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Research Article

A Computational Method for Volterra Integro-Differential Equation

Erkan ÇİMEN

Van Yuzuncu Yil University, Faculty of Education, Department of Mathematics, Van, Turkey

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Abstract
In this paper, we examine the initial value problem for a linear first order Volterra integro-differential equation. In order to solve the problem computationally, we present a novel finite difference method, which is based on the method of integral identities with the use of the basis functions and interpolating quadrature rules with remainder term in integral form. Furthermore, as a consequence of error analysis the method is proved to be first-order convergent in the discrete maximum norm. Finally, an example is provided to support our theoretical results.

Keywords: Error Estimate, Finite Difference Method, Initial Value Problem, Volterra Integro-Differential Equation

Volterra İntegro-Diferansiyel Denklem İçin Bir Çözüm Yöntemi

Öz
Bu çalışmada, lineer birinci mertebeden Volterra integro-diferansiyel denklem içeren başlangıç değer problemini ele almaktayız. Bu problemi nümerik olarak çözmek için yeni bir sonlu fark metodu veriyoruz. Bu metot, kalan terimi integral biçiminde olan interpolasyon quadratür formülleri ve üstel baz fonksiyonunu içeren integral özdeşliklerinden meydana gelmektedir. Ayrıca, bu metodun hata analizinin bir sonucu olarak, ayrık maksimum normda birinci mertebeden yakınsaklığı ispatlandı. Son olarak, elde edilen teorik sonuçları destekleyen nümerik örnek verildi.

Anahtar Kelimeler: Başlangıç-Değer Problemi, Hata Değerlendirmesi, Sonlu Fark Metodu, Volterra İntegro-Diferansiyel Denklem

1. Introduction

In this work, we are concerned with the following Volterra integro-differential initial value problem:

$$Lu := u'(t) + a(t)u(t) = f(t) + \int_0^t K(t,s)u(s)ds,\ t \in I, \quad (1.1)$$

$$u(0) = A \quad (1.2)$$

where $I = (0, T]$ and A is a given constant. $a(t)$, $f(t)$ and $K(t,s)$ are given sufficiently smooth functions on $\bar{I} = [0, T]$ and $\bar{I} \times \bar{I}$, respectively and moreover

$a(t) \geq \alpha > 0.$

Volterra integro-differential equations (VIDEs) arise widely in physics, chemistry, biology and engineering applications modelled by initial value problems (Jerri, 1999; Gaetano and Arino, 2000; Song and Baker, 2004). In recent years, it seems that the studies on these problems are being taken more and more into consideration, both in terms of modeling and solution of them. Existence and uniqueness of solution to VIDEs is discussed in books Volterra (1959); Hackbusch (1995); Lakshmikantham and Rao (1995); Jerri (1999); Burton (2005) and the references therein. Furthermore, there are many researchers who have investigated asymptotic expansion and numerical

Corresponding Author: cimenerkan@hotmail.com

approaches for this type equations (Chang, 1982; Hackbusch, 1995; Lakshmikantham and Rao, 1995; Kythe and Puri, 2002; Rahman, 2007; Babolian and Shamloo, 2008; Mehdiyeva et al., 2011; Wazwaz, 2011). The authors in Hoppensteadt et al. (2007); Fazeli and Hojjati (2015); Okayama (2018) are studied the numerical solutions of VIDEs by using various methods, such as collocation method, Runge-Kutta method, Sinc-Nyström method. Nevertheless, authors in Amiraliyev and Sevgin (2006); Amiraliyev and Yilmaz (2014); Kudu et al. (2016) are suggested numerical solutions of VIDEs by using fitted difference method. In this paper, we are present a novel finite-difference scheme on a uniform mesh to approximate (1.1-(1.2). This approach based on the method of integral identities with the use of interpolating quadrature rules with the weight and remainder terms in integral form. As a consequence of the method, in local truncation errors contain only first order derivative of the exact solution and hence eases analysis of the convergence. Our aim of this paper is to approximate for solving (1.1)-(1.2) using a numerical method based on the technique in work of Amiraliyev and Yilmaz (2014). The remainder paper is organized as follows. In Section 2, we put forward a priori estimates of the exact solution of (1.1)-(1.2). In Section 3, we introduce finite difference discretization of the problem (1.1)-(1.2). The error analysis of the approximate solution of the problem is present in Section 4. Furthermore, we prove convergence of our method in the discrete maximum norm. The algorithm for solving the difference problem and the numerical results is formulated in Section 5. Finally, a summary of the main conclusions of the paper are also given.

