised in the problem, the closer it comes to model building.

The *reasoning competency*, firstly, consists of being able to follow and assess mathematical reasoning that is a chain of arguments in support of a claim—orally as well as in writing. In particular it concerns the understanding of what a mathematical proof is, the basic ideas of a proof, and when claims do or do not constitute a proof. For example this comprises being able to understand the logic behind a counter example as well as being able to distinguish between what may be referred to as "structure and detail", i.e. main lines of argument and mathematical technicalities, as discussed by Ottesen (2009). Secondly, it consists of the ability to actually devise and carry out actual and valid mathematical proofs (Niss and Jensen 2002, p. 54).

The four competencies regarding mathematical language and tools (II) are:

5. representation competency,
6. symbols and formalisms competency,
7. communication competency,
8. aids and tools competency.

On the one hand, the *representation competency* comprises being able to understand, i.e. decode, interpret, and distinguish between, as well as utilize different representations of mathematical objects, phenomena, problems, or situations (including symbolic, algebraic, visual, geometric, graphic, diagrammatic, tabular, verbal, or material representations) (Niss and Jensen 2002, p. 56). On the other hand, the competency includes being able to understand the mutual relations between different representational forms of the same

object, knowing about their strengths and weaknesses, and being able to choose and switch between them in given situations (Niss and Jensen 2002, p. 57).

The *symbols and formalisms competency* deals with being able to decode symbol and formal language, translate back and forth between mathematical symbols and natural language, and being able to treat and utilize mathematical symbolism. Furthermore, it comprises having an insight into the nature of the rules of formal mathematical systems, e.g. axiomatic systems (Niss and Jensen 2002, p. 58). The competency differs from the representation competency in that it focuses on the character, status, and meaning of the symbols as well as the way these are used, including the rules for such usage. In addition to this, the competency is also about handling of formal mathematical systems, whether or not these have a symbolic form.

The *communication competency*, firstly, consists of being able to study and interpret others' written, oral, or visual mathematical expressions or texts. Secondly, it consists of the capability to express oneself in different ways and at different theoretical or technical levels about mathematical matters, in written, oral, and/or visual form (Niss and Jensen 2002, p. 60). Since written, oral, and visual communication in and with mathematics make use of various representations, the communication competency is closely connected to the representation competency—and also the symbol and formalism competency since such communication often relies on mathematical symbols and terms. However, the communication competency "goes further than the others since the communication happens between the sender and receiver, and their situations, backgrounds and prerequisites need to be taken into account for communication in the same way that purpose, message and media must" (Niss and Jensen 2002, p. 60). It should also be noticed that communication *about*, as opposed to *in* and *with*, mathematics does not necessarily need to include specific mathematical forms of representation.

The last of the eight competencies, the *aids and tools competency*, firstly consists of having knowledge of the existence and properties of the diverse forms of relevant tools used in mathematics and having an insight into their possibilities and limitations in different sorts of contexts, and secondly, it comprises being able to reflectively use such aids (Niss and Jensen 2002, p. 62). The aids and tools discussed here not only include those related to IT, i.e. calculators and computers (including arithmetic programs, graphic programs, computer algebra and spreadsheets), they include also tables, slide rules, abacuses, rulers, compasses, protractors, logarithmic and normal distribution paper, etc. The competency is about being able to deal with and relate to all such tools and aids.

## 6 Two Empirical Examples of Competence Development

The two empirical examples that we use to illustrate and exemplify scenario 1, i.e. how history of mathematics can be used to develop students' mathematical competence while at the same time having the students work on mathematical ideas that are relevant to the mathematics curriculum, without resorting to Whig history, are taken from a cohort of mathematics projects made over the past 30 years by students at Roskilde University, Denmark. The study programs at Roskilde University are rather unique, so we shall make a brief presentation of the educational context in which these projects are carried out. All bachelor's and master's programs are project organized in the sense that in each semester the students work in groups of 4–8 students on a project that runs throughout the entire

semester.[14] The students are supposed to spend half of their study time each semester on these projects. The other half is spend on ordinary course work. The projects are student directed and problem oriented. This means that the students themselves formulate a problem that they want to solve. The project work is then to solve the problem, including designing, setting up and carrying out experiments or other kinds of investigations needed for a solution, write a report of 50–100 pages, and defend their work at an oral exam with external evaluation. The problems for the project work are chosen by the students in dialogue with a professor who supervises the group in their project work.

The problems for the students' project work are not defined with respect to a curriculum in the ordinary sense but by project themes. In the mathematics study program students have to perform three projects. The requirement for one of them is that the students should work with a problem that deals with the nature of mathematics and its 'architecture' as a scientific subject such as its concepts, methods, theories, foundation, etc., in such a way that the status of mathematics, its historical development, or its place in society gets illuminated. Not all the projects performed under this constraint deal with the history of mathematics but quite a few do, including the projects that we use in this paper. The reason for choosing the two history-of-mathematics projects to be described and analyzed below is that the students, who worked with these projects, happened to investigate aspects of differential equations and integration. These are topics that are included in the core curriculum of advanced upper secondary school mathematics and mathematics and science studies in universities, and therefore the project reports can be used to illustrate and exemplify our scenario 1, although they were not designed and performed for that particular purpose, but within the existing practice described above.

In the following we will not go into comprehensive discussions, neither of the history in question nor of the projects themselves. We shall only summarize enough to illustrate how and in what sense history of mathematics can be used for the learning of aspects of core curriculum mathematics without becoming Whig. To this end we adopt the competence-based understanding of mathematics education and a multiple perspective approach to the history of the practice of mathematics.[15]

### 6.1 A Student Project on Physics' Influence on Differential Equations

The project on physics' influence on the development of differential equations was made by five students, who formulated the following problem for their project work, which they answered in a written, coherent report of 90 pages (Nielsen et al. 2005):[16]

> How did physics influence the development of differential equations? Was it as problem generator? Did physics play a role in the formulation of the equations? Did physics play a role in the way the equations were solved? (Paraphrased from Nielsen et al. 2005, p. 8)

In order to answer them, the students analyzed Johann Bernoulli's solution of the catenary problem and the brachistochrone problem, respectively; and Jakob Bernoulli's solution to the brachistochrone problem. These three sources from the 1690s are well

[14] The project work for the 2 year science introductory program and its educational purposes are described in Blomhøj and Kjeldsen (2009). The integration of history and philosophy of mathematics within the 2 year science program is described and discussed in Kjeldsen and Blomhøj (2009).

[15] A discussion of the first student project (Sect. 6.1) with respect to developing mathematical competence and turning meta-discursive rules into objects of reflections may be found in Kjeldsen (forthcoming a) and Kjeldsen and Blomhøj (Accepted), respectively.

[16] All quotes from the projects have been translated from Danish into English by us.

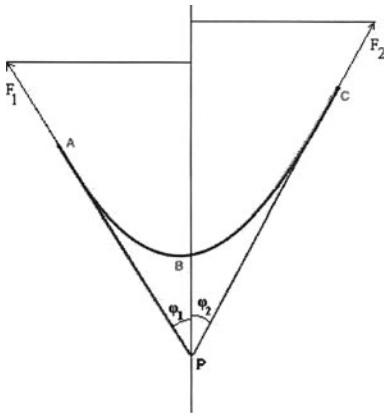
known and generally considered to be crucial for the development of a mathematical theory of differential equations.

To illustrate how history of mathematics can serve as a tool to develop students' mathematical competence we will present three examples from the part of the students' report where they deal with Johan Bernoulli's derivation of a differential equation for the catenary and his construction of a solution. These extracts of the students' report will then be analyzed with respect to their potential to develop the students' mathematical competencies.

The catenary problem is to describe the curve that hangs freely between two points. The students read the solution that Johann Bernoulli presented in a series of lectures for Marquis de l'Hôpital, supported by an English translation of some extracts in Bos (1975). To be sure, working with the original sources in the set up of this project work was rather difficult and challenging for the students.

The first example is an extract from the students' report where they explain five hypotheses or assumptions from statics that Bernoulli used to describe the curve (Nielsen et al. 2005, pp. 21–23) [In the following the students use the letter $P$ to denote both the point where the weight of the chain acts as well as the value of the weight itself]:

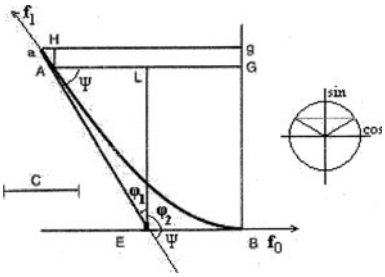
1. The chain is completely flexible; 2. The forces $F_1$ and $F_2$ that carry the weight of the chain in $A$ and $C$ are the same as the force required to carry the weight of the chain $ABC$ in $P$. These forces are imagined to be spanned by weightless cords $AP$, $CP$, along the tangents to the curve in $A$ and $C$ (see figure). 3. If the chain hangs freely from $A$ and $C$, and if a point $F$ on the chain is fixed and the piece $AF$ is removed, then the shape of $FBC$ does not change. 4. The force $F_0$ required in $B$ to keep the chain in position $B$ is independent of the position of $A$. 5. With respect to the figure, the following equations hold: (i) $F_1/F_2 = \sin\phi_2/\sin\phi_1$; (ii) $P/F_2 = \sin(\phi_1 + \phi_2)/\sin\phi_1$; (iii) $P/F_1 = \sin(\phi_1 + \phi_2)/\sin\phi_2$




In order to understand how Bernoulli reached these hypotheses the students looked in books on statics, and they combined the physics with results from trigonometry. We will not give more details on this just mention that the students derived (i), (ii), and (iii) above using the sine and cosine relations.

