ESSAYS IN INTERTEMPORAL ECONOMICS

by

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Submitted to the Department of
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ABSTRACT

These essays seek to advance understanding of the properties of intertemporal equilibrium models, with a view toward providing more rigorous foundations for monetary theory. Much of best-known monetary theory of recent decades relies heavily upon casual references to general equilibrium theory as justification for simple conclusions about money, even though there is no role for money in standard general equilibrium models. These essays show that when the standard models are modified, by assuming finite-lived agents or transaction costs, familiar qualitative results about general equilibria need no longer hold.

The first two essays are concerned with the properties of dynamic models in which some agents are finite-lived while others are infinite-lived. Perfect foresight equilibria of overlapping generations models are known to have very different properties from those of models in which all agents are infinite-lived. For example, equilibria of overlapping generations models need not be Pareto-efficient, may involve valued fiat money, and may be locally indeterminate; none of these are possible in the case of a model with only infinite-lived agents. These essays consider a stationary economy with agents of both types; production and non-depreciating productive assets are also allowed for. The first essay establishes the existence of perfect foresight equilibrium and of steady state equilibrium

under quite general conditions. It is shown that monetary equilibria are not possible if any infinite-lived agents or non-depreciating assets are present.

The second essay considers the local determinacy of perfect foresight equilibrium near a steady state of such an economy. It is shown that the mere presence of infinite-lived agents, production, and non-depreciating assets does not rule out the possibility of indeterminate equilibrium. It is shown, however, that a sufficiently large population of effectively infinite-lived agents, a sufficiently large number of independent production activities in use, a sufficient amount of substitution in production, or a sufficient amount of substitution among goods in the demands of the finite-lived consumers would suffice to guarantee that any equilibria that exist are locally determinate.

The third essay presents a model of a "precautionary" demand for money, arising from the conjunction of transaction costs upon the premature liquidation of an investment project and uncertainty upon the part of agents about the timing of future expenditures. The effects of perfectly anticipated monetary policy in such a model are considered; it is shown that the optimal rate of inflation is positive, contrary to the famous Friedman argument. This is because if the demand for money results from the sort of need for liquidity modelled here, the effects of perfectly anticipated inflation are not simply those of a distorting tax on cash balances. It is shown that the redistributive aspect of inflationary policy can effectively provide insurance against an unexpected need for liquidity, which insurance the agents are unable to arrange by other means. For sufficiently low rates of inflation, this welfare gain will outweigh the deadweight loss due to the distorting tax aspect of inflationary policy.

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Although he taught at M.I.T. for only a semester, Frank Hahn's lectures provided a great deal of inspiration for this dissertation. It was he who taught me to discriminate between the aspiration to rigorous formalization and the casual invocation of abstract theory in policy debates, and who convinced me that a deeper understanding of intertemporal equilibrium models could contribute to the reconstruction of monetary theory. His comments on the third essay in particular were helpful.

I would also like to thank my class-mate, Walter Muller, co-author of two of these essays. That this has been the most fruitful collaborative effort in which I have ever engaged owes much to Walter's gracious manner. Discussions with him have greatly improved my understanding of the topics treated here, as well as my use of the English language.

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Stationary Overlapping Generations Economies with Production and Infinite-Lived Consumers:

I. Existence of Equilibrium.

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and

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I. Introduction

A relatively well-developed literature exists on overlapping generations models of pure exchange economies. Under reasonably general conditions, Balasko and Shell (1980,1981a) have established the existence of perfect foresight Walrasian equilibrium, either with or without fiat money as a store of value. In the special case of a single two period lived consumer per generation, with a log-linear utility function, Balasko and Shell (1981b) have established that non-monetary equilibrium is unique.

Kehoe and Levine (1983) show that in the case of stationary economies, steady state perfect foresight equilibria - that is, equilibria in which the relative prices of goods each period remain constant and the consumption by each generation is the same - exist both with and without valued fiat money, and that under generically valid regularity conditions the number of steady states of each type is odd. They also show that near steady states, perfect foresight equilibria need not be determinate - that is, there may be a continuum of valid perfect foresight equilibria converging to the steady state - and that the dimension of the continuum of

equilibria is robust to perturbations of the parameters of the economy.

Since pure exchange economies with only finite-lived consumers admit equilibria with valued fiat money without having to introduce transactions costs or other ad hoc reasons for holding money, they have been the subject of most of the work in the literature. However, there are a number of natural extensions to the model that are interesting in their own right. Diamond (1965) and Gale (1972) have given preliminary treatments of overlapping generations models with production. Wilson (1981) has established the existence of perfect foresight equilibria for pure exchange economies with both finite-lived and infinite-lived agents.[1] These extensions, however, have not been treated as thoroughly as the pure exchange case with finite-lived consumers alone.

In this paper we develop a stationary overlapping generations model with production, with infinite-lived as well as finite-lived consumers, and with nondepreciating assets called "land" that yield a stationary stream of consumption goods. We show that Wilson's method of proof of the existence of equilibrium in the pure exchange case can be modified to establish existence of equilibrium for

our model and we also prove the existence of a non-monetary steady state equilibrium. In a companion paper, Muller and Woodford (1983), we examine the question of local determinacy of equilibrium near such steady states.

II. The Model

We analyze a stationary overlapping generations model that generalizes the one in Kehoe and Levine (1983). Since there are many similarities, we use their notation whenever possible. There are n goods in each time period. Each generation t=1,2,3... of finite-lived agents is identical, and consists of a finite number of agents who consume in period t and t+1.[2] There is also a generation 0 that consists of a finite number of agents who consume only in period 1. In addition, there is a finite set of infinite-lived agents that live in every period. Each infinite-lived agent has an additively separable utility function and an endowment which is the same in each period. Finally, there is a constant returns to scale production technology available in each period that allows production spanning two periods.

The consumption and savings decisions of the different types of consumers in generation t, for t=1,2,3..., are aggregated into excess demand functions $y(p_t,p_{t+1})$ in period t and $z(p_t,p_{t+1})$ in period t+1 where p_t denotes the price vector for goods in period t. The aggregate excess demand of generation 0 is denoted $z_0(p_1)$. We assume the excess demand functions to have the following properties.

(A.1) (Differentiability) $y,z:R^{2n}_{++}\to R^n$, $z_0:R^n_+\to R^n$ are smooth C^1 functions.

(A.2) (Walras's Law) $p_t'y(p_t,p_{t+1})+p_{t+1}'z(p_t,p_{t+1})=0$ for t=1,2,3...; $p_1'z_0(p_1)=0$.[3]

(A.3) (Homogeneity) y,z,z_0 are homogenous of degree 0.

(A.4) (Non-Satiation)[4] $||y+z||\to\infty$ as (p_t,p_{t+1}) approaches any element of $\partial R^{2n}_+\setminus\{0\}$, $||z_0||\to\infty$ as p_1 approaches any element of $\partial R^n_+\setminus\{0\}$.

(A.5) (Boundedness) There exist finite n-vectors $\bar{y},\bar{z} >>0$ such that $y(p_t,p_{t+1})>>-\bar{y}$, $z(p_t,p_{t+1}) >>-\bar{z}$ for all $(p_t,p_{t+1}) \in R^{2n}_{++}$, and $z_0(p_1) >>-\bar{z}$ for all $p_1 \in R^n_+$.

The vectors $\bar{y}$ and $\bar{z}$ in (A.5) represent the aggregate endowment of a generation in the first and second periods of its life respectively.

Conditions (A.1) - (A.5) are the only properties that y and z must satisfy in general. Debreu (1974) has demonstrated that, for any y and z that satisfy (A.2) and (A.3), there exists a generation of 2n utility maximizing consumers whose aggregate excess demand functions y* and z* agree with y and z on any set of positive relative prices uniformly bounded away from zero. There is a minor technical complication in that y* and z* may not agree with y and z as some relative prices approach zero. As we argue below, however, we can ignore this qualification when studying the behavior of the excess demand functions near steady states.

One might think that the excess demand functions require arguments representing the rates of return available by holding land or engaging in production.