Notation: Henceforth, C indicates a generic positive constant. In addition, some particular fixed constants of this type are denoted by subscripting C. We shall use $\|g\|_\infty = \max_{t\in\bar{I}}|g(t)|$, for any $g \in C(\bar{I})$.

2. The Properties of The Exact Solution

Here, we give some important properties for the solution of our problem, which are needed in later sections for the analysis of appropriate the numerical solution.

Lemma 2.1: Let $a(t)$, $f(t) \in C(\bar{I})$ and $K(t,s) \in C(\bar{I}\times\bar{I})$. Then for the solution u of the problem (1.1)-(1.2) the following estimates hold:

$$\|u\|_\infty \le C_0, \tag{2.1}$$

$$\|u'\|_\infty \le C \tag{2.2}$$

where

$$C_0 = (|A| + \alpha^{-1}\|f\|_\infty)e^{\alpha^{-1}\bar{K}T},$$

$$\bar{K} = \max_{\bar{I}\times\bar{I}}|K(t,s)|.$$

Proof: From (1.1), we have

$$u(t) = u(0)e^{-\int_0^t a(\eta)d\eta} + \int_0^t [f(s) + \int_0^s K(s,\xi)u(\xi)d\xi]e^{-\int_s^t a(\eta)d\eta}ds,$$

$$|u(t)| \le |u(0)|e^{-\alpha t} + \int_0^t [|f(s)| + \int_0^s |K(s,\xi)||u(\xi)|d\xi]e^{-\alpha(t-s)}ds$$

$$\le |A|e^{-\alpha t} + \alpha^{-1}\|f\|_\infty(1-e^{-\alpha t}) + \alpha^{-1}\bar{K}(1-e^{-\alpha t})\int_0^t |u(\xi)|d\xi$$

$$\le |A| + \alpha^{-1}\|f\|_\infty + \alpha^{-1}\bar{K}\int_0^t |u(\xi)|d\xi.$$

Now, applying the Gronwall's inequality to this inequality, we get

$$|u(t)| \le (|A| + \alpha^{-1}\|f\|_\infty)e^{\alpha^{-1}\bar{K}t}$$

which leads to (2.1). Also, from (1.1) we have

$$|u'(t)| \le |a(t)||u(t)| + |f(t)| + \int_0^t |K(t,s)||u(s)|ds$$

$$\le \|a\|_\infty C_0 + \|f\|_\infty + \bar{K}C_0\int_0^t ds$$

$$\le \|f\|_\infty + C_0(\|a\|_\infty + \bar{K}T)$$

which immediately leads to (2.2)

3. The Difference Scheme

We introduce the uniform mesh on the I:

$$\omega_N = \{t_i = i\tau, i = 1,2,\dots,N;\ \tau = T/N\},$$

$$\bar{\omega}_N = \omega_N \cup \{0\}.$$

In order to simplify the notation, we define $g_i = g(t_i)$ for any function $g(t)$ and y_i represents an approximation of $u(t)$ at t_i also $g_{i-\frac{1}{2}} = g(t_i - \frac{\tau}{2})$. Also, for any mesh function g_i defined on ω_N we use

$$g_{\bar{t},i} = \frac{g_i - g_{i-1}}{\tau},$$

$$\|g\|_{\infty,\omega_N} = \max_{1\le i\le N}|g_i|.$$

We give the difference approximation of Eq. (1.1), by using the following identity:

$$\tau^{-1}\int_{t_{i-1}}^{t_i} Lu(t)\varphi_i(t)dt = \tau^{-1}\int_{t_{i-1}}^{t_i}[f(t) + \int_0^t K(t,s)u(s)ds]\varphi_i(t)dt,\ 1 \le i \le N \tag{3.1}$$

with the basis functions

$$\varphi_i(t) = e^{-\int_t^{t_i} a(s)ds}, t_{i-1} \le t \le t_i$$

which is the solution of the problem

$$\begin{cases} -\varphi_i'(t) + a(t)\varphi_i(t) = 0, t_{i-1} < t \le t_i, \\ \varphi_i(t_i) = 1. \end{cases} \tag{3.2}$$

The relation (3.1) is rewritten by taking into account the equation (1.1), we get

$$\tau^{-1}\int_{t_{i-1}}^{t_i} u'(t)\varphi_i(t)dt + \tau^{-1}\int_{t_{i-1}}^{t_i} a(t)u(t)\varphi_i(t)dt = \tau^{-1}\int_{t_{i-1}}^{t_i} f(t)\varphi_i(t)dt + \tau^{-1}\int_{t_{i-1}}^{t_i}[\int_0^t K(t,s)u(s)ds]\varphi_i(t)dt. \tag{3.3}$$

Next, using formulas (2.1) and (2.2) from Amiraliyev and Mamedov (1995) on each interval (t_{i-1}, t_i) left hand side (3.3) and taking into account (3.2) we have following precise relation

$$\tau^{-1}\int_{t_{i-1}}^{t_i} u'(t)\varphi_i(t)dt + \tau^{-1}\int_{t_{i-1}}^{t_i} a(t)u(t)\varphi_i(t)dt = A_i u_{\bar{t},i} + B_i u_i.$$

where

$$A_i = \tau^{-1}\int_{t_{i-1}}^{t_i}\varphi_i(t)dt + \tau^{-1}\int_{t_{i-1}}^{t_i}(t - t_i)a(t)\varphi_i(t)dt,$$

$$B_i = \tau^{-1}\int_{t_{i-1}}^{t_i} a(t)\varphi_i(t)dt. \tag{3.4}$$

For the integral term from (3.3), after applying the appropriate quadrature rules, we have

$$\tau^{-1}\int_{t_{i-1}}^{t_i}\left[\int_0^t K(t,s)u(s)ds\right]\varphi_i(t)dt = \tau^{-1}\int_{t_{i-1}}^{t_i}\varphi_i(t)dt\int_0^{t_{i-\frac{1}{2}}} K\left(t_{i-\frac{1}{2}}, s\right)u(s)ds + R_i^{(1)}$$

$$= [\tau^{-1}\int_{t_{i-1}}^{t_i}\varphi_i(t)dt]\tau\sum_{j=0}^{i-1} K\left(t_{i-\frac{1}{2}}, s_j\right)u_j + R_i^{(1)} + R_i^{(2)}$$

where

$$R_i^{(1)} = \tau^{-1}\int_{t_{i-1}}^{t_i} dt\varphi_i(t)\int_{t_{i-1}}^{t_i}\frac{\partial}{\partial t}\left[\int_0^t K(t,s)u(s)ds\right] \times [T_0(s-t) - T_0(t_{i-\frac{1}{2}} - t)]dt \tag{3.5}$$

$$R_i^{(2)} = \sum_{j=1}^{i-1}\int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}}[t_{j+\frac{1}{2}} - t - \tau T_0(t_j - t)] \times \times \frac{d}{dt}\left[K\left(t_{i-\frac{1}{2}}, t\right)u(t)\right]dt \tag{3.6}$$

and

$$T_0(\lambda) = 1,\ \lambda > 0;\ T_0(\lambda) = 0, \lambda \le 0.$$

Hereby, we write the exact relation for $u(t_i)$:

$$\ell u_i := A_i u_{\bar{t},i} + B_i u_i = F_i + \tau C_i\sum_{j=0}^{i-1} K\left(t_{i-\frac{1}{2}}, t_j\right)u_j + R_i \tag{3.7}$$

with

$$C_i = \tau^{-1}\int_{t_{i-1}}^{t_i}\varphi_i(t)dt,$$

$$F_i = \tau^{-1}\int_{t_{i-1}}^{t_i} f(t)\varphi_i(t)dt,$$

$$R_i = R_i^{(1)} + R_i^{(2)}$$

where A_i, B_i and $R_i^{(k)}$ $(k = 1,2)$ are determined by (3.4), (3.5), (3.6), respectively.

By virtue of (3.7) we suggest the following difference scheme for approximating (1.1)-(1.2):

$$\ell y_i := A_i y_{\bar{t},i} + B_i y_i = F_i + \tau C_i\sum_{j=0}^{i-1} K\left(t_{i-\frac{1}{2}}, t_j\right)y_j, \tag{3.8}$$

$$y_0 = A. \tag{3.9}$$

4. Stability Bound and Convergence

In order to examine the convergence of the method, the error function $z_i = y_i - u_i$, $(1 \le i \le N)$ which is the solution of the following discrete problem

$$\ell z_i = R_i, 1 \le i \le N, \tag{4.1}$$

$$z_0 = 0. \quad . \tag{4.2}$$

Lemma 4.1: Let $|G_i| \le \overline{G}_i$ and $\overline{G}_i$ be a nondecreasing function.

$$\ell v_i := A_i v_{\bar{t},i} + B_i v_i = G_i, 1 \le i \le N \quad (4.3)$$

$$v_0 = A \quad . \quad (4.4)$$

Then the solution of difference problem (4.3)-(4.4) holds:

$$|v_i| \le |A| + \alpha^{-1}\overline{G}_i, 1 \le i \le N.$$

Proof: The proof is almost identical that of (Amiraliyev and Yilmaz, 2014). ■

Lemma 4.2: Let $K(t,s) \in C^1(\bar{I} \times \bar{I})$. Then for the truncation error R_i, the following estimate holds

$$\|R\|_{\infty,\omega_N} \le CN^{-1}.$$

Proof: Since $0 < \varphi_i(t) \le 1$

$$\left|R_i^{(1)}\right| \le \int_{t_{i-1}}^{t_i} \left|\frac{\partial}{\partial t}\left[\int_0^t K(t,s)u(s)ds\right]\right| dt$$

and after applying Leibniz rule, we get

$$\left|R_i^{(1)}\right| \le \int_{t_{i-1}}^{t_i} |K(t,t)u(t)|\, dt + \int_{t_{i-1}}^{t_i} \left|\left[\int_0^t \frac{\partial}{\partial t}K(t,s)u(s)ds\right]\right| dt.$$

Hence the estimate $\left|R_i^{(1)}\right| \le C\tau$ is easily obtained.

Next, from (3.6)

$$\left|R_i^{(2)}\right| \le 2\tau \sum_{j=1}^{i-1} \int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \left[\frac{d}{dt}K\left(t_{i-\frac{1}{2}},t\right)u(t) + K\left(t_{i-\frac{1}{2}},t\right)u'(t)\right] dt$$

taking into account Lemma 2.1, we have

$$\left|R_i^{(2)}\right| \le 2\tau^2(N-1)\left(C_0\left|\frac{\partial}{\partial s}K(t,s)\right| + C\overline{K}\right) \le C\tau. \quad ■$$

Lemma 4.3: The solution of (4.1)-(4.2) satisfies the following estimate

$$\|z\|_{\infty,\omega_N} \le C\|R\|_{\infty,\omega_N}. \quad (4.5)$$

Proof: From the solution of (4.1)-(4.2):

$$\ell z_i := A_i z_{\bar{t},i} + B_i z_i = \tau C_i \sum_{j=0}^{i-1} K\left(t_{i-\frac{1}{2}}, t_j\right) z_j + R_i$$

and taking account Lemma 4.1

$$|z_i| \le \alpha^{-1}\tau \sum_{j=0}^{i-1} \overline{K}|z_j| + \alpha^{-1}\|R\|_\infty.$$

From here by using the difference analogue of Gronwall's inequality, we get

$$|z_i| \le \alpha^{-1}\|R\|_\infty e^{\alpha^{-1}\overline{K}t_i}$$

and this immediately leads to (4.5). ■

Finally, we give the main convergence result.