The second example is an extract from the students' report that shows their understanding of Bernoulli's derivation of the differential equation $dy{:}dx = a{:}s$ for the catenary. Bernoulli based his derivation on the five hypotheses from statics, the infinitesimals $dx = gG$ and $dy = Ha$, and the length of the curve, called $s$ (see the figure and the extract below). The students explained it as follows (Nielsen et al. 2005, pp. 27–29) [The letter $a$ is used in several different ways below: In the extract of the source, $a$ is used to denote the force $F_0$ and to denote the letter on the diagram facing $H$. The students then used $a$ to denote the generic angle in the formula $\sin a = \sin(180^\circ - a)$].

Bernoulli considered the part of the chain from *A* to *B*. … Since the weight of the chain is evenly distributed along the chain one can choose a unity of weight such that the weight of *AB* is equal to the length of the curve *s*. … Furthermore, Bernoulli sets $F_0 = a$, since $F_0$ is a constant according to hypothesis 4. The tangents to the curve (the forces $F_1$ and $F_0$) meet in the new *P* here called *E*. This means that the mass of the piece of the chain *AB* is placed in *E* in hypothesis 2. According to hypothesis 5 [(ii)] the equation can now be expressed in the following way:




$$\frac{weight_of_AB}{F_0} = \frac{s}{a} = \frac{\sin(\phi_1 + \phi_2)}{\sin\phi_1}$$

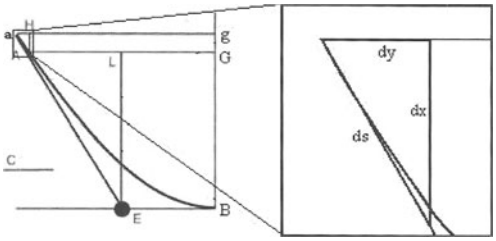
Since $\sin a = \sin(180^\circ - a)$ (see the figure):

$$\sin\Psi = \sin(\phi_1 + \phi_2) = \frac{EL}{EA}$$

From the same figure we see that $\sin\phi_1 = \frac{AL}{EA}$ so by combining these three expressions we get the following expression:

$$\frac{s}{a} = \frac{\sin\Psi}{\sin\phi_1} = \frac{\frac{EL}{EA}}{\frac{AL}{EA}} = \frac{EL}{AL}$$

We now see that the ratio between the two sides *EL* and *AL* corresponds to the ratio between the two smaller sides of a right angle triangle. The same ratio can be expressed from a much smaller but similar triangle, namely the ratio *dx*/*dy* (see the figure) [$BG = x$, $GA = y$, $Gg = dx$, $Ha = dy$]. Hence we get the expression used by Johann Bernoulli to express the catenary:




$$\frac{dy}{dx} = \frac{a}{s}$$

This whole derivation began with an analysis of how the forces affect the catenary, and from there through tools from geometry a single expression could be derived.

The third example we will give is the students' work with Bernoulli's solution of the equation for the catenary. First the students struggled to understand and explain how Bernoulli could rewrite the equation $dy/dx = a/s$ into the form $dy = adx\Big/\sqrt{x^2 - a^2}$ hereby getting rid of *s*. The students used three pages of calculations and algebraic manipulations in their report to explain this new formulation of the equation, in the course of which they also realized that Bernoulli was disregarding the constant of integration, when he developed the equation. The students explained how Bernoulli dealt with the "missing" constant as follows:

We remember that Bernoulli disregarded the constant of integration in his further development of the differential equation. He get around this now by substituting *x* with $x + a$. Both the problem with the parallel shift of the curve and the way to solve it was quite normal at the time. The constant of

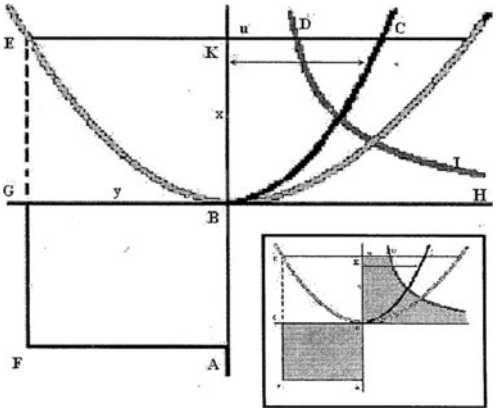
> integration was not included in the formulas and the procedure for solving the problem was to make an inspection of the figure at a suitable place in the calculations. … the equations then becomes $dy = adx/\sqrt{x^2 + 2ax}$ (Nielsen et al. 2005, p. 34).

The students then solved this equation by separation in the modern way and they continued:

> Johann Bernoulli and his colleagues in the 17th century did not know the logarithmic and the exponential functions. Instead he interpreted the integral geometrically as an area below a curve. Hence, his solution was in the form of a geometrical construction (Nielsen et al. 2005, p. 36).

As the students' wrote in the quote above, Bernoulli solved the equation by constructing the curve. He drew the figure below (except for the figure with the shaded areas which is a figure added by the students). He let $BA = a$, and described an equilateral hyperbola $BC$ with centre $A$ and vertex $B$. He then constructed a curve $DI$ such that $BA$ is the middle proportional between $KC$ and $KD$ everywhere. Below is an extract with the students' explanation of Bernoulli's construction (Nielsen et al. 2005, pp. 37–39):

> A hyperbola has the expression
>
> $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
>
> Since the hyperbola $BC$ is equilateral $a = b$. We manipulate the above and get: [*we are omitting the students manipulations*] $y = \sqrt{x^2 - a^2}$. Since $x = x + a$ we know from before that the above can be written as $y = \sqrt{x^2 + 2ax}$. The curve $DI$ is now performed as the *middle proportional* between $KC$ and $KD$. This *middle proportional* means according to [here the students have a reference. They had never met this term before] that: $KC/BA = BA/KD$. Since … $BA = a$ and $KC$ is our hyperbola, we can use this to find an expression for $KD$: $KC/BA = BA/KD$ hence $KD = (BA)^2/KC$ and then
>
> $$KD = a^2 \Big/ \sqrt{x^2 + 2ax}$$




> We notice that $KD$ is the expression for the curve $DI$ with respect to the $y$ variable. Bernoulli then reasons as follows: He constructs a rectangle $GBAF$ the area of which is equal to the area bounded by $IDKBH$ (the gray areas in the added picture in the figure). Notice that $BA$ is constant. Any point on the catenary can be constructed with a $x$ ($BK$) and the corresponding value of $y$ ($BG$) determined by the line segment $FG$'s position. Bernoulli does not prove that this is correct. The correctness of the construction rests on the assumption that an area below a curve can be described with a rectangle. This is problematic since $DI$ tends to infinity when $x$ tends to zero. … Even though the method is unorthodox [from a modern point of view] it gives the right result. Constructing the curve demands a fair amount of geometrical intuition and cannot be extended into a general method.

If we analyze the students' project work and their report with respect to the eight mathematical competencies which span mathematical competence, we will see that the students actually worked with issues relating to all eight competencies. Here we will only mention five aspects, for a discussion of all eight competencies with respect to this particular project work we refer to Kjeldsen (forthcoming a). The students' *problem*

*solving* skills where trained extensively in their work with Bernoulli's text on the catenary. As emphasized above, the students had to fill in all the 'gaps' in Bernoulli's presentation in the course of which the students derived several intermediate results themselves. This is present in all three examples, maybe most clearly in example 1 where the students worked with Bernoulli's five hypotheses from statics and in example 3 in their explanation of Bernoulli's geometrically construction of the solution to the differential equation of the catenary. Parts of the students' *mathematical modeling* competency were developed through their struggle to understand Bernoulli's mathematization of the physical description of the catenary, and to justify his five assumptions from statics. The students had problems understanding and following Bernoulli's arguments. They had to derive the five assumptions from statics themselves, as mentioned above this was not an easy task, as they wrote in their report:

> One example [of the amount of work it took us to work through Bernoulli's text] that we can mention is his [Bernoulli's] remark after hypotheses 5. … "*these rules can be found in any textbook in statics*". Also here it took us quite some work to realize how these hypotheses could be derived from the parallelogram of forces and trigonometric derivations. (Nielsen et al. 2005, p. 39)

Their *mathematical reasoning* competency was developed, in particular through their struggle to clarify the apparent lack of rigor in Bernoulli's way of arguing compared with modern standards. Bernoulli's way of argumentation is not in accordance with what is considered 'proper' or rigorous mathematical reasoning today. His use of the so-called infinitesimal triangle, which was a standard way of arguing at Bernoulli's time, puzzled the students, who were well aware that the point $a$ in the infinitesimal line segment $Ha$ does not lie on the tangent through $A$, but on the curve of the catenary. Hence, Bernoulli incorporated an error when he, in example 2 above, assumed that this forms a triangle similar to triangle $ALE$. At first, this created cognitive hurdles for the students that challenged them to reflect upon the differences between how mathematicians reasoned with equations between differentials and how differential equations are reasoned with in their textbook.

The students' ability to distinguish between and utilize different *representations* of e.g. mathematical objects was especially challenged in example 3 above during their study of Bernoulli's construction of the solution to the differential equation of the catenary. Here they experienced a representation of a solution that is quite different from the analytical representation they usually see in modern textbooks. The fact that Bernoulli constructed the curve geometrically raised a discussion among the students—and with their supervising professor (one of the authors)—of conceptual aspects of solutions to a differential equation. Finally, in their written report (90 pages), the students' explained and analyzed the mathematics in its historical context from the perspective of how another scientific discipline, physics, influenced this episode in the history of mathematics, and thereby answered their problem formulation. In writing the report, the students developed their competency to *communicate* in, with, and about mathematics far beyond the opportunities given when students work with differential equations from ordinary textbook materials.