This is not the case. Consider first a monetary economy. In equilibrium it must be true that the available rate of return through any such arbitrage activities is no higher than the rate of return on holding money - a gross nominal return of exactly one - or money would not be held. Therefore the excess demands in any equilibrium are exactly those that would have been calculated assuming that the only means of saving is by holding cash and the equilibrium conditions derived using $y(p_t, p_{t+1})$ and $z(p_t, p_{t+1})$ are the correct ones. The argument is similar in a non-monetary economy. In equilibrium all production opportunities must generate non-positive profits, so any production process in use makes zero profits and has a gross nominal return equal to one. Further, there can be no arbitrage profit possible from holding land in equilibrium, so the gross nominal return must be exactly one. Again the excess demands in equilibrium are exactly those that would have been calculated assuming that the only means of saving is by holding cash, and the equilibrium conditions derived using $y(p_t, p_{t+1})$ and $z(p_t, p_{t+1})$ are the correct ones.

There are also a finite number of infinite-lived

agents,[5] denoted h=1,2,...,H. Infinite-lived consumer h has a utility function

$$V^h = \sum_{t=1}^{\infty} (\beta_h)^{t-1} u^h(c_t^h) \tag{2.1}$$

where c_t^h is the consumption vector for h in period t and $0 \leq \beta_h < 1$ is the discount factor for h. The single period utility function u^h satisfies the following assumptions.

(B.1) (Differentiability) For each h, $u^h: R^n_{++} \to R$ is C^2.

(B.2) (Monotonicity, Concavity) For each h, $Du^h >> 0$ and D^2u^h is negative definite, for all $c_t^h \in R^n_{++}$.

(B.3) (Boundary)[6] For each h, $||Du^h|| \to \infty$ as c_t^h approaches any element of ∂R^n_+, but $||Du^h||$ is bounded for c_t^h in any bounded subset of R^n_{++}.

Infinite-lived consumer h has an n-vector of endowments a^h in each period, where $a^h >> 0$. Given a price sequence $\{p_t\}_1^{\infty}$, the consumption demands of h are chosen so as to maximize V^h subject to the lifetime budget constraint[7]

$$(2.2) \qquad \sum_{t=1}^{\infty} p_t' c_t^h \leq \sum_{t=1}^{\infty} p_t' a^h.$$

The production technology is specified by a cone $Q \subset R^{2n}$, where $(q_1, q_2) \in Q$ means that a feasible production plan exists that yields net output q_1 in the first period of operation and q_2 in the second. The cone Q satisfies the following assumptions.[8]

(C.1) Q is closed and strictly convex.

(C.2) If $(0,q) \in Q$, then $q \leq 0$.

(C.3) (Boundedness) There exists $\hat{p} >> 0$ such that $\hat{p}'(q_1+q_2) \leq 0$ for all $(q_1, q_2) \in Q$.

(C.4) If $(q_1, q_2) \in Q$, then $(q_1, 0) \in Q$.

(C.5) (Free Disposal) If $(q_1, q_2) \in Q$, $q_3 \leq q_1$ and $q_4 \leq q_2$, then $(q_3, q_4) \in Q$.

While (C.1) and C.5) are familiar assumptions, the others require some comment. (C.2) says that single

period production - aside from simply disposing of goods - takes place in the first period of operation. Note that this assumption places no restrictions on the production possibilities in the economy.

Assumption (C.3) is necessary in order to guarantee the existence of steady state equilibrium. For the proof of equilibrium, the following weaker version suffices:

(C.3') There exists $(p_1, p_2) >> 0$ belonging to Q^*, the dual cone of Q; i.e., such that $p_1'q_1 + p_2'q_2 < 0$ for all $(q_1, q_2) \in Q$.

(C.3') is plainly necessary for equilibrium to exist, for otherwise, no matter what prices prevailed, agents would wish to operate some production activity at an unbounded level. But the stronger assumption (C.3) is necessary if a steady state equilibrium is to exist, regardless of the rate of time preference of the infinite-lived consumers. For if (C.3) did not hold, there would exist some $(q_1, q_2) \in Q$ such that $q_1 + q_2 \geq 0$, $q_1 + q_2 \neq 0$. It would then be impossible for the zero profit condition to hold with the same goods prices each period. Furthermore, there

would be some threshold $\bar{\beta}<1$ such that for any $\bar{\beta}<\beta<1$, and any $p \in R^n_{++}$, production activity (q_1,q_2) would yield positive profits at prices $p_1=p$, $p_2=\beta p$. Therefore it would be impossible for there to exist a steady state equilibrium with $\bar{\beta}<\beta<1$. But if an infinite-lived consumer existed with $\beta_h>\bar{\beta}$, there could exist no steady state with $\beta\leq\bar{\beta}$ (as will be discussed in Section V), and so there could exist no steady state.

Assumption (C.4) is necessary in order to characterize the profit opportunities of the infinite horizon economy in terms of the profitability of elements of Q. Without an assumption of this sort, there could exist a price sequence $\{p_t\}_1^\infty$ such that $p_t'q_1+p_{t+1}'q_2\leq 0$ for all $(q_1,q_2) \in Q$, and for all $t=1,2,3...$, and yet there exist a production plan $\{q_{1t},q_{2t}\}_1^\infty$ that makes positive profits. For example, consider an economy with two goods and let Q consist of any positive scalar multiple of $(q_1,q_2)=(1,-1;-2,2)$, or any vector strictly less than such a vector. Let $p_{1t}=2(\frac{3}{4})^{t-1}$, $p_{2t}=(\frac{3}{4})^{t-1}$. Then at prices (p_t,p_{t+1}), vector (q_1,q_2) earns negative profits, $-\frac{3}{2}(\frac{3}{4})^{t-1}$. But there exist feasible production plans

with positive profits, such as $(q_{1t},q_{2t})=(2^{t-1}, -2^{t-1}; -2^{t},2^{t})$ for t=1,2,3,..., which yields total profits of +1. This problem arises when an activity makes positive profits in the initial period of operation and negative in the second. Even if the negative profits in the second period are larger than the positive profits in the first period, it may be possible in an infinite-horizon economy to take the profits of the first period, then cover the expenditure in the second period by operating the process again, and so on forever. In fact, it could be possible to make infinite positive profits in this way, even though the production activity (q_{1t},q_{2t}) chosen each period yields negative profits by itself. (C.4) rules out these possibilities. It says that it is always possible to stop work in progress. Therefore, at equilibrium prices there can be no production plan that makes positive profits in the first period. If (C.4) is satisfied, then $(p_t,p_{t+1}) \in Q^*$ for t=1,2,3,..., suffices to guarantee that there exists no production plan $\{q_{1t},q_{2t}\}_1^\infty \in Q$ with $(q_{1t},q_{2t}) \in Q$ for t=1,2,3,..., which earns positive profits.

At some points it is convenient for us to specify the production technology in terms of profit functions rather than the cone Q. (This is especially true of the sequel.) It is well-known that a production cone of the sort described above generates a continuous profit function $\Psi(p_t, p_{t+1})$, where $\Psi(p_t, p_{t+1})$ denotes the maximum profits attainable through any activity $(q_{1t}, q_{2t}) \in Q$, subject to the constraint $||q||=1$.[9] Since a profit function is always convex, it can be represented in the form

$$\Psi(p_t, p_{t+1}) = \max_j \Psi_j(p_t, p_{t+1})$$

where the Ψ_j, $j=1,\ldots,m$, are a finite number of smooth (C^1) profit functions.[10] Let the functions $\Psi_j(p_t, p_{t+1})$, for $j=1,\ldots,m$, describe the same production technology as the cone Q. These functions will be homogeneous degree one, convex, and will satisfy

$$\Psi_j(\hat{p}, \hat{p}) < 0 \tag{2.3}$$

for all j, where $\hat{p}$ is the vector defined in (C.3).

We will use the following notation for the derivatives of the profit functions.

$$B_1^j(p_t,p_{t+1})' \equiv D_1\Psi_j(p_t,p_{t+1})$$

$$B_2^j(p_t,p_{t+1})' \equiv D_2\Psi_j(p_t,p_{t+1})$$

$$B_1(p_t,p_{t+1}) \equiv [B_1^1 \ldots B_1^m]$$

$$B_2(p_t,p_{t+1}) \equiv [B_2^1 \ldots B_2^m]$$

In equilibrium, one will have to have $p_t'B_1^j(p_t,p_{t+1}) + p_{t+1}'B_2^j(p_t,p_{t+1}) \leq 0$ for all j. Then any production plan $(q_{1t},q_{2t}) \in Q$ which earns zero profits at prices (p_t,p_{t+1}) will have to be a linear combination of the vectors (B_1^j,B_2^j). We will therefore write

(2.4a) $$q_{1t} = B_1 b_t$$

(2.4b) $$q_{2t} = B_2 b_t$$

where the non-negative vector b_t is called the vector of "activity levels," since B_1 and B_2 play the role of activity analysis matrices.