Theorem 4.1: Let u be the solution of (1.1)-(1.2) and y the solution (3.8)-(3.9). Then

$$\|y - u\|_{\infty,\bar{\omega}_N} \le CN^{-1}.$$

Proof: By combining the previous Lemmas 4.2 and 4.3, we can immediately prove. □

5. Algorithm and Numerical Results

In this section, we present numerical results obtained by applying the numerical method (3.8)-(3.9) to the particular problem. We rewritten difference scheme (3.8)

$$y_i = \frac{A_i}{A_i+\tau B_i} y_{i-1} + \frac{\tau F_i}{A_i+\tau B_i} + \frac{\tau^2 C_i}{A_i+\tau B_i} \sum_{j=0}^{i-1} K\left(t_{i-\frac{1}{2}}, t_j\right) y_j$$

from hence, for $1 \le i \le N$ with together $y_0 = A$, we get any y_i.

Example 5.1 Now, we consider the test problem

$$u'(t) + 2u(t) = 1 - t + \int_0^t (t-s)u(s)ds,\ t \in (0,1],$$
$$u(0) = 1.$$

The exact solution is given by

$$u(t) = e^{-t}.$$

We define the exact error E_i^N, the computed maximum pointwise error E^N as follows, respectively:

$$E_i^N = |y_i - u_i|,\ E^N = \max_{0 \le i \le N} E_i^N$$

where y is the numerical approximation to u for various values of N. For Example 5.1 the computational results obtained by present method are given in the Tables 1-3.

Table 1. Computational results for $N = 64$

t_i	$u(t_i)$	$y(t_i)$	E_i^N
0.000	1.0	1.0	0.0
0.125	0.8824969	0.8825552	5.83341E-5
0.250	0.7788008	0.7790124	2.11611E-4
0.375	0.6872893	0.6877284	4.39125E-4
0.500	0.6065307	0.6072557	7.25078E-4
0.625	0.5352614	0.5363190	1.05760E-3
0.750	0.4723666	0.4737945	1.42799E-3
0.875	0.4168620	0.4186921	1.83006E-3
1.000	0.3678794	0.3701391	2.25968E-3

Table 2. Computational results for $N = 128$

t_i	$u(t_i)$	$y(t_i)$	E_i^N
0.000	1.0	1.0	0.0
0.125	0.8824969	0.8825256	2.86513E-5
0.250	0.7788008	0.7789057	1.04941E-4
0.375	0.6872893	0.6875077	2.18464E-4
0.500	0.6065307	0.6068919	3.61281E-4
0.625	0.5352615	0.5357889	5.27430E-4
0.750	0.4723666	0.4730791	7.12533E-4
0.875	0.4168620	0.4177755	9.13484E-4
1.000	0.3678794	0.3690076	1.12820E-3

Table 3. Maximum pointwise errors for Example 5.1

N	E^N	N	E^N
32	4.53203E-3	256	5.63682E-4
64	2.25968E-3	512	2.81736E-4
128	1.12820E-3	1024	1.40842E-4

6. Conclusions

In this work, we have developed a novel method for approximating solution of the initial-value problem for a linear first order Volterra integro-differential equation. The method was based on an exponentially difference scheme on a uniform mesh. As results from the method, first order convergence in the discrete maximum norm was obtained. However, using the method, we were solved a numerical and the obtained results were displayed in Tables 1-3. These results were achieved to show the efficiency and accuracy of the our method. Theoretical results represented undergoing research more complicated Volterra integro-differential equations, such as delay Volterra integro-differential equation.

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