### 6.2 A Student Project on Infinity and 'Integration' in Antiquity

The project on infinity and 'integration' in Antiquity was made by four students. They were interested in the discussions of infinity in Antiquity. They wondered whether, and if so to what extent, there was a connection between the mathematicians' use of the method of exhaustion to avoid infinity in their proofs and philosophy, in particular Aristotle's

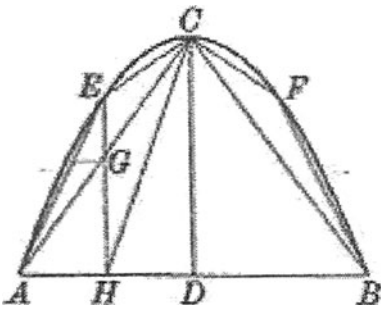
rejection of actual infinity. The students formulated the following problems for their project work:

> How close were the Greeks at the time of Archimedes at developing a general theory of integration? Why didn't they take 'the last step'? To what extend can this be linked with philosophy especially Aristotle's dogma of infinity where actual infinity was tabooed? (Holm et al. 2004, p. 6)[17]

The students studied Aristotle's ideas about infinity by reading parts of book III and book VI of Aristotle's *Physics* in an English translation. They studied the use of the method of exhaustion in mathematics in Antiquity using Euclid's proof of the theorem that 'circles are to one another as the squares of their diameters' from book XXII of Euclid's *Elements* (Heath 1956). They based their discussions of the influence of philosophy on mathematics at the time on their readings of Aristotle, and of two texts of Archimedes on the quadrature of the parabola: one where he explained how he found his result by means of mechanics, and one where he proved the result by means of geometry.

In order to exemplify how the students' work with the original sources served as a means for developing mathematical competence we will present extracts from two examples from the students' report where they deal with Archimedes' two texts on the quadrature of the parabola.

Based on their readings of Archimedes' proof in Heath's book *The Works of Archimedes* from 1912, the students explained Archimedes' proof of the quadrature of the parabola in the following way (Holm et al. 2004, pp. 55–58):

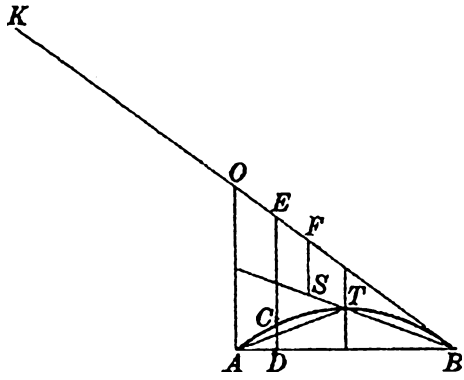
> On the figure we see the segment $ABC$ of a parabola with vertex C and a chord $AB$. We form the triangle $ABC$. In the following a triangle will be designated with the symbol Δ, which means that the triangle $ABC$ will be designated $\Delta ABC$. $\Delta ABC$ is now divided in two equal triangles $\Delta ADC$ and $\Delta BDC$ with help of the median $CD$. Let $H$ designate the midpoint of the line segment $AD$ and construct the median $HE$ which is parallel to $CD$. By using an elementary theorem about conic sections, saying that $EG = \frac{1}{2}\, GH$, we get the relation:
>
> $$\Delta AEC = {}^1\!/_2 \Delta AHC = {}^1\!/_8 \Delta ABC$$
>
> 
> 
>
> where the first equality is true because $EG = \frac{1}{2}GH$, and thereby $\Delta AEG : \Delta AGH = 1 : 2$ and $\Delta GEC : \Delta HGC = 1 : 2$, from which we get that $\Delta AEC : \Delta AHC = 1 : 2$. The second equality is true, because $\Delta ABC$ consists of precisely four triangles with equal heights and base which all have the same area as $\Delta AHC$. The above relation gives that $\Delta AEC + \Delta BFC = {}^1\!/_4 \Delta ABC$. This is obvious because $\Delta AEC$ and $\Delta BFC$ are mirror images of each other. If we continue bisecting the chord $AB$ of the segment of the parabola with medians that intersect the parabola with the corresponding filling out of the segment of the parabola with triangles, i.e. the construction of the inscribed polygon, we reach the below sequence of areas as the number of sides in the inscribed polygon increases:
>
> $$\Delta ABC, {}^1\!/_4 \Delta ABC, {}^1\!/_{4^2} \Delta ABC, \ldots$$

We then jump to the first part of the students' explanation of how Archimedes used the double-proof-by-contradiction method of exhaustion to demonstrate that the quadrature of the segment of the parabola is ${}^4\!/_3 \Delta ABC$ (Holm et al. 2004, p. 58):

[17] The students' formulation of their problem for their project work is always a source of critique and discussions that in most cases go on throughout the entire semester. In this case, e.g. the students' formulation of the second question can be interpreted as though they have a platonic view of mathematics.

I. It is first assumed that the segment of the parabola is less than $K$ [$K = {}^4/_3\Delta ABC$]. By using theorem 23[18] it can be seen that, if $X$ is the last element of the sequence, and if one chooses $X$, such that $X$ is smaller than the difference between the segment of the parabola and $K$, then: $A + B + C + \cdots + X + {}^1/_3X = {}^4/_3A = K$. It can now be seen that since $K$ exceeds $A + B + C + \cdots + X$ with an area, that is smaller than $X$, and $K$ also exceeds the area of the segment of the parabola with an area, that is greater than $X$, it follows that: $A + B + C + \cdots + X > \Sigma$ [here $\Sigma$ denotes the area of the segment of the parabola], and hence we have the contradiction, because the sum $A + B + C + \cdots + X$ is the sum of [$\Delta ABC + {}^1/_4\Delta ABC + {}^1/_{4^2}\Delta ABC + \cdots + {}^1/_{4^n}\Delta ABC$] which is smaller than $\Sigma$.

The following is the students' account of how Archimedes found the quadrature of the parabola by means of mechanics (Holm et al. 2004, p. 61):




Let $ATB$ be the segment of the parabola that is bounded by the line segment $AB$ and has vertex $T$. This is shown in the figure. Now the tangent that touches the parabola in $B$ is constructed, and the tangent is prolonged to the point $K$. Also the ordinates $AO$ and $DE$ are constructed. The ordinate $AO$ falls on the tangent in $O$, which bisects the line $BK$, meaning that $OK = OB$. Then an elementary theorem from the antique theory of conic sections, which state the following, is used: $CD{:}ED = AD{:}AB$, which also gives that $CD{:}ED = OE{:}OB$, and since $OK = OB$ the relation can be written as $CD{:}ED = OE{:}OK$. This relation holds for all ordinates. If one now imagine, that an arbitrary chosen line segment $CD$ is displaced to the lever's end point, $K$, then the line segment $CD$ is in equilibrium with the line segment $ED$, if $ED$ is suspended in $E$. If the same is done with 'all the ordinates' of the segment of the parabola, then if the parabola is suspended in $K$, it is in equilibrium with $\Delta BOA$. The point of equilibrium is $S$, the point of intersection of the medians, shown in the figure, which means that one can interpret the triangle as been suspended in $F$, since $OF = 1/3\ OB$. Since $OK = 3\ OF$, it follows from the principle of the lever that the segment of the parabola $= {}^1/_3\Delta BOA = {}^4/_3\Delta TAB$, hence the quadrature of the parabola is found.

Analyzing the students' project work in the frame work of mathematical competence we will, as was the case with the previous project work, realize that they worked with aspects related to all eight competencies. Here we will emphasize four competencies: The students' *problem solving* competency was developed through their work with the mathematical sources, but especially through their work with and struggle to understand how Archimedes found a candidate for the quadrature of the parabola by means of mechanics. Their *reasoning* competency was developed throughout the whole project, and especially through their work with and use of the method of exhaustion with its strategy of double-proof-by-contradiction. In order to argue for their conclusion (see the next section) that it is difficult to reject completely an influence from philosophy, they compared the mathematical reasoning of Archimedes both in his demonstration of the quadrature of the parabola by means of the method of exhaustion, and in his explanation of how the result can be found by resorting to mechanical arguments, with our modern way of reasoning in the theory of integration. In their discussion they showed ability to assess the different

[18] Theorem 23 of Archimedes' *Quadrature of the Parabola* states that: "given a series of areas $A$, $B$, $C$, $D$, … $Z$ of which $A$ is the greatest, and each is equal to four times the next in order, then $A + B + C + \cdots + Z + 1/3\ Z = 4/3\ A$".

kinds of mathematical reasoning used in the various sources and today. Their *symbols and formalisms* competency was trained e.g. through their work with Euclid's proof that circles are to one another as the square of their diameters, and Archimedes' two "proofs" of the quadrature of the parabola. Especially in Archimedes' mechanical arguments for the quadrature of the parabola they experienced the force of symbols in mathematics. They experienced mathematical formalism especially through their work with the sources using the method of exhaustion and the mathematical rigor connected to it. Especially in Euclid's *Elements* they received first hand experiences with an axiomatic-deductive mathematical system and body of knowledge. Finally, their skills to *communicate* in, with and about mathematics were all developed through their written documentation (86 pages) of the results of their project work, and their argumentations for the answers to the questions of their problem formulation which (as will be seen in the next section) were anchored in the mathematical sources of (among others) Euclid and Archimedes, in the writings of Aristotle, and in recent literature from philosophy and the history of mathematics.