Any (z_0,z,y,V^h,a^h,Q) satisfying (A.1)-(A.5), (B.1)-(B.3), and (C.1) - (C.5) specify a stationary overlapping generations economy. The assets we call "land" can be represented in this economy as the endowments of infinite-lived agents with discount factors of $\beta_h=0$, that is, who wish to consume only in period one. Such agents are effectively members of finite-lived generation 0, but hold initial title to a non-depreciating asset which yields a vector of consumption goods each period, which asset they sell in order to finance period one consumption. The equilibrium concept of interest to us is then:

Definition. A perfect foresight equilibrium of economy (z_0,z,y,V^h,a^h,Q) is a price sequence $\{p_t\}_1^\infty$, an activity sequence $\{b_t\}_1^\infty$, and consumption sequences $\{c_t^h\}_1^\infty$, for $h=1,2,\ldots,H$, such that for each h, $\{c_t^h\}_1^\infty$ maximizes (2.1) subject to (2.2),

$$(2.3)\quad z_0(p_1)+y(p_1,p_2) + \sum_{h=1}^{H} c_1^h - \sum_{h=1}^{H} a_1^h - B_1(p_1,p_2)b_1 = 0,$$

$$(2.4)\quad z(p_{t-1},p_t)+y(p_t,p_{t+1}) + \sum_{h=1}^{H} c_t^h - \sum_{h=1}^{H} a_t^h - B_1(p_t,p_{t+1})b_t - B_2(p_{t-1},p_t)b_{t-1} = 0, \text{ for all } t = 2,3,\ldots, \text{ and}$$

$$(2.5)\quad p_t'B_1(p_t,p_{t+1})+p_{t+1}'B_2(p_t,p_{t+1}) \leq 0,\ b_t \geq 0, \text{ and } p_t'B_1(p_t,p_{t+1})b_t + p_{t+1}'B_2(p_t,p_{t+1})b_t = 0, \text{ for all } t=1,2,\ldots$$

Equations (2.3) and (2.4) require that supply equal demand in each period, while (2.5) requires that all production activities yield non-positive profits in equilibrium and that the production plan actually in use make zero profits.

III. Existence of Perfect Foresight Equilibrium

By extending the proof of Wilson (1981), we can easily establish the existence of a perfect foresight equilibrium in an overlapping generations model with production and infinite-lived consumers. Since the proof

is so similar to that of Wilson, we only indicate here how his proof must be modified and refer the reader to his paper for details. Following Wilson, we let the demands of the finite-lived generations be described in terms of the preference sets and endowments of individual members of those generations, rather than aggregating the demands of all members of a given generation into a single excess demand function.

Let consumers, their preferences, and their endowments be as in Wilson's paper.[11] His assumptions I-VI hold for a stationary economy of the type described in section II.[12] We add a production technology, as described in section II. Index the agents in the economy by the elements $\alpha \in A$. We have assumed that there are a finite number of agents in each finite-lived generation. Let x^α and ω^α represent the consumption allocation and endowment - both infinite dimensional vectors - of agent α. An allocation $\{x^\alpha\}_{\alpha \in A}$ is called attainable if there exists a $\{q_{1t}, q_{2t}\}_1^\infty$ such that $(q_{1t}, q_{2t}) \in Q$ for all $t=1,2,3\ldots$ and $\sum_{\alpha \in A} x^\alpha = z+\omega$ where $\omega = \sum_{\alpha \in A} \omega^\alpha$ and $z = (q_{11}, q_{12}+q_{21}, q_{13}+q_{22}, \ldots)$. A price

sequence $\{p_t\}_1^\infty$, an allocation $\{x^\alpha\}_{\alpha \varepsilon A}$, and a feasible production plan $\{q_{1t}, q_{2t}\}_1^\infty$ are a competitive equilibrium if

(i) $\{x^\alpha\}_{\alpha \varepsilon A}$ is attainable using $\{q_{1t}, q_{2t}\}_1^\infty$,

(ii) for each agent α, x^α is optimal in the budget set defined by $\{p_t\}_1^\infty$ and ω^α, and

(iii) $(p_t, p_{t+1}) \varepsilon Q^*$, and $p_t' q_{1t} + p_{t+1}' q_{2t} = 0$, for all $t \geq 1$.

As in Wilson, the method of proof is to consider a sequence of subeconomies - called "truncated economies" - with only a finite number of agents and goods. We first establish existence of a competitive equilibrium in the truncated economy, and then demonstrate that a limiting allocation, price sequence, and production activity sequence is an equilibrium for the full economy. Let the truncated economy E_t, for $t=1,2,3...$, consist of

(i) agents: all H infinite-lived agents and all finite-lived agents in generations 0 through t; we call this set of agents A_t;

(ii) goods: each of the n goods in periods 1 through t; assets corresponding to the period t+1 endowments of each member of finite-lived generation t, one asset for each member, and H additional assets, where the h'th asset, h=1,2,...H, is a claim to the vector a^h in period t+1 and all subsequent periods;

(iii) production technology: any production plan yielding net outputs $\{q_{1s},q_{2s}\}_1^{t-1}$ is feasible if $\{q_{1s},q_{2s}\} \in Q$ for s=1,2...t-1.

Agents' preferences over the truncated set of goods are the preference relations naturally induced on this set of goods by their preference relations over the entire set. Agents' endowments are the same as in the full economy, but restated as bundles of goods from the truncated set. Therefore, each truncated economy is fully specified with

a finite set of agents and goods. Since the assumptions of Theorem 1 in McKenzie (1981) are satisfied, a competitive equilibrium $\{p_s^t\}_1^t$, $\{x^{\alpha t}\}_{\alpha \varepsilon A_t}$, $\{q_{1s}^t, q_{2s}^t\}_1^{t-1}$ exists.

In each competitive equilibrium of a truncated economy,

$$\sum_{\alpha \varepsilon A_t} x^{\alpha t} = \omega^t + z^t \tag{3.1}$$

where $\omega^t = \sum_{\alpha \varepsilon A_t} \omega^\alpha$ and $z^t = (q_{11}^t, q_{12}^t + q_{21}^t, \ldots, q_{2t-1}^t, 0, 0 \ldots)$.

It is possible to place bounds on feasible values of z^t that are valid for all t. Note that (C.3') and (C.4) imply the existence of an n-vector $\pi >> 0$ such that

$$\pi' q_1 \leq 0 \tag{3.2}$$

for all $(q_1, q_2) \varepsilon Q$. Furthermore for i,j = 1,2,...,n, one can define

(3.3) $u_{ij} \equiv \min\{u \geq 0 \mid \exists (q_1, q_2) \in Q \text{ s.t. } q_{1(i)} \geq -u, q_{2(j)} = 1\}$

Then (C.2) implies that for each j,

$$\max_i u_{ij} > 0$$

Feasible production in any period cannot use more of any input than the total endowment of all agents in the truncated economy, which is a subset of the full set of agents, so

(3.4a) $$q_{11}^t \geq -\bar{y} - \bar{z} - a$$

(3.4b) $$q_{2s}^t + q_{1s+1}^t \geq -\bar{y} - \bar{z} - a$$

for all $s \geq 1$ and $t \geq 1$, where $a \equiv \sum_{h=1}^{H} a^h$.