### 6.3 Discussion of the Examples for Scenario 1

We have used the two project reports as empirical examples to illustrate our claim of it being possible to integrate history of mathematics into mathematics education in situations where history is not on the agenda, i.e. where it is not part of the curriculum, in such a way that history can function as a tool for learning core curriculum mathematics without resorting to a Whiggist approach to history. We are well aware that these projects have been made within a rather unique educational structure that allows the students to conduct a history of mathematics project as part of the educational requirements. Nevertheless, our analyses of parts of these projects within the framework of a multiple perspective approach to history and the competence-based understanding of mathematics education show that, by choosing perspectives which address the intended learning outcomes, it is possible to have students work with pieces of original historical texts as historians, and at the same time develop their mathematical competence.[19]

In both projects the students' questions, their problem formulation, pertain to the practice of mathematics in a specific episode of the history of mathematics from Antiquity and the seventeenth century, respectively. In the project dealing with the seventeenth century the students read the sources to investigate *how* physics played a role in the development of differential equations, *what* function physics served in the practice of formulating problems that called for differential equations, *what* role, if any, physics played in the derivation of the equations, and in the way the equations were solved. In the project dealing with Antiquity the students studied the history of infinity and (among other things) the quadrature of the parabola from the perspective of philosophy's influence on a particular area of mathematics in Greece.

For both projects it follows that a Whiggist approach to history would not provide answers to the students' problem formulation. If the content of the sources is formulated and explained by modern mathematics, it becomes impossible to answer the students' questions. The students worked with the sources as historians. They asked historical questions to a specific

[19] A framework for how such an approach can be implemented in secondary mathematics education on a small scale was presented at the workshop *Does history have a significant role to play for the learning of mathematics? Multiple perspective approach to history, and the learning of meta level rules of mathematical discourse* at the ESU 6 in Vienna, July 19–23, 2010. Interested readers are thus referred to the forthcoming proceedings from this conference.

episode in the history of mathematics. They read, analyzed, and interpreted the sources from the practice of mathematics in a non-Whiggist way from the perspective of (for the first project) how another scientific discipline influenced mathematicians' formulation of problems and the methods they used to solve them, and (for the second project) techniques of proofs, the understanding of infinity, and the understanding of the concept of quadrature, and the similarities to and differences with our modern concept of the integral.

Regarding the project on physics' influence on the development of differential equations the students concluded that physics did play a decisive role in generating problems that motivated the introduction of differential equations, that physics also influenced the way in which these equations were derived for the catenary and the brachistochrone, and finally that physics was significant for Johann Bernoulli's solution of both problems, but that physics played no role for Jakob Bernoulli's solution of the brachistochrone. The students who worked on the project on integration and infinity in Antiquity read recent research literature in history of mathematics on the subject and used this, their readings of Archimedes' texts, and their readings of Aristotle's work to give some answers to the questions in their problem formulation. The following quote illustrate some of the students' considerations of a possible influence from philosophy on the mathematics in Antiquity, and exhibits how the students' conclusions are anchored in their reading of the mathematical sources (Holm et al. 2004, p. 83–84):

> The problematique of mathematical rigor in the theory of integration is related to actual infinity which is avoided by the use of the method of exhaustion. In the theory of integration actual infinity is present in the concept of limit, where infinitely many infinitely small infinitesimals are dealt with. In the 17th century there were a lot of worries about these problems, which were not solved until the end of the 19th century. Many researchers assume in analogy with this that Greek mathematicians had similar worries, even a definitely fear of infinity. […] A possible worry of the use of actual infinity among the Greek mathematicians is often related to Aristotle's theory of infinity, where one could only talk about infinity in the sense of potential infinity. The rejection of actual infinity by Aristotle and in mathematics through the use of the method of exhaustion takes place roughly at the same time. That this connection is caused by an influence from philosophy has been rejected by the modern researchers we have treated in this report. […] the arguments [of the modern researchers] can be disputed. Therefore we find that it is difficult to reject an influence from philosophy on the development of the method of exhaustion completely.

None of the groups of students used the methodological ideas of epistemic objects and techniques explicitly, but these would have served as a useful aid in both projects in structuring the students' analyses and reflections, since the students addressed the issues of what the objects under investigation were in their respective episodes in the history of mathematics and what kind of techniques the mathematicians used in their investigations. As we will see in the empirical material that are analyzed in our treatment of scenario 2, these constructs are excellent tools for analyzing original sources, and excerpts of such, with respect to the dynamics of the production of mathematical knowledge in the texts.

We have already mentioned that in both these projects history served as a tool for developing the students' mathematical competence within all of the eight specific competencies, and in the previous sections we have documented this with respect to five, respectively four, of the specific competencies, and these will not be repeated here. In the projects the students discussed mathematical notions of differential equations and integration, respectively. Both projects were driven by a wish to understand the emergence of concepts that are central in traditional curriculum descriptions of mathematics. The students who worked with physics' influence on the development of mathematics project were not used to develop differential equations for physically given systems themselves. This forced them to think about differential equations in a new setting, and hereby connections

were created between the students' conception of differential equations and their historical experiences, which broadened their own conception of differential equations. Also, their discussion of the constant of integration and Bernoulli's "way around" that supported their own understanding of this phenomenon, which in general is a cognitive obstacle for many students (Blomhøj and Kjeldsen 2010).

The analyses of the projects show that history can be used as a tool for the learning of core curriculum mathematics without turning Whig if we choose perspectives that attend to the function and/or nature of explicit mathematical entities, e.g. certain concepts, problems, methods of proofs, and/or perspectives that question research approaches in concrete historical episodes. In both projects these learning outcomes were initiated among other things by the students' discussions and reflections about the differences between how these entities and approaches were presented, conceived, and used in the historical texts, and how it is done in modern textbooks. Through such activities, connections between the students' historical experiences and their experiences from having been taught their modern textbook version of the material in class can be created in the learning process. Such connections challenge the students to use and reflect upon their mathematical conceptions in new situations, providing opportunities for them to learn new aspects and to broaden their understandings of, in these particular cases, differential equations and the concept of integration. Therefore, it is indeed of didactical interest to find and analyze original sources from different episodes in the history of mathematics with regard to their potential to challenge students' mathematical conceptions.

## 7 Theoretical Constructs for Dealing with Scenario 2

The theoretical construct to be used when considering anchoring of history in the in-issues is the use of the KOM-report's 'second order' competencies, the so-called *types of overview and judgment*. In addition, the idea of structuring the students' discussions and reflections around a use of 'general topics and issues', which are certain kind of perspectives from which the history of mathematics can be viewed, also play a significant role for dealing with scenario 2.

### 7.1 The KOM-Report's 'Overview and Judgment'

Besides listing and discussing the eight mathematical competencies, the authors of the KOM-report also point out three 'second order' competencies, referred to as types of *overview and judgment*. These are:

1. the actual application of mathematics in other subject and practice areas;
2. the historical evolvement of mathematics, internally as well as in a societal context;
3. the nature of mathematics as a subject.

To possess overview and judgment within these three areas means to have active insights and "a set of viewpoints, which provide overview and judgment about the connection of mathematics to circumstances and allotting in nature, society, and culture" (Niss and Jensen 2002, p. 66).

The first form of overview and judgment concerns actual application of mathematics to extra-mathematical purposes within areas of everyday life, society, or other scientific disciplines. The application is brought about through the creation and utilization of mathematical models. As opposed to the modeling competency, which deals with active modeling

skills, this type of overview and judgment is of a more broad and generalized form, almost of a sociological or science-philosophical nature (Niss and Jensen 2002, pp. 67–68).

The third type of overview and judgment concerns the fact that mathematics as a subject area has its own characteristics, as well as the characteristics themselves. Some of these, mathematics has in common with other subject areas, while others are unique (Niss and Jensen 2002, pp. 69–70).

For our purpose in this paper it is of course the second type concerning historical development which is the most interesting. About this the KOM-report states:

> The object for this type of overview and judgment is the fact that mathematics has evolved in time and space, in culture and society. […]
> The type of overview and judgment should not be confused with knowledge of 'the history of mathematics' viewed as an independent topic. The focus is on the actual fact that mathematics has evolved in culturally and socially determined environments, and on the driving forces and mechanisms which are responsible for this evolution. On the other hand, it is obvious that if overview and judgment regarding this evolution is to have *solidness* they must rest on concrete examples from the history of mathematics. (Niss and Jensen 2002, p. 68, italics added)

From this quote, it is clear that the type of overview and judgment concerning the historical development of mathematics focuses on the use of history as a goal as opposed to that of history as a tool. Also, the talk of 'solidness' in this quote underpins that the treatment of history should not be detached from the mathematical in-issues of the concrete examples.

For the upper secondary level the overview and judgment concerning history is deepened even further:

> In the teaching of mathematics at the upper secondary level the students must acquire knowledge about the historical evolution within selected areas of the mathematics […]. The central forces in the historical evolution must be discussed including the influence from different areas of application. Through this, the students must develop a knowledge and an understanding of mathematics as being created by human beings and, in fact, having undergone an historical evolution – and not just being something which has always been or suddenly arisen out of thin air. (Niss and Jensen 2002, p. 268)

As examples of insights to be taken into account when considering the second type of overview and judgment, the KOM-report mentions the following questions:

(a) How has mathematics developed through the ages?
(b) What were the internal and external forces and motives for development?
(c) What types of actors were involved in the development?
(d) In which social situations did it take place?
(e) What has the interplay with other fields been like? Etc. (Niss and Jensen 2002, pp. 68–69, lettering of questions added)

### 7.2 General Topics and Issues

One way of framing students' discussions and reflections on meta-issues is through a use of *general topics and issues* from the history of mathematics (and historiography[20]). This notion builds on the idea that by identifying concrete historical examples illustrating somewhat general features, students will be able to learn something about the historical

[20] If including epistemic objects and techniques as a general topic and issue in itself, see Jankvist (2009c) and Jankvist (2010a).

development of mathematics in general, even though they may only be exposed to a single, but exemplary, case. Furthermore, if these general topics and issues are introduced and treated carefully, they can assist in bringing about an anchoring of the meta-issues of history in the related in-issues.

One example of such a general issue or topic is that mentioned in the KOM-report of the driving forces and mechanisms responsible of the evolution and development of mathematics. When discussing such matters one may distinguish between the inner and the outer driving forces. *Inner driving forces* are the forces and mechanisms which are responsible for developing the mathematics from the inside. These can be, for example, intriguing questions, unsolved puzzles and riddles, unproven conjectures, etc. which drive mathematicians in their research. *Outer driving forces* are the forces and mechanisms which influence the discipline of mathematics from the outside. Examples are societal needs, money, and, not least, war to mention a few. For example, the U.S. funding of research after World War 2 and during the Cold War was a major (outer) driving force for several scientific and technological disciplines at the time.

A second topic or issue is the discussion of *pure versus applied mathematics*. These disciplines have not always been in clinch (e.g. both Euler and Gauss were pure as well as applied mathematicians), but during the 20th century pure and applied mathematics—or mathematicians—seemed to drift apart.[21] Pure mathematicians would pride themselves of studying abstractions which had no practical applications whatsoever and applied mathematicians developed a more hostile attitude towards pure mathematics and pure mathematicians (Davis 1994). Funding is of course an issue in the dispute, but also the application of mathematics in warfare has played a role. One who was indeed very much in opposition to this, and therefore to applied mathematics, was the English number theorist G.H. Hardy. In an address to the British Association for the Advancement of Science in 1915, Hardy said: "A science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life" adding that the study of prime numbers did neither, while "the greatest mathematicians of all ages have found in it a mysterious attraction impossible to resist" (Wells 2005, p. 120). The end of the first world war and the beginning of the second did not change Hardy's views in any way, as is evident from his book *A Mathematician's Apology* from 1940 (Hardy 1992).

A third general topic and issue in the history of mathematics (and science in general) is that of *multiple discoveries* (or inventions) (e.g. Merton 1973). There are several examples of this in the history of mathematics, where two persons, or groups of people, have published similar results independently of each other, sometimes at approximately the same time in history—then known as simultaneous discoveries. One of the more well-known examples of this in the history of mathematics is of course the development of the infinitesimal calculus by Liebniz and Newton, respectively. The topic of multiple discoveries also includes the discussion of attribution of credit.[22]

[21] For an example of relationships between research in pure and applied mathematics of the 20th century, see Kjeldsen (2010b).

[22] Yet another example of a general topic and issue (also related to the third type of overview and judgment) could be that of the old discussion of mathematics being discovered or invented (e.g. Hersh 1997). See Jankvist (2009c) for a more thorough discussion of the above mentioned general topics and issues.

## 8 Two Empirical Examples of Anchoring

As with the examples used to illustrate development of students' competencies, we will not go into deep discussions of the data or the historical cases (for this, readers are referred to Jankvist 2009c; Jankvist 2010a). We will, however, provide some examples of anchoring when having students work with history of mathematics in a setting of 'general topics and issues'. The examples are taken from two historical teaching modules which were implemented in a Danish upper secondary mathematics class in 2007. The history of mathematics has been on the agenda of the Danish upper secondary mathematics program since 1987, where it was one of three so-called 'aspects' more or less resembling the KOM-project's three types of overview and judgment (Jankvist 2009c). Despite good intentions these aspects did not always find their way into the classrooms. This however seemed to change with the new upper secondary school reform, which was implemented in 2006–2007. In this the mathematics program consist of 2⁄3 fixed curriculum and 1⁄3 which is free for the schools and teachers to choose.[23] Along with more strict requirements regarding the inclusion of history this freedom in curriculum provides a unique opportunity of having the students work with arbitrary episodes in the history of mathematics which are illustrative in the sense of using history as a goal—this being the original purpose of including history in the upper secondary mathematics program in the first place (UVM 1987, 2007).

The historical episode dealt with in the first module was that of the early development of error-correcting codes, more precisely works of Claude Shannon, Richard Hamming, and Marcel Golay. The second module was on the creation of public-key cryptography (Whitfield Diffie and Martin Hellman), the RSA implementation of this (Ronald Rivest, Adi Shamir, and Leonard Adleman), and the history of the number theory that is used in RSA. Both of these historical developments are examples of multiple discoveries. Based on Shannon's display of Hamming's so-called (7,4)-error correcting code (Shannon 1948), Golay was able to make the generalization of this into the entire family of Hamming-codes (Golay 1949) and discover a few additional and different ones himself. Due to the wish of the Bell Laboratories wanting to patent Hamming's codes, his own publication of these was delayed a couple of years (Hamming 1950). Regarding the other case, Diffie and Hellman published the idea of a public-key cryptography system in 1976, but were unable to find a mathematical function to fit the description (Diffie and Hellman 1976). Rivest, Shamir and Adleman found one the following year and patented a system based on this discovery (Rivest et al. 1978). However, at this time the British Government Communication Headquarter (GCHQ) had already known about the system and its implementation for a couple of years, but not published anything since it had been labeled classified information (Singh 1999). Furthermore, the historical development of public-key cryptography is a classical example of pure mathematics finding an unexpected application, since the first mathematical function used to make this work was based on a use of prime numbers and number theory, thus making it possible to relate this to the viewpoints of G.H. Hardy as mentioned earlier. Both cases can be discussed in terms of inner and outer driving forces, but in particular the second is interesting in this regard since the development of number theory mainly was driven from within mathematics itself, whereas the development of public-key cryptography was initiated by a growing need for secure communication.

[23] The fixed curriculum is tested on a national written exam and the remaining 1⁄3 is tested on a local oral exam (together with parts of the fixed curriculum).

The implementation of the two teaching modules consisted of approximately 15 90-min lessons each, following specially prepared written teaching materials displaying the in-issues as well as the meta-issues of the chosen historical cases.[24] Each of the modules included prepared *essay assignments* for the students to work on in groups, in particular one large final one that was to be carried out at the end of the module. The essay assignments dealt with meta-issues concerning the historical case of the module and were centered around the general topics and issues as explained above. We will not go into detailed descriptions of the students' work on every general topic and issue, for this the readers are referred to Jankvist (2009c). Instead we shall provide a selection of data from the students' hand-in essays illustrating the presence of anchoring in their work. For both modules we shall show how the methodological idea of epistemic objects and techniques was used in the students' analyses of excerpts of original sources in a way that ensured that their historical reflections rested on the subject matter of the involved mathematics.

### 8.1 A Teaching Module on the Early History of Error-Correcting Codes

Our selected example from the first module is a direct use of the idea of epistemic objects and techniques. In one of the essay assignments the students were asked to consider a series of selected excerpts from Hamming's 1950-paper on error-correction and then try to identify some of the mathematical techniques used by Hamming, and then in terms of these discuss how they were used to develop the codes, i.e. the objects under investigation. One of the objectives of this essay assignment was for the students to make a distinction between the already available techniques, i.e. the well-established mathematics that Hamming was using, and the techniques that he himself had to create in the process.

One group which did rather well in making this distinction answered:

> Hamming uses generalized concept of distance; elements of linear algebra; geometrical models; and unity $n$-dimensional squares.
> A metric which is called a distance $D(x, y)$ is the definition of Hamming distance.
> Since $x$ and $y$ must be different ($x \neq y$), we can find how similar the two codes are, because we talk about a more generalized concept of distance.
> Linear algebra is concerned with the study of addition and proportionality, and it is the concept of linearity that binds the concepts of addition and proportionality together.
> We use geometrical figures to understand the $n$-dimensions, since these exceeds what can be understood physically, i.e. 1st, 2nd, and 3rd dimension.
> $n$-dimensional cube is the same as the metric space. (Group $5_1$)

As an example of a group that did not do it as clearly, and therefore mixed in some of Hamming's own developed techniques, we have:

> Hamming uses these methods to study his codes:
>
> $$D(x, y) = 0 \text{ if and only if } x = y$$
> $$D(x, y) = D(y, x) > 0 \text{ if } x \neq y$$
> $$D(x, z) + D(y, z) > D(x, y) \text{ (triangle inequality).}$$
>
> $t$-error detecting and $t$-error correcting [codes] express how many errors may be corrected in a given code. He also used Hamming-distance and nearest neighbor decoding to correct the errors that occurred. The packing radius shows which codes within the given radius that may be 'interpreted' correctly. The syndrome is, among other things, used to locate the errors. When he [Hamming] had difficulties visualizing it, he used a geometrical setting to better visualize the 1s and 0s and thereby achieve understanding. (Group $2_1$)

[24] The two materials, Jankvist (2008b) and Jankvist (2008c), are available online as IMFUFA Texts no. 459 and 460, respectively. See http://milne.ruc.dk/ImfufaTekster/

We don't go into details of the technical terms in the quotes, which the students were familiar with due to the module; *t*-error detection/correction and the syndrome are examples of not already available techniques for Hamming when he began his study of error-correcting codes.

Although not every statement in the above quotes may be entirely correct, they still show clear evidence of some anchoring being present. As a fact, all (six) groups of students in the class were able to identify the generalized concept of a metric (in one form or another) as one of the techniques used by Hamming in his studies of the codes. A more anecdotical and superficial treatment of this piece of history of mathematics would not have enabled the students to address these meta-issues in this manner, on a basis of their knowledge of the related in-issues.

## 8.2 A Teaching Module on the History of Public-Key Cryptography

For the second module we provide two selected examples, again one on a direct use of the idea of objects and techniques and then one on the general topic and issue of pure versus applied mathematics.