Equations (3.2) and (3.4a) imply

$$q_{11}^t \leq r_1 < \infty$$

for all t, where elements of r_1 are given by

$$r_{1(j)} = \frac{\sum_{k \neq j} \pi_k (\bar{y}+\bar{z}+a)_{(k)}}{\pi_j}$$

And (3.3) and (3.4a) imply

$$q_{21}^t \leq v_1 < \infty$$

for all t, where the elements of v_1 are given by

$$v_{1(j)} = \min_i \left(\frac{\bar{y}_i + \bar{z}_i + a_i}{u_{ij}} \right)$$

Equation (3.4b) then implies

$$q_{12}^t \geq -\bar{y}-\bar{z}-a-v_1$$

for all t, which in turn allows upper bounds for q_{12}^t and q_{22}^t to be calculated in the same manner as those for q_{21}^t. Repeating this process yields upper and lower bounds

(3.5a) $$-\bar{y}-\bar{z}-a \leq q_{11}^t \leq r_1$$

(3.5b) $$-\bar{y}-\bar{z}-a-v_{s-1} \leq q_{1s}^{t} \leq r_s, \text{ for } s \geq 2$$

(3.5c) $$-\bar{y}-\bar{z}-a-r_{s+1} \leq q_{2s}^{t} \leq v_s, \text{ for } s \geq 1$$

for all t, where the elements of r_s and v_s are defined recursively by

(3.6a) $$r_{s(j)} = \sum_{k \neq j} \frac{\pi_k(\bar{y}+\bar{z}+a+v_{s-1})(k)}{\pi_j}$$

(3.6b) $$v_{s(j)} = \min_i\left(\frac{\bar{y}_i+\bar{z}_i+a_i+v_{s-1(i)}}{u_{ij}}\right)$$

Therefore, one has an upper bound

(3.7) $$z^t \leq \hat{z} < \infty$$

for all t, where $\hat{z} = (r_1, r_2+v_1, r_3+v_2, \ldots)$. (3.7) and

$\omega^t \leq \omega$ imply $\sum_{\alpha \in A_t} x^{\alpha t} \leq \omega + \hat{z} < \infty$ for all t. Since $0 \leq x^{\alpha t}$ for each agent α,

$$0 \leq x^{\alpha t} \leq \omega + \hat{z} < \infty . \tag{3.8}$$

We have shown that the consumption allocations and production plans for each truncated equilibrium lie in compact sets, and using the methods in Wilson's proofs of Lemmas 3 and 4, one can also establish that the prices $\{p_s^t\}$ lie in a compact set. By Tychonoff's theorem (Munkres, 1975, pg. 232), there exists a subsequence of truncated economies such that

$$\{p_s^t\} \to \{\bar{p}_s\}$$

$$\{x^{\alpha t}\} \to \{\bar{x}^{\alpha}\}$$

$$\{q_{1s}^t, q^t{}_{2s}\} \to \{\bar{q}_{1s}, \bar{q}_{2s}\} .$$

It remains only to show that these limiting values are a competitive equilibrium for the full economy. Since

(3.1) holds for each truncated equilibrium, it follows that $\sum_{\alpha \in A} \bar{x}^{\alpha} = \omega + \bar{z}$, where $\bar{z} = (\bar{q}_{11}, \bar{q}_{12}+\bar{q}_{21}, \bar{q}_{13}+\bar{q}_{22}, \ldots)$. Since $(q_{1s}^t, q_{2s}^t) \in Q$ for each t, and Q is closed, it follows that $(\bar{q}_{1s}, \bar{q}_{2s}) \in Q$ for all s. Hence $\{\bar{x}^{\alpha}\}_{\alpha \in A}$ is attainable for the full economy. Similarly, one can show that $(\bar{p}_s, \bar{p}_{s+1}) \in Q^*$ and $(\bar{q}_{1s}, \bar{q}_{2s})$ makes zero profits for $s=1,2,3,\ldots$ The proof of Wilson's Theorem 1 now establishes that for each finite-lived agent α, $\bar{x}^{\alpha}$ is optimal in the budget set defined by $\{\bar{p}_s\}_1^{\infty}$ and ω^{α}, and for each infinite-lived agent α, that $\bar{x}^{\alpha}$ is optimal among all consumption allocations costing the same amount at prices $\{\bar{p}_s\}_1^{\infty}$, and that $0 < \bar{p} \cdot \omega^{\alpha} \leq \bar{p} \cdot \bar{x}^{\alpha} < \infty$. The proof of his Theorem 2 establishes that $\bar{p} \cdot \omega^{\alpha} = \bar{p} \cdot \bar{x}^{\alpha}$, if there exists a finite set of agents who are endowed with more than a fraction $\varepsilon > 0$ of each good in each period. In our model the set of infinite-lived agents has this property, since the economy is stationary and $a^h >> 0$. Hence we have the following proposition.

Proposition 3.1: For any stationary economy satisfying the assumptions of section II, a perfect foresight Walrasian equilibrium exists.

IV. Monetary Equilibria

If a constant quantity of fiat money exists in our stationary economy - that is, if a certain amount is assumed to exist at the beginning of period 1 in the hands of the initial old and there is no money creation or destruction thereafter - the conditions for perfect foresight equilibrium stated in section II are unchanged, but the specification of z_0 must be modified. z_0 is no longer homogeneous of degree 0, and it in this case satisfies

$$p_1' z_0(p_1) = M \tag{4.1}$$

instead of the equation in (A.2), where M is the quantity of fiat money. In most cases of interest to us, monetary equilibria are impossible. That is, one cannot have a solution to the equilibrium conditions in which (4.1) holds with $M>0$ and finite prices. Specifically, we prove the following.

Proposition 4.1: In the case of stationary economies satisfying the assumptions of section II with the

aforementioned changes made in the specification of z_0, monetary equilibrium is only possible if there are no infinite-lived consumers and there is no land.

Proof: Let $\{p_t\}_1^\infty$, $\{q_{1t},q_{2t}\}_1^\infty$, and $\{c_t^h\}_1^\infty$ be a monetary equilibrium. Define

$$S_t=p_t'[q_{1t}-y(p_t,p_{t+1})] - \sum_{s=1}^{t} p_s'\left(\sum_{h=1}^{H}(c_s^h-a^h)\right).$$

(2.3), (2.4) and (4.1) imply that $S_t=M$ for all $t\geq 1$. Suppose there is at least one infinite-lived agent. For all infinite-lived agents to be satisfying their budget constraints one must have

$$\liminf_{T\to\infty} \sum_{t=1}^{T} p_t'\left(a-\sum_{h=1}^{H} c_t^h\right)\geq 0$$

which means that for any $\varepsilon>0$, there exists a $T(\varepsilon)$ such that

$$\sum_{s=1}^{t} p_s'\left(a-\sum_{h=1}^{H} c_s^h\right)>-\varepsilon$$

for all $t \geq T(\varepsilon)$. Therefore for any $\varepsilon > 0$, there must exist a $T(\varepsilon)$ such that

$$p_t'[q_{1t} - y(p_t, p_{t+1})] > M - \varepsilon$$

for all $t \geq T(\varepsilon)$. (C.4) implies that

$$-p_t' y(p_t, p_{t+})_1 > M - \varepsilon$$

for all $t \geq T(\varepsilon)$. Using (A.5) one has

$$p_t' \bar{y} > M - \varepsilon \tag{4.2}$$

for all $t \geq T(\varepsilon)$. For an infinite-lived agent's budget set to be defined, the value of his endowment, $\sum_{t=1}^{\infty} p_t' a^h$, must converge, which, since $a^h >> 0$,[13] means

$$\sum_{t=1}^{\infty} p_t < \infty \tag{4.3}$$

in all components. Therefore

$$\sum_{t=1}^{\infty} p_t' \bar{y} < \infty .$$

However, for $0<\varepsilon<M$, (4.2) implies

$$\sum_{t=1}^{\infty} p_t' \bar{y} = \infty$$

which is a contradiction.