When asked about objects and techniques in this module one group answered the following, which also indicate a degree of anchoring being present:

> The RSA algorithm rests upon three mathematical theorems: the Chinese remainder theorem, Fermat's little theorem, and Euler's theorem. Congruences are a fantastic tool [technique] when dealing with integers of several hundred digits as you do in RSA. Carl Friedrich Gauss (1777–1855) defined congruences, moduli, and remainders in his work *Arithmeticae*. [*A definition of congruence is provided*.] The security in the RSA system builds upon prime factoring being a difficult calculation problem. If $p \mid ab \rightarrow p \mid a \vee p \mid b$. The fundamental theorem of arithmetic [*the theorem is written up*]. Euler's $\varphi$-function [*a definition is provided*]. Inverse elements: In the RSA system the decryption algorithm is an inverse element of the encryption system [a definition of inverse elements is provided]. (Group $2_2$)

The idea, of course, was to look at public-key cryptography and the RSA algorithm as being the objects under investigation, and then have the students discuss the techniques used by Diffie and Hellman, and Rivest, Shamir, and Adleman in their work.

In another essay assignment the students were to discuss some of G. H. Hardy's views on mathematics, based on a selective reading of his *A Mathematicians Apology* (the students read about two thirds of the English edition of the book as part of their preparation for this assignment). In particular, the students were asked to account for Hardy's characteristic and view of pure and applied mathematics, respectively; to discuss his statements about number theory using their own knowledge of the development of RSA; and to relate this to the importance of conducting basic research. A somewhat typical answer to this was that of group $3_2$:

> He [Hardy] likes abstract mathematics, the kind of mathematics you cannot conceive immediately, mathematical rules where human aspects cannot be applied to anything. He sees it [mathematics] more as a [pure] science, and not as something which has concrete applications. Hardy believes prime numbers and number theory to be one of the purest forms of mathematics. It has no practical applications. Therefore it cannot be used [applied] nor misused. If Hardy had lived longer he would probably have become disappointed in seeing how number theory was applied in RSA.
> "A science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life." This is a quote by Hardy where he obviously is being ironic since he doesn't believe the study of primes to be neither. RSA may tell us that basic research in mathematics is quite important since RSA couldn't have been realized had it not been for the basic research from 200 years earlier. This doesn't fit too well with Hardy's statements about number theory and primes. (Group $3_2$)

Another group answered:

> Applied mathematics builds upon pure mathematics. For example, RSA could not have been developed before they received help from the number theoretician Adleman, who was a pure mathematician. Hardy points to that pure mathematics may only be conceived in one way. That is to say prime numbers only can be prime numbers no matter who is looking at them. However, applied mathematics is or can be relative. Hardy believes that number theory is pure mathematics because it has no application. He says that it is so pure that it cannot be used for anything. But here he is wrong, because it is used repeatedly today, e.g. in RSA. (Group $1_2$)

Needless to say, none of the students' answers are flawless. But again, what is interesting is that the students include elements of the taught and learned mathematical in-issues (e.g. primes), even though they were not directly asked to do so and the assignment did not have the same degree of 'built-in insurance' of anchoring as the epistemic objects-and-techniques essays did. We shall discuss possible reasons for this in the section below.

### 8.3 Discussion of the Examples for Scenario 2

Both of the historical cases used in the examples to illustrate scenario 2 build on mathematics which is outside the normal upper secondary mathematics curriculum. In the actual setting this was possible due to the 1/3 free curriculum, which may of course not be the case in other educational settings. Nevertheless, it provided the opportunity to choose historical cases which illustrate several aspects of using history as a goal, without being limited by curriculum constrictions in one's choice. Thus, it is the historical insights, which one wishes the students to come to possess that decides the mathematics involved, and not the other way round. We shall structure the discussion of the examples for scenario 2 in three parts, each focusing on a different aspect of the students' development of insights regarding history: firstly we consider the development of insights in relation to the specific modules; secondly we do so in relation to the role of general topics and issues in the final essay assignments; and thirdly we address the role of epistemic objects and techniques in this development, in particular in regard to anchoring the students' meta-issue insights in related in-issues of the two teaching modules.

Which of the KOM-report's insights regarding history, and thereby also in terms of history as a goal, did the students then develop as a result of these modules?[25] First of all, by means of concrete examples the students experienced how two branches of discrete mathematics, error-correcting codes and public-key cryptography, evolved during the 20th century (insight a). Furthermore, as part of the second module, the students were introduced to excerpts on number theory from the works of Euclid, Sun Zi, Fermat, Gauss, Euler, and others and thereby saw how this field of mathematics went from being solely pure mathematics to finding its application in the RSA-algorithm. Second, both historical cases provided the students with insights into how external forces motivated the development of these branches of discrete mathematics, and the second module provided the students with insights into the internal driving forces as well (insight b). Regarding error-correcting codes, Hamming became annoyed with the Bell Laboratories' computers dropping his calculations when they *detected* an error, and came up with a more efficient method that was able to correct the occurring errors and carry on with the calculations. Regarding cryptography, Diffie and Hellman as well as Rivest and Shamir were all driven

[25] The students' insights and possible changes in beliefs and views were evaluated by giving them questionnaires and conducting individual interviews before, in between, and after the implementations of the two modules. For an elaboration of this, see Jankvist (2009c).

by the desire to find a way around the old 'key-distribution problem', saying that whenever two parties have to exchange a message safely they will first have to meet up and exchange a (private) key for encryption and decryption. By means of excerpts from the original papers it was shown to the students, how none of these researchers were blind to the external needs of cryptography avoiding this problem, they mention things such as the beginning Internet, electronic mail, and digital signature. And with a selected overview of the development of number theory through the ages, this further provided the students with an idea of the power of internal driving forces within a field of research. Third, besides the involvement of university researchers these historical cases also illustrate how other actors influence the development of mathematics; for example a private, although government supported, corporation such as the Bell Labs and not least secret agencies such as the British GCHQ and the American NSA (insights c and d). But equally important, the students realized that mathematics is developed by human beings, and furthermore that this development is not caused singlehandedly by a few geniuses, but that much of it is due to groups of researchers working together to solve the same problem. Fourth, that the people making up these groups may have different educational backgrounds and that they therefore may complement each other in their work (insight e). For example, both Rivest and Shamir being computer scientists relied on the help of Adleman, who was a mathematician, to test and shoot down their ideas for a mathematical function to fit the public-key cryptography system. And finally, the choice of historical episodes from the twentieth century (1940s and 1970s) had as an effect that it became easier for the students to realize that mathematics is a field that still develops in modern times, and is not just made up of rigid truths in their textbook based on findings from the seventeenth and eighteenth centuries (Jankvist 2009b).

The use of general topics and issues in the final essay assignments of the modules played a role in contributing to the students' development of the above mentioned insights. For example, having them look for and identify inner and outer driving forces in the second module assisted them in their group discussions of what drove the development of public-key cryptography and number theory, respectively (insight b), and thus having them try to act as historians on a smaller scale. Regarding how mathematics evolves through the ages (insight a), the general topic and issue of pure versus applied mathematics assisted the students in structuring their discussion of the need for basic research and Hardy's *Apology* (cf. the quotes from students' essays above). Furthermore, it provided them with a way of differentiating between different purposes for developing mathematics, and it provided them with a new vocabulary; before the second module students would talk about 'applications of mathematics' but not about 'applied mathematics' as a discipline in itself. The general topic and issue of multiple discoveries provided the students with insights into some of the more sociological aspects of the development of mathematics (and science in general), e.g. the need for recognition of one's work, the need for accuracy in attribution of credit, and not least the outer (and cultural) circumstances which may influence this, such as patent delays, classification of research, etc. (insights c and d). But most importantly, the idea of general topics and issues provided a setting and frame for the students to perform their discussions and reflections within, and it acted as a guide line in choosing the two historical cases in the first place. Of course it is important to weigh the difficulty of the mathematics involved, but when using history as a goal it is equally important to try and identify exemplary historical cases from which the students may make generalizations regarding the development of mathematics in a certain historical episode. And it seems a reasonable firsthand assumption, that the more general topics and issues a certain case includes, the more exemplary it is for the purpose of generalization.

Besides helping the students realize that the development of mathematics is dependent on time and space as well as culture and society, the methodological aid of epistemic objects and techniques in particular assisted in anchoring the students' discussions of the original sources in the mathematical in-issues of the historical cases. And more so, it helped *ensure* a presence of anchoring, because it is more or less impossible to identify what constitutes an epistemic technique in an original source, or excerpts thereof, without actually knowing some of the mathematics involved in these. Nevertheless, and as mentioned earlier, despite the other general topics and issues not having this 'built-in insurance' of anchoring, the essay assignments based on these still resulted in student answers with some degree of anchoring being present. One explanation for this may be the strong focus on the mathematical in-issues of the historical cases in the teaching materials. Somehow there seems to be a spin-off effect, meaning that if the students are presented to the meta-issues in the teaching material in a way that is already more or less anchored in the related mathematical in-issues, then this is somehow transferred into their own way of dealing with the meta-issues. But in any case, whether the anchoring in the essays came about by the multiple-perspective approach, the use of general topics and issues, or because it was present in the material, it qualified the history-as-a-goal insights of the students in the way that these were more than just anecdotical. For example, the students were not only able to say that Hamming used a bunch of math to develop his codes, they were able to say what math, e.g. that he used the notion of a metric as an epistemic technique and furthermore explain what this metric looks like in the discrete setting for two binary codes $x$ and $y$. Nor were the students only able to say that there is a lot of number theory behind RSA, they had worked with prime numbers and knew that it is difficult to prime factor large integers, they knew which historical theorems and other epistemic techniques that were used to make the RSA-algorithm function, and they were aware of the fact that even though these results had been available for centuries it required yet a historical component for all the pieces to fall into place and for the key-distribution problem to be solved, namely the computer.

## 9 Concluding Discussion

As mentioned in the beginning of the paper, when looking at the distribution of empirical studies on using history in mathematics education according to their initial purpose, the vast majority falls within the category of using history as a tool. Furthermore, most of these address the use of history as a cognitive tool rather than a motivational tool or an evolutionary tool (Jankvist 2009c).