The argument is the same if there is land, as long as the land produces all of the goods in each period. However, if not all goods are produced by land, and no infinite-lived consumers exist, the proof is slightly more complicated. Recall that (C.3) guarantees that there exists $\hat{p}>>0$ such that $(\hat{p},\hat{p}) \in Q^*$. Therefore

$$\hat{p}' q_{11} + \sum_{t=1}^{T} \hat{p}'(q_{1t+1}+q_{2t}) < \hat{p}' q_{1T+1}$$

for all T. (C.4) then implies

$$\hat{p}' q_{11} + \sum_{t=1}^{T} \hat{p}'(q_{1t+1}+q_{2t}) < 0$$

for all T. (2.3) and (2.4) now imply that for all T

$$\hat{p}'(z_0(p_1)+ y(p_1,p_2)-a) + \sum_{t=2}^{T} \hat{p}'(y(p_t,p_{t+1}) + z(p_{t-1},p_t)-a) \leq 0.$$

Furthermore,

$$\sum_{t=1}^{T-1} \hat{p}'(y(p_t,p_{t+1})+z(p_t,p_{t+1})-a) \leq -\hat{p}'(z_0(p_1)+y(p_T,p_{T+1})-a) \leq \hat{p}'(\bar{z}+\bar{y}+a) < \infty$$

for all T. Therefore for any $\delta>0$, there must exist a sequence of periods S such that for any $t \in S$,

$$\hat{p}'(y(p_t,p_{t+1})+z(p_t,p_{t+1})-a) < \delta.$$

Because of (A.5) there is an upper bound ω such that

$$y(p_t,p_{t+1})+z(p_t,p_{t+1})-a < \omega < \infty$$

for all $t \in S$. Because of (A.4) there exists $\upsilon>0$ such that

$$p_t > \upsilon \left(p_t' \bar{y} \right) e$$

for all $t \in S$, where e is a vector all of whose components are 1's. If $M>0$, (4.2) then implies

(4.4) $$p_t > \upsilon (M-\varepsilon) e$$

for all $t \in S$. Now if even one good is produced by land, (4.3) must hold for the corresponding component, but that in turn contradicts (4.4).

Q.E.D.

Proposition 4.1 establishes conditions under which a monetary equilibrium cannot exist and since monetary equilibrium is not our principal interest here, we do not pursue the question of its existence any further. It should be noted, however, that the assumption of a stationary economy is crucial to Proposition 4.1. Otherwise, the infinite-lived agent's budget set may be well-defined without (4.3) holding, so that it is possible for (4.2) to hold for all t. One-good examples

of this sort are given in Wilson (1981) and Tirole (1983).

V. Existence of Steady State Equilibrium.

Our principal interest in this paper (and the reason for our restriction to the case of stationary economies) is in steady states, and equilibria near steady states. By a steady state is meant an equilibrium in which $p_t=\beta^t p, b_t=b, c^h_t=c^h$, for all t.

Definition. A perfect foresight steady state equilibrium of economy (z,y,V^h,a^h,Q) is a price vector p, a scalar β, an activity vector b, consumption vectors c^h for $h=1,\ldots,H$, and non-negative scalars λ^h for each h such that $c^h\neq 0$, satisfying

$$(5.1)\quad z(p,\beta p)+y(p,\beta p)+\sum_{h=1}^{H} c^h - a - B_1(p,\beta p)\,b - B_2(p,\beta p)\,b=0,$$

$$(5.2)\quad p'B_1(p,\beta p)+\beta p'B_2(p,\beta p)\leq 0,\ b\geq 0,$$

$$(5.3)\quad p'B_1(p,\beta p)\,b+\beta p'B_2(p,\beta p)\,b=0,$$

(5.4) $\beta_h \leq \beta < 1$ for $h=1,2,\ldots,H$, with $\beta_h=\beta$ for every h such that such that $c^h \neq 0$, and

(5.5) $Du^h(c^h)=\lambda^h p$ for every h such that $c^h \neq 0$.

Note that the possible steady states are determined solely by the stationary features of the economy. Therefore a steady state equilibrium is not an equilibrium in the sense of section II unless the initial conditions are carefully chosen to be consistent with it, although other equilibrium paths may approach a steady state asymptotically. Muller and Woodford (1983) provide a local answer to the question of when a steady state is approached by equilibrium paths consistent with a given set of initial conditions.

Conditions (5.1), (5.2), (5.3), and (5.5) are the steady state forms of the equilibrium conditions given in section II, and the requirement that $\beta_h=\beta$ for every h such that $c^h \neq 0$ follows from the first order conditions for utility maximization in the case of a steady state. If not all infinite-lived agents have the same rate of time discount, steady state is only possible if some have accumulated debts so great that they no longer consume.

It might be thought that no aspect of the preferences of these latter agents matters for the steady state, only the relation that exists between their debts and their endowments. This is in a strict sense true, but a steady state with $\beta_h > \beta$ for some agent (whose debts exactly cancel the lifetime value of his endowment) can only be reached from initial conditions which allow the economy to start in steady state. That is, there are no equilibria converging to such a steady state asymptotically, even from initial conditions very close to those consistent with the steady state. This is because a perturbation in the initial conditions that increases agent h's lifetime budget, even infinitesimally, causes the equilibrium path to diverge from the steady state, since agent h wishes to accumulate wealth as long as the rate of return in the economy is higher than his rate of time discount.

In the case of a steady state with $\beta_h < \beta$, on the other hand, a perturbation of the initial conditions which slightly reduced agent h's debts would lead him to choose a course of wealth de-cumulation, so that he would approach the level of debt at which it ceases to be possible for him to consume. Hence neither his

consumption, nor the deviation in the level of his debts from the steady state level, would ever be appreciable, and the equilibrium path could remain close to the steady state path and converge to it asymptotically. Because these considerations are so familiar, we rule out steady states with $\beta_h > \beta$ in (5.4).

Order the infinite-lived consumers so that $\beta_1 > \beta_2 \ldots > \beta_H$. Then (5.4) requires that $\beta \geq \beta_1$ at a steady state. There are then two possible kinds of steady states: those in which $\beta > \beta_1$ and those in which $\beta = \beta_1$. In the former case the steady state is determined exactly as in an economy where there are no infinite-lived agents, but there is land yielding the vector a of goods each period. The infinite-lived agents have sold claims to all of their future endowments to the finite-lived agents for immediate consumption and the finite-lived agents pass these claims down from generation to generation. In the latter case the rate of time discount of the least impatient infinite-lived agent determines the rate of return for the whole economy.

Either type of steady state, considered alone, might fail to exist. This is easily shown by a simple example that provides an intuitive grasp of the relation between

the two types of steady states. Consider the case of a single good and no production activities. A steady state is a β satisfying

$$z(\beta)+y(\beta)+\sum_{h=1}^{H} c^h-a=0,$$
$$\beta_1 \leq \beta < 1, \quad \text{and}$$
$$(\beta-\beta_h)\, c_h=0.$$

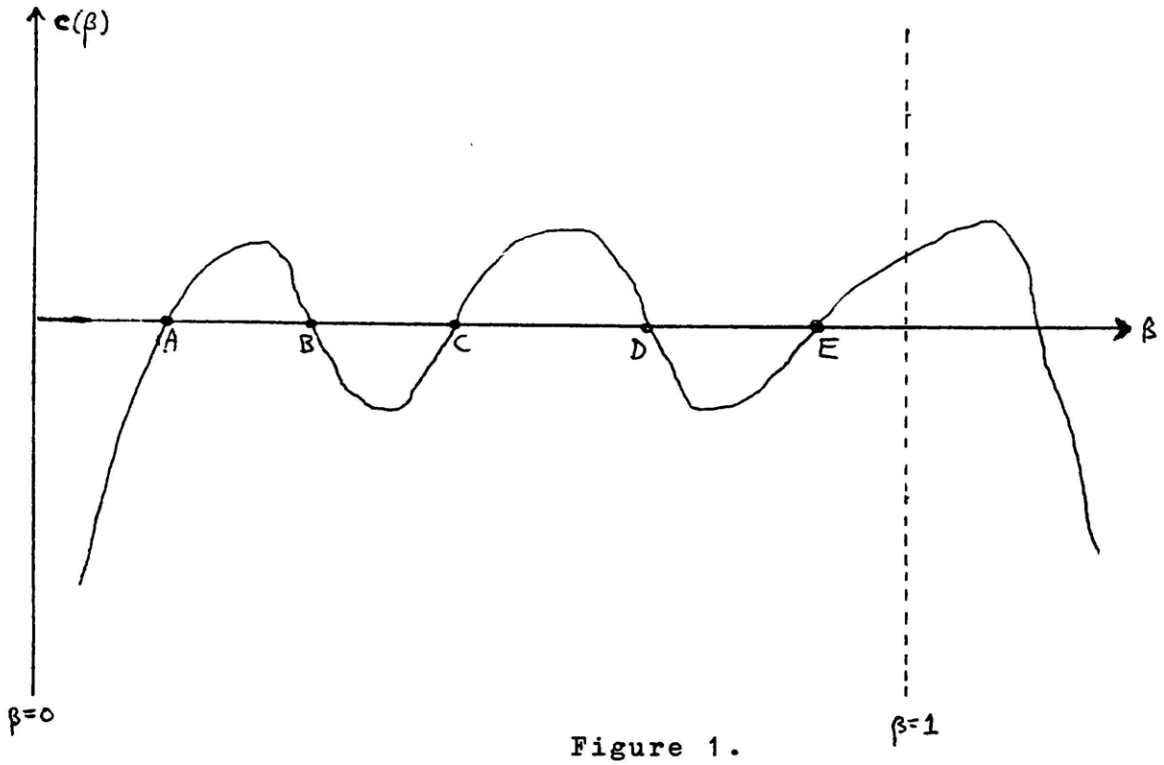


Figure 1.

Define $c(\beta)=a-z(\beta)-y(\beta)=a-(1-\beta)z(\beta)$, and plot $c(\beta)$, as shown in Figure 1. (A.4) implies $c(\beta)$ is negative near

$\beta=0$ and for β sufficiently large. (A.2) implies that $c(\beta)$ is positive at $\beta=1$. Therefore, there are an odd number of values of β between zero and one for which $c(\beta)=0$. Each such point corresponds to a steady state of the economy consisting of finite-lived agents, no infinite-lived agents, and land yielding the vector a. These values of β are also steady states of the economy with infinite-lived agents whose endowments sum to the vector a, if they satisfy $\beta \geq \beta_1$. Steady states of the other type are values of β such that $\beta=\beta_1$ and $c(\beta)>0$. Note that in the case of steady states of the second type, the relative amounts of steady state consumption on the part of the agents with $\beta_h=\beta_1$ are completely indeterminate - the c^h's may take any positive values summing to $c(\beta)$.

In the above example neither type of steady state is guaranteed to exist, but one or the other always does. To see this, note that if β_1 is to the right of point E, then a steady state of the first type does not exist, but one of the second type does, and if β_1 is to the left of point E, then a steady state of the first type exists. This result is true in general. Further, the indeterminacy of allocation of the steady state

consumption among infinite-lived consumers with $\beta_h=\beta_1$ is also true in general. Let J be the number of infinite-lived consumers with $\beta_h=\beta_1$, and let $\sigma_1,\ldots,\sigma_J$ be an arbitrary set of non-negative weights summing to 1. For any set of weights one chooses, if there exists a steady state with infinite-lived agents consuming, then there exists a steady state in which the relative values of the steady state consumption of these J agents is according to the set of weights $\sigma_1,\ldots,\sigma_J$. Specifically, one can prove the following result.

Proposition 5.1: Let $\sigma_1,\ldots,\sigma_J$ be any non-negative weights summing to one, assigned to the $J\leq H$ infinite-lived consumers with $\beta_h=\beta_1$. There exists at least one steady state for the economy (y,z,V^h,a^h,Q) satisfying

$$\sigma_j p'c^h=\sigma_h p'c^j \text{ for all } h,j\leq J. \tag{5.6}$$

Proof: The method of proof is to construct a continuous mapping of a closed, bounded, and convex set into itself and to show that any fixed point of the map is a steady

state. Before proceeding, we state a series of definitions that are needed for the proof. Let

$$F=\{ f \in R_{+}^{n} \mid f=q_1+q_2+\bar{y}+\bar{z}+a \text{ for some } (q_1,q_2)\in Q\}$$

where Q is the production cone. Define

$$S_\varepsilon=\{\pi \mid \pi' e=1 \text{ and } \pi \geq \varepsilon e\}$$

where $\varepsilon>0$ is chosen such that for some small $\eta>0$

(i) $e'(z(p,\beta p)+y(p,\beta p)-a-q_1-q_2)\geq \eta>0$
for all $p \in \partial S_\varepsilon, \beta_1<\beta<1$, and $(q_1,q_2) \in Q$
such that $q_1+q_2 \geq -\bar{y}-\bar{z}-a$; and

(ii) $\hat{p} \in S_\varepsilon$, where $\hat{p}$ is defined in (C.3).

(A.4), (A.5) and (C.3) assure that this is possible. It is necessary to work with S_ε instead of the set of normalized prices because we need a closed set on which the excess demand functions are defined.

It is worth noting at this point that, for purposes of the analysis of equilibrium in the vicinity of steady states that we pursue in the sequel, (A.1) - (A.5) completely characterize excess demand functions derived from utility maximization. If y and z satisfy (A.1) - (A.3), then there exist y^* and z^*, derived from utility maximization by 2n consumers, that agree with y and z on S_ε. Our definition of S_ε implies that every steady state of (z,y,V^h,a^h,Q) lies in its interior. Utilizing a result of Mas-Collel (1977), one can show that there exist y^* and z^* such that every steady state of (z^*,y^*,V^h,a^h,Q) must also lie in the interior of S_ε. Consequently, the only steady states of (y^*,z^*) are those of (y,z). Furthermore, S_ε can be chosen large enough so that (y,z) and (y^*,z^*) agree on any open neighborhood of these steady states.

Let $\Omega = \{(\pi,\gamma,c) \mid \pi \in S_\varepsilon, \beta_1 \leq \gamma \leq 1, \text{ and } c \in F\}$. It is apparent from these definitions that Ω is a closed, bounded, and convex set. The mapping of Ω into itself takes (π,γ,c) to (p,β,d). It is defined in two steps. Given (π,γ,c), (p,β) is chosen to minimize

$$\left\| \begin{matrix} p-\phi_1(\pi,\gamma,c) \\ \beta-\phi_2(\pi,\gamma,c) \end{matrix} \right\|^2$$

subject to

(5.7) $$\beta_1 \leq \beta \leq 1$$

(5.8) $$p'e=1$$

(5.9) $$p \geq \varepsilon e$$

(5.10) $$p'(B_1(\pi,\gamma\pi)+B_2(\pi,\gamma\pi)) \leq (1-\beta)\pi' B_2(\pi,\gamma\pi)$$

where

$$\phi_1(\pi,\gamma,c) = \pi + z(\pi,\gamma\pi) + y(\pi,\gamma\pi) + c - a$$

$$\phi_2(\pi,\gamma,c) = \gamma + \pi' z(\pi,\gamma\pi) - \theta(\pi,\gamma)$$

$$\theta(\pi,\gamma) = \begin{cases} \dfrac{\pi'a}{1-\gamma} & \text{for } \beta_1 \leq \gamma \leq \hat{\gamma} \\ \dfrac{\pi'a}{1-\hat{\gamma}} & \text{for } \hat{\gamma} \leq \gamma \leq 1. \end{cases}$$

Here $\hat{\gamma}$ is a value very close to 1. The truncation is necessary to get a function that is defined on the closed interval $[\beta_1,1]$. Given (p,β) and the vector of Lagrange multipliers associated with (5.10) - call them b - d is chosen as follows. For each h,h=1,...,J, find the d^h that maximizes $u^h(d^h)$ subject to

$$(5.11)\quad p'd^h \leq \sigma^h \max[\, p'a - p'z(p,\beta p) - p'y(p,\beta p) + p'(B_1(p,\beta p) + B_2(p,\beta p))b, 0]$$

Define

$$d = \alpha \sum_{h=1}^{J} d^h$$

where α is the largest number between 0 and 1 such that

$$\alpha \sum_{h=1}^{J} d^h \in F.$$

The mapping is continuous and it takes Ω into itself, so Brouwer's fixed point theorem implies that there exists a $(p,\beta,d) \in \Omega$ that is mapped into itself.