The fact that hitherto no strong arguments for using history as a cognitive tool (or a motivational tool for that matter) have been made in comparison with arguments for other possible 'tools' for teaching and learning mathematics is, we believe, a major reason for the lack of impact of 'history in mathematics education' in mathematics education research in general. For a mathematics education researcher, who is mainly familiar with mathematics and perhaps some psychology, resorting to the history of mathematics is not likely to be the first obvious choice. This is because a use of history of mathematics requires more than just mathematical knowledge; it requires historical knowledge as well. And if one is not at least remotely familiar with parts of the history of mathematics, there are traps to fall in—the main one being adopting a Whiggist brand of history when incorporating it in teaching activities, as pointed out by Fried (2001). So, if one is not already familiar with

history of mathematics why should one then bother with all the effort, if the pursued educational ends can be reached by, perhaps, more accessible 'tools'?

As illustrated for scenario 1, a change in perspective towards mathematical competencies could make the educational benefits of using history as a tool stand out clearer. When addressing the teaching and learning of mathematics through a competence-based view of mathematics education, rather than only a concept development view, it becomes more evident what the straightforward gains of using history as a tool are. Namely, that students will develop their mathematical competencies when they come into contact with historical texts and sources, when they have to grasp these, relate them to modern presentations of the same material, communicate their findings, etc. And depending on the historical cases, the mathematical competencies can be developed on a basis of the curriculum mathematics that they had to encounter with anyway. Perhaps it might even be argued that the competence-based view of mathematics education seems a more natural setting for discussing the benefits of using history as a tool than one solely centered round the understanding of mathematical concepts.

Regarding scenario 2, we are aware that the educational programs where history of mathematics is included in terms of using history as a goal probably are few in number. However, and even though it may be on a small scale, there does seem to be an acknowledgement of broadening the scope of students' mathematical education, also in the sense of bringing in more elements related to the humanities, including history. As for the dilemma pointed to by Fried (see earlier), he himself states that it should not be taken as an argument for disregarding history of mathematics in mathematics education, rather "the dilemma should be taken as a challenge to reconsider the nature of mathematics education so that the history of mathematics as a serious pursuit becomes an integral part of what it means to be mathematical educated" (Fried 2008, p. 4). A similar point of view makes up the basis for the KOM-report's type of overview and judgment regarding the historical development of mathematics. Therefore it is sensible to evaluate students' developed historical insights according to the description and examples provided by the KOM-report (others may of course be thought of), as was illustrated by the examples for scenario 2. As may also be deduced from a careful reading of the KOM-report, the historical examples used in teaching should be exemplary, which the use of general topics and issues assisted in accomplishing, and as mentioned more explicitly the students' historical insights should possess some kind of solidity or anchoring. And as illustrated by the examples, the elements of the multiple-perspective approach to the history of mathematics concerning epistemic objects and techniques were especially helpful in bringing about such a kind of anchoring.

So to sum up briefly: based on our presentation of theoretical constructs and the empirical examples used to illustrate their use when it comes to the integration of history in mathematics education, it seems reasonable to argue that there indeed appears to be something to gain for the field by looking closer at the suggested new avenues, i.e.

1. the avenue of focusing on students' development of mathematical competencies when using history as a tool in a setting, where history of mathematics is not an integral part of the mathematics curriculum, and
2. the avenue of investigating ways to anchor students' meta-issue discussions and reflections in the related mathematical in-issues when using history as a goal in a setting, where historical insights are considered an integral part of the mathematics program.

Furthermore, and equally important, by adopting the multiple-perspective approach to the history of mathematics, it is clear that these avenues can be pursued without making the use of history neither anachronical nor anecdotical, i.e. without adopting a Whiggist brand of the history of mathematics when using it as a tool and without making the treatment of the historical meta-issues superficial and detached from the mathematical content when using it as a goal, respectively.[26]

But one thing is pointing out and arguing for what appears to be promising avenues for new research, another thing is to actually conduct research within these avenues. And it is only by the conduction of actual research, and in this case *empirical* research in particular, that these avenues may have an impact on 'the impact of 'history in mathematics education' in mathematics education research' in general. And furthermore, when conducting such empirical studies the field of mathematics education research has a lot to offer in the form of theoretical constructs and methods for designing teaching experiments, for gathering and producing data, and not least for analyzing the data. As examples of the latter we refer to Jankvist (2009c) and Jankvist (forthcoming), who has used the theory of commognition by Sfard (2008) to analyze different degrees of anchoring in students' meta-issue discussions and reflections as based on the various discursive tracks that these may follow, and Kjeldsen and Blomhøj (Accepted) who discuss history as a method for the learning of meta-discursive rules in mathematics, also within Sfard's theory of commognition. In doing so, stronger ties are also built between the field of history in mathematics education and the field of mathematics education research, ties which appear necessary for any future impact.

Thus, what we suggest is to apply the two suggested avenues in order to have a new look at some of the standing questions and claims within the field of history in mathematics education, to investigate and test these empirically, and in doing so applying the available and relevant literature from mathematics education research. By doing so we may help increase the impact and at the same time attempt to remedy or balance the current situation of the International Study Group on the Relations between the History and Pedagogy of Mathematics (HPM), which may be phrased as: "Too much H and M, not enough P".[27]

## References

Barbin, E., Stehlikova, N., & Tzanakis, C. (2007). Europan summer universities on the history and epistemology in mathematics education. In: Furinghetti, F., Kaijser, S. & Tzanakis, C. (Eds.), *Proceedings HPM2004 & ESU4*. Uppsala Universitet, revised edition, pp. xxix–xxxi.

Blomhøj, M., & Kjeldsen, T. H. (2009). Project organised science studies at university level: Exemplarity and interdisciplinarity. *ZDM—International Journal on Mathematics Education, 41*(1-2), 183–198.

Blomhøj, M., & Kjeldsen, T. H. (2010). Learning mathematics through modeling: The case of the integral concept. In B. Sriraman, L. Haapasalo, B. D. Søndergaard, G. Palsdottir, & S. Goodchild (Eds.), *The*

[26] In the present paper we have tried to illustrate how to establish if students posses and/or develop any of the eight mathematical competencies or the type of overview and judgment regarding history. However, the question of how to judge the degree of possession of a competency by someone is a different matter. In the KOM-report the authors distinguish between three dimensions that can be used to evaluate to what degree a person possesses a given competency: *coverage*, *span of action*, and *technical level*. Due to the KOM-report still being fairly new and because its aspects are still in the process of finding their way into actual teaching and mathematics programs, research on these aspects are only just emerging, and we will not go into further details on this issue here.

[27] Phrased by Victor Katz at the HPM Americas' annual meeting in Washington DC, 2010. See minutes: http://www.hpm-americas.org/wp-content/uploads/2010/04/Minutes.pdf

*first sourcebook on Nordic research in mathematics education* (pp. 569–581). Charlotte: Information Age Publishing.
Bos, H. J. M. (1975). *The calculus in the eighteenth century II: Techniques and applications. History of mathematics origins and development of the calculus 5*. Milton Keynes: The Open University Press.
Calinger, R. (Ed.). (1996). *Vita mathematica–historical research and integration with teaching*. Washington: The Mathematical Association of America.
Corry, L. (2004). Introduction: The history of modern mathematics–writing and rewriting. *Science in Context, 17*(1/2), 1–21.
Davis, C. (1994). Where did twentieth-century mathematics go wrong? In S. Chikara, S. Mitsuo, & J. W. Dauben (Eds.), *The intersection of history and mathematics, no. 15 in science networks (historical studies)* (pp. 129–142). Basel: Birkhäuser Verlag.
Diffie, W., & Hellman, M. E. (1976). New directions in cryptography. *IEEE Transactions on Information Theory, IT-22*, 644–654.
Epple, M. (2000). Genisis, Ideen, Institutionen, mathmatische Werkstätten: Formen der Mathematikgeschichte–Ein metahistorischer Essay. *Mathematische Semesterberichte, 47*, 131–163.
Epple, M. (2004). Knot invariants in Vienna and Princeton during the 1920s: Epistemic configurations of mathematical research. *Science in Context, 17*(1/2), 131–164.
Fasanelli, F. & Fauvel, J. G. (2007), The international study group on the relations between the history and pedagogy of mathematics: The first twenty-five years, 1967-2000. In F. Furinghetti, S. Kaijser, & C. Tzanakis (Eds.), *Proceedings HPM2004 & ESU4*. Uppsala Universitet, revised edition, pp. x–xxviii.
Fauvel, J. (Ed.). (1990). *History in the mathematics classroom. The IREM Papers*. Leicester: The Mathematical Association.
Fauvel, J. (1991). Using History in Mathematics Education. *For the Learning of Mathematics*, *11*(2), 3–6. Special Issue on History in Mathematics Education.
Fauvel, J., & van Maanen, J. (Eds.). (2000). *History in mathematics education—The ICMI study*. Dordrecht: Kluwer.
Fried, M. N. (2001). Can mathematics education and history of mahtematics coexist? *Science & Education, 10*, 391–408.
Fried, M. N. (2008). *History of mathematics and the future of mathematics education.* Mathematics Education: an ICMI perspective (WG5). Presented at the centennial celebration of ICMI in Rome and available at: http://www.unige.ch/math/EnsMath/Rome2008/.
Furinghetti, F. (2004). History and mathematics education: A look around the world with particular reference to Italy. *Mediterranean Journal for Research in Mathematics Education*, *3*(1–2), 1–20. Special double issue on the role of the history of mathematics in mathematics education (proceedings from TSG 17 at ICME 10).
Golay, M. J. E. (1949). Notes on digital coding. *Proceedings of the IRE, 37*, 657.
Gulikers, I., & Blom, K. (2001). 'A historical angle', A survey of recent literature on the use and value of the history in geometrical education. *Educational Studies in Mathematics, 47*, 223–258.
Hamming, R. W. (1950). Error detecting and error correcting codes. *Bell System Technical Journal, 29*, 147–160.
Hardy, G. H. (1992). *A Mathematician's apology*. Cambridge: Cambridge University Press, canto edition. Foreword by C. P. Snow.
Heath, T. L. (1912). *The works of Archimedes*. New York: Dover Publications, Inc.
Heath, T. L. (1956). *The thirteen books of Euclid's elements*. New York: Dover Publications, Inc.
Hersh, R. (1997). *What is mathematics really?*. Oxford: Oxford University Press.
Holm, A. M., Frier, B., Nielsen, L. H. & Pedersen, L. M. (2004). *Uendelighed og 'integration' i antikken*. IMFUFA, Roskilde University. English translation of title: *Infinity and 'integrations' in Antiquity*.
Jahnke, H. N., Knoche, N., & Otte, M. (Eds.). (1996). *History of mathematica and education: Ideas and experiences*, No. 11 in Studien zur Wissenschafts-Sozialund Bildungsgeschichte der Mathematik. Göttingen: Vandenhoeck & Ruprecht.
Jankvist, U. T. (2008a). Den matematikhistoriske dimension i undervisning—gymnasialt set. *MONA, 4*(1), 24–45. English translation of title: The dimension of the history of mathematics in teaching—The case of upper secondary level.
Jankvist, U. T. (2008b). *Kodningsteoriens tidlige historie—et undervisningsforløb til gymnasiet*, No. 459 in Tekster fra IMFUFA. Roskilde: IMFUFA. English translation of title: The early history of error correcting codes—A teaching module for upper secondary school.
Jankvist, U. T. (2008c). *RSA og den heri anvendte matematiks historie—et undervisningsforløb til gymnasiet*, No. 460 in Tekster fra IMFUFA. Roskilde: IMFUFA. English translation of title: RSA and the history of the applied mathematics in the algorithm—A teaching module for upper secondary school.