The Kuhn-Tucker conditions for the maximizations involved in the mapping, evaluated at the fixed point (p,β,d), are

$$z(p,\beta p)+y(p,\beta p)+d-a-\left(B_1(p,\beta p)+B_2(p,\beta p)\right)b+\lambda_2 e+\lambda_1=0 \tag{5.11}$$

$$X(p,\beta,b)-\delta+\rho=0 \tag{5.12}$$

$$p'\left(B_1(p,\beta p)+\beta B_2(p,\beta p)\right)\leq 0 \tag{5.13}$$

$$p'\left(B_1(p,\beta p)+\beta B_2(p,\beta p)\right)b=0 \tag{5.14}$$

$$\lambda_1'(p-\varepsilon e)=0 \tag{5.15}$$

$$\rho(\beta-\beta_1)=0 \tag{5.16}$$

(5.17) $$\delta(1-\beta)=0$$

(5.18) $$p'd = \alpha \max[\, p'a-p'z(p,\beta p)-p'y(p,\beta p) + p'(B_1(p,\beta p)+B_2(p,\beta p))\, b, 0]$$

(5.19) If $d \neq 0$, then $Du^h(d^h)=\lambda^h p$ for every h such that $\sigma_h \neq 0$,

where

$$X(p,\beta,b)=p'z(p,\beta p)-\theta(p,\beta)-p'B_2(p,\beta p)b.$$

Note that ρ, δ, λ_2, and the λ^h are Lagrange multipliers associated with various constraints in the maximization problems. All of them must be non-negative except for λ_2, which is unconstrained since it is associated with an equality constraint. Similarly, λ_1 and b are non-negative vectors of Lagrange multipliers.

Now (5.12) and (5.17) imply

(5.20) $$(1-\beta)X(p,\beta,b) = (1-\beta)\delta-(1-\beta)\rho \leq 0.$$

The definition of $\theta(p,\beta)$ implies

$$p'a-(1-\beta)\theta(p,\beta)\geq 0. \tag{5.21}$$

(A.2) and (5.14) imply that (5.18) can be written

$$p'd = \alpha \max[\, p'a-(1-\beta)\theta(p,\beta)-(1-\beta)X(p,\beta,b),0].$$

(5.20) and (5.21) imply

$$p'd = \alpha[\, p'a-(1-\beta)\theta(p,\beta)-(1-\beta)X(p,\beta,b)]. \tag{5.22}$$

Multiplying (5.11) on the left by $(e'-np')$ and using (5.15) and (5.22) yields

$$\begin{aligned}
0=e'[\, & z(p,\beta p)+y(p,\beta p)+d-a-(\, B_1(p,\beta p)+B_2(p,\beta p))\, b \\
& -n(1-\beta)X(p,\beta,b)+n(\, p'a-(1-\beta)\theta(p,\beta))-np'd \\
& +(e'-np')\lambda_1 .
\end{aligned}$$

$$\begin{aligned}
0=e'[\, & z(p,\beta p)+y(p,\beta p)-a-(\, B_1(p,\beta p)+B_2(p,\beta p))\, b] \\
& +e'd+n(1-\alpha)[\, p'a-(1-\beta)\theta(p,\beta)-(1-\beta)X(p,\beta,b)] \\
& +(1-n\varepsilon)e'\lambda_1 .
\end{aligned}$$

Now if $\lambda_1 \neq 0$, $p \in \partial S_\varepsilon$ and the first term of the above

equation is at least as large as $\eta>0$. However, the second term is non-negative by (5.18); the third is non-negative by (5.22); and the fourth is non-negative, since $\varepsilon\leq 1/n$, in order for S_ε to be non-empty. This is a contradiction, so $\lambda_1=0$ and p is necessarily in the interior of S_ε.

Multiplying (5.11) on the left by p' yields

$$(5.23)\quad (1-\alpha)[\,(1-\beta)X(p,\beta,b)-p'a+(1-\beta)\theta(p,\beta)\,]+\lambda_2=0$$

which implies $\lambda_2\geq 0$ because of (5.20) and (5.21). Now (5.11) implies

$$d \leq a+(B_1(p,\beta p)+B_2(p,\beta p))\,b-z(p,\beta p)-y(p,\beta p),$$
$$d << a+\bar{z}+\bar{y}+(B_1(p,\beta p)+B_2(p,\beta p))\,b.$$

Therefore d is in the interior of F, and $\alpha=1$. Then (5.23) implies $\lambda_2=0$.

It can also be shown that for $\hat{\gamma}$ chosen close enough to 1, $\beta\leq\hat{\gamma}$. To discover how to choose $\hat{\gamma}$, we must digress slightly. Consider the equation

$$(5.24) \qquad \left(B_1(\pi,\gamma\pi)+B_2(\pi,\gamma\pi)\right)g >> -\bar{z}-\bar{y}-a.$$

Let $G=\{g\geq 0 \mid g$ satisfies (5.24) for some $(\pi,\gamma)\in S_\varepsilon x[\beta_1,1]\}$. (C.3) implies that there exists $\hat{g}$ such that $\hat{g}\geq g$ for all $g \in G$. This in turn implies a lower bound $-v_1$ on $\pi' B_2(\pi,\gamma\pi)g$ for all $g \in G$ and $(\pi,\gamma) \in S_\varepsilon x[\beta_1,1]$. One can also establish an upper bound v_2 on $\pi' z(\pi,\gamma\pi)$ for all $(\pi,\gamma) \in S_\varepsilon x[\beta_1,1]$ and a value $\hat{\gamma}<1$ such that

$$(5.25) \qquad \frac{\pi' a}{1-\hat{\gamma}} > v_1 + v_2$$

for all $\pi \in S_\varepsilon$.

Let $\hat{\gamma}$ be chosen so as to satisfy (5.25). Now suppose that at the fixed point (p,β,d), $\hat{\gamma}<\beta\leq 1$. Then $\rho=0$ and (5.12) implies $X(p,\beta,b)\geq 0$. However,

$$X(p,\beta,b) = p'z(p,\beta p) - \frac{p'a}{1-\hat{\gamma}} - p'B_2(p,\beta p)b$$

and (5.25) implies $X(p,\beta,b)<0$, which establishes a contradiction.

Thus the fixed point (p,β,d) and the associated non-negative vector of Lagrange multipliers b must satisfy

(i) $z(p,\beta p)+y(p,\beta p)+d-a-[B_1(p,\beta p)+B_2(p,\beta p)]b=0,$

(ii) $p'[B_1(p,\beta p)+\beta B_2(p,\beta p)]\leq 0,$

(iii) $p'[B_1(p,\beta p)+\beta B_2(p,\beta p)]b=0,$

(iv) $d \geq 0,\ p'd(\beta-\beta_1)=0,\ \beta-\beta_1\geq 0,$

(v) $\beta\leq\gamma<1,$

(vi) $d = \sum_{h=1}^{J} d^h,$ and

(vii) The constructed d^h's satisfy (5.4), (5.5) and (5.6).

Therefore it is a steady state equilibrium satisfying (5.6). Note that the impossibility of consumption by infinite-lived agents in a steady state with $\beta>\beta_1$ turns out to be a Kuhn-Tucker duality relationship, as does the impossibility of an activity with negative profits being

operated in steady state.

Q.E.D.

This establishes the existence of a non-monetary steady state for any stationary economy satisfying the assumptions of section II. As is shown in section IV, a monetary equilibrium, and hence a monetary steady state, is impossible if either land or infinite-lived consumers exist in the economy. However it is possible to have a monetary steady state in the case of only finite-lived consumers and production satisfying the axioms of section II.

Proposition 5.2: A monetary steady state exists for an economy of the type described in Section II, if there are no infinite-lived agents and no land in the economy.

Proof: The conditions for a steady state in this economy are (5.1)-(5.3) with the terms involving land and infinite-lived agents dropped. The nominal excess savings of the economy in any steady state is

$$S=p'[B_1(p,\beta p)b-y(p,\beta p)].$$

(5.1), (5.3) and (A.2) imply $(1-\beta)S=0$. Therefore $\beta=1$ for any monetary steady state - that is, a steady state in which $S\neq 0$. (A.1) - (A.5) imply that $z(p,p)+y(p,p)$ has the properties of an excess demand function in an n-good static general equilibrium model. (C.1) - (C.5) imply that $[B_1(p,p)+B_2(p,p)]$ has the properties of the transpose of a Jacobian matrix of m C^1 profit functions in an n-good static general equilibrium model. Since an equilibrium is known to exist in the corresponding static model, a monetary steady state exists.

Q.E.D.

This shows that it is not the case that the mere existence of other assets in an economy rules out the possibility of a monetary equilibrium.[14] As noted above, however, our main interest is not in monetary equilbria, but in the local determinacy of non-monetary equilibria of economies with infinite-lived consumers or non-depreciating assets. This is the subject treated in Muller and Woodford (1983).

FOOTNOTES

[1] All consumers are undoubtedly mortal. But Barro (1974) has argued that consumers who care about the welfare of their descendants behave in such a way that the infinite collection of members of a single lineage may be treated like a single, infinite-lived, consumer.