Jankvist, U. T. (2009a). A categorization of the 'whys' and 'hows' of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261.
Jankvist, U. T. (2009b). History of modern applied mathematics in mathematics education. *For the Learning of Mathematics, 29*(1), 8–13.
Jankvist, U. T. (2009c). *Using history as a 'Goal' in Mathematics Education*'. Ph.D. thesis, IMFUFA, Roskilde University. No. 464 in Tekster fra IMFUFA, Roskilde. http://milne.ruc.dk/Imfufa Tekster/pdf/464.pdf.
Jankvist, U. T. (2010a). An empirical study of using history as a 'goal'. *Educational Studies in Mathematics, 74*(1), 53–74.
Jankvist, U. T. (2010b) Essay review: A century of mathematics education: ICMI's first hundred years. *Historia Mathematica*. (Available online).
Jankvist, U. T. (forthcoming). Anchoring students' meta-perspective discussions of history in mathematics. (Under review).
Jensen, B. E. (2003). *Historie–livsverden og fag*. Copenhagen: Gyldendal.
Katz, V. (Ed.). (2000). *Using history to teach mathematics—An international perspective*, No. 51 in MAA notes. Washington: The Mathematical Association of America.
Kjeldsen, T. H. (2009a). Abstraction and application: New contexts, new interpretations in twentieth-century mathematics. In E. Robson & J. Stedall (Eds.), *The Oxford handbook of the history of mathematics* (pp. 755–780). New York: Oxford University Press.
Kjeldsen, T. H. (2009b). Egg-forms and measure-bodies: Different mathematical practices in the early history of the modern theory of convexity. *Science in Context, 22*(1), 85–113.
Kjeldsen, T. H. (2010a). History in mathematics education—Why bother? Interdisciplinarity, mathematical competence and the learning of mathematics. In B. Sriraman & V. Freiman (Eds.), *Interdisciplinarity for the 21st century: Proceedings of the 3rd international symposium on mathematics and its connections to arts and sciences, Moncton 2009* (pp. 17–48). Charlotte, NC: Information Age Publishing.
Kjeldsen, T. H. (2010b). History of convexity and mathematical programming: Connections and relationships in two episodes of research in pure and applied mathematics of the 20th century. In Rajendra Bhatia (Ed.), *Proceedings of the International Congress of Mathematicians 2010 - invited lectures* (Vol. 4, pp. 3233–3257). New Delhi: Hindustan Book Agency.
Kjeldsen, T. H. (forthcoming a). A multiple perspective approach to the history of the practice of mathematics in a competency based mathematics education: history as a means for the learning of differential equations. To appear in Katz, V., & Tzanakis, C. (Eds.), *Recent developments on introducing a historical dimension in mathematics education*, The Mathematical Association of America.
Kjeldsen, T. H. (forthcoming b). Uses of history in mathematics education: Development of learning strategies and historical awareness. To be published in *CERME 7, Proceedings of the seventh Congress of the European Society for Research in Mathematics Education.*
Kjeldsen, T. H., & Blomhøj, M. (2009). Integrating history and philosophy in mathematics education at university level through problem-oriented project work. *ZDM Mathematics Education Zentralblatt für Didaktik der Mathematik, 41*(1–2), 87–103.
Kjeldsen, T. H., & Blomhøj, M. (Accepted). Beyond motivation—History as a method for the learning of meta-discursive rules in mathematics. *Educational Studies in Mathematics*.
Kjeldsen, T. H., Pedersen, S. A., & Sonne-Hansen, L. M. (2004). Introduction. In T. H. Kjeldsen, S. A. Pedersen, & L. M. Sonne-Hansen (Eds.), *New trends in the history and philosophy of mathematics*, Vol. 19 of studies in philosophy (pp. 11–27). Odense: University Press of Southern Denmark.
Menghini, M., Furinghetti, F., Giacardi, L., & Arzarello, F. (2009). *The first century of the international commission on mathematical instruction (1908–2008)—Reflecting and shaping the world of mathematics education*. Roma: Istituto della Enciclopedia Italiana fondata da Giovanni Treccani.
Merton, R. K. (1973). *The sociology of science—Theoretical and empirical investigations*. Chicago: The University of Chicago Press. Chapter 16. Previously published as "Singletons and Multiples in Scientific Discovery" in Proceedings of the American Philosophical Society 105, no. 5, 1961 pp. 470–486.
Nielsen, K. H. M., Nørby, S. M., Mosegaard, T., Skjoldager, S. I. M. & Zacho, C. M. (2005). *Fysikkens indflydelse på udvikling af differentialligninger og den efterfølgende teoriudvikling*. IMFUFA, Roskilde University. English translation of title: Physics' influence on the development of differential equations and the following development of theory.
Niss, M. (2003). Mathematical competencies and the learning of mathematics: The Danish KOM project. In A. Gagatsis & S. Papastavridis (Eds.), *3rd Mediterranean conference of mathematical education* (pp. 115–124). Athens: Hellenic Mathematical Society.
Niss, M. & Jensen, T. H. (Eds.). (2002). *Kompetencer og matematiklæring—Ideer og inspiration til udvikling af matematikundervisning i Danmark*. Undervisningsministeriet. Uddannelsesstyrelsens

temahæfteserie nr. 18. English translation of title: Competencies and Learning of Mathematics—Ideas and Inspiration for the Development of Mathematics Education in Denmark.

Ottesen, S. T. (2009). *Relating university mahtematics teaching practices and students' solution processes*, No. 463. In Tekster fra IMFUFA. Roskilde: IMFUFA. Ph.D. thesis in the didactics of mathematics from IMFUFA, Roskilde University.

Rheinberger, H.-J. (1997). *Toward a history of epistemic things: Synthesizing proteins in the test tube*. Stanford: Stanford University Press.

Rivest, R. L., Shamir, A., & Adleman, L. (1978). A method for obtaining digital signatures and public-key cryptosystems. *Communications of the ACM, 21*(2), 120–126.

Schubring, G. (2008). *The debate on a "geometric algebra" and methodological implications*. Paper presented at the HPM 2008 satelitte meeting of ICME 11, Mexico.

Sfard, A. (2008). *Thinking as communicating*. New York: Cambridge University Press.

Shannon, C. E. (1948). A mathematical theory of communication I, II. In: D. Slepian (Ed.), *Key Papers in The Development of Information Theory*, pp. 5–18; 19–29. New York: IEEE Press.

Shell-Gellasch, A., & Jardine, D. (Eds.). (2005). *From calculus to computers–Using the last 200 years of mathematics history in the classroom*, No. 68 in MAA Notes. Washington: The Mathematical Association of America.

Singh, S. (1999). *The code book—The secret history of codes and code-breaking*. London: Forth Estate.

Smestad, B. (2002). Matematikkhistorie i grunnskolens lærebøker: En kritisk vurdering. *Alta* pp. 1–61.

Smestad, B. (2003). Historical topics in Norwegian textbooks. In O. Bekken & R. Mosvold (Eds.), *Study the masters* (pp. 163–168). Göteborg: Nationellt Centrum för Matematikutbildning, NCM.

Swetz, F., Fauvel, J., Bekken, O., Johansson, B., & Katz, V. (Eds.). (1995). *Learn from the masters*. Washington: The Mathematical Association of America.

Tzanakis, C., & Arcavi, A. (2000). Integrating history of mathematics in the classroom: Ananalytic survey. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education* (pp. 201–240). Dordrecht: Kluwer/The ICMI Study. Chapter 7.

UVM (Undervisningsministeriet) (1987). 'Bekendtgørelse af 1987'. English translation of title: Ministerial order of 1987.

UVM (Undervisningsministeriet) (2007). Vejledning: Matematik A, Matematik B, Matematik C. Bilag 35, 36, 37. English translation of title: Ministerial order of 2007—Mathematics levels A, B, and C.

Wells, D. (2005). *Prime numbers—The most mysterious figures in math*. Hoboken, NJ: Wiley.