[2] The restriction to two-period lives is not significant, given that we allow an arbitrarily large (finite) number of goods and an arbitrary number of types of finite-lived consumers. By redefining time periods, and relabeling goods and generations, it is always possible to put an overlapping generations model into this form. See Balasko, Cass, and Shell (1980).

[3] This identity for $z_0(p_1)$ holds only in the case of the non-monetary economy which we describe first. In the case of a monetary economy (section IV), this must be modified. However, it is be shown that monetary equilibria are possible only if neither land nor infinite-lived consumers exist, so that our primary interest is in non-monetary equilibria. In the sequel, we also modify this condition in order to allow for initial claims by or against members of generation 0, representing debts incurred before the start of the model.

[4] Note that this assumes that each good is demanded by some finite-lived consumer. This does not rule out the treatment of multi-period finite lives in the manner mentioned in footnote 2. But it is an obstacle to the treatment of multi-period production processes, which would require the introduction of intermediate goods which are not included in any agent's final consumption demands. While (A.4) could be weakened so as to allow treatment of such cases, we avoid this here in order to simplify the exposition.

[5] In general, one may suppose that there are H different types of infinite-lived consumers, with an arbitrary

number N_h of each type. For most purposes we reduce notational clutter by assuming there to be only one agent of each type.

6 Note that this assumes that each infinite-lived consumer demands all goods. As discussed in Footnote 4, this assumption would be too strict if multi-period production processes were to be treated, but it allows some simplification of the proofs to follow.

7 This is no constraint if $\sum_{t=1}^{\infty} p_t$ does not converge, so the demands of infinite-lived consumers are undefined for such price sequences. But it is obvious that no such sequence could be a perfect foresight equilibrium.

8 The assumption that Q is a cone means that there are constant returns to scale in all activities. But as is well known, allowing decreasing returns to scale production is only a trivial generalization, since any such technology can be represented as a constant returns to scale technology by the inclusion of an additional "factor" required for the production activity, assumed to be in fixed supply, which "factor" receives the profits from the production activity as its rent.

9 The constraint is necessary because of the assumption of constant returns to scale in production.

10 In the sequel, we assume these functions to be C^2 as well, but this is not necessary for any of the proofs in this paper.

11 Note that the assumption made in section II, that the preferences of the infinite-lived consumers are additively separable, is nowhere required in this section.

[12] Our assumptions that all goods are demanded by both finite and infinite-lived consumers, and that infinite-lived agents have strictly positive endowments of each good, are used to guarantee Wilson's Assumption VI (Irreducibility). In fact, these assumptions are considerably stronger than is necessary to satisfy Assumption VI. And Assumption VI is itself stronger than is necesary for existence of equilibrium in the case of an economy with production, as it represents a specialization of McKenzie's (1981) concept of irreducibility to the case of a pure exchange economy. Hence one could weaken any of these assumptions (e.g., to allow for multi-period production processes), at the cost of having to substitute alternative arguments at those places in Wilson's proof of his Lemmas 3 and 4 and his Theorem 1 where he relies upon Assumption VI. Because our proof is so closely parallel to Wilson's, we have not made such changes, in order not to have to repeat here all the steps in his proof.

[13] If we did not assume $a \gg 0$, the theorem would still be true. See the proof in the case of land, below.

[14] In fact, this was recognized by Gale (1972), although the point seems not yet to be widely known.

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Stationary Overlapping Generations Economies with Production and Infinite-Lived Consumers:

II. Determinancy Of Equilibrium

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I. Introduction

A desirable property for the equilibria of any economic model is local determinancy. That is, one would like the specification of endowments, technology, and preferences in the model to suffice to uniquely determine the equilibrium values of all economic variables, at least in a certain neighborhood. This is necessary to perform the familiar sort of comparative statics exercises, asking for the effect of various parameter changes on equilibrium values. Under reasonably general regularity conditions, the competitive equilibria of static general equilibrium models have this property. See Debreu (1970) and Kehoe (1983). However, this need not be the case for perfect foresight competitive equilibria of dynamic general equilibrium models.[1] In particular, Kehoe and Levine (1983) have shown that perfect foresight equilibria of stationary overlapping generations exchange economies need not be locally determinate. The dimension of indeterminacy is only limited by the number of goods in each period. In a stationary economy with n goods each period, they show that the manifold of equilibrium paths converging asymptotically to a given steady state, consistent with

given initial conditions, may have dimension as large as n-1 in the case of non-monetary equilibria and as large as n in the case of monetary equilibria. They also show that robust examples exist of every dimension of indeterminacy between zero and the upper limit.

On the other hand, Kehoe and Levine (1982a) show that perfect foresight equilibria of infinite-horizon exchange economies with a finite number of infinite-lived consumers are locally unique. It is therefore interesting to inquire as to the determinacy of perfect foresight equilibria for models that contain both finite and infinite-lived consumers. It is known that the addition of even a single agent who consumes forever can sharply restrict the possible perfect foresight equilibria of an economy; for example, in the long run the real rate of return cannot exceed the infinite-lived agent's rate of time discount, even if the only equilibria of the economy minus that agent would have involved higher rates of return. If the addition of a single infinite-lived consumer renders perfect foresight equilibria locally determinate regardless of the size of the infinite-lived agent's endowment in each period relative to the size of the total endowment in each

period, then the results of Kehoe and Levine are not of much interest; for Barro (1974) has argued -- against drawing policy conclusions from properties of overlapping generations models, in another context -- that finite-lived agents whose descendants' welfare enters into their utility functions behave like infinite-lived consumers, and therefore that conclusions drawn from models of economies with infinite-lived consumers are more robust. If only some fraction of the population need behave in this way to rule out indeterminacies, then the point seems irrefutable, but if the entire population must behave as infinite-lived consumers in order to guarantee determinacy of perfect foresight equilibria, the results of Kehoe and Levine remain of great importance.

Accordingly, the present paper considers the local determinacy of steady state perfect foresight equilibria for stationary economies in which both finite and infinite-lived consumers coexist. The Kehoe-Levine model is also generalized to include production and nondepreciating assets called "land" that yield pure rents. As with infinite-lived consumers, the addition of even a small quantity of land to a stationary overlapping generations exchange economy imposes sharp restrictions

on the set of perfect foresight equilibria -- monetary equilibria are impossible, and in the long run the real rate of return must be positive. It is therefore interesting to know if it rules out indeterminacy. Production as well creates links between periods that do not exist in the Kehoe-Levine model. Aggregate consumption in each period is no longer determined by agents' aggregate endowment, which allows greater scope for infinite-lived consumers to substitute future for present consumption. One might think that this would increase the likelihood of a small population of infinite-lived consumers, with relatively small endowments, determining the behavior of equilibrium paths to such an extent that they are locally determinate.

Our model is introduced in Muller and Woodford (1983), where we prove the existence of perfect foresight equilibria and steady states. Here we show that even with all the generalizations above, indeterminacy is possible; indeed the dimension of indeterminacy can still be as large as the number of goods per period in the economy. On the other hand, we find that indeterminacy is impossible in any of the following extreme cases:

(i) if the infinite-lived consumers are much more numerous or consume much more than the finite-lived consumers in each period;

(ii) if the production technology allows a great amount of substitution;

(iii) if the number of independent production activities in use in equilibrium is equal to the number of goods in each period; or

(iv) if in the excess demands of the finite-lived consumers, substitution effects are much larger than income effects.

It follows that for indeterminacy to be a problem, the economy must not be close to any of these extremes -- the infinite-lived consumers must be a sufficiently small fraction of the economy, the production technology must be sufficiently inflexible, the tastes of the finite-lived consumers must be sufficiently inflexible, and the number of independent production activities in use in the steady state must be less than the number of goods in each period. This seems to exhaust the conclusions about the conditions for determinacy that may be drawn without restricting one's attention to specific functional forms.

II. Equilibria Converging to a Steady State

Our analysis is restricted to perfect foresight equilibria close to a steady state equilibrium, and converging to it asymptotically. We define a steady state as in the previous paper.

Definition. A perfect foresight steady state equilibrium of the economy (y, z, V^h, a^h, Q) is a price vector p, a scalar β, an activity vector b, consumption vectors c^h for $h=1,\ldots,H$, and non-negative scalars λ^h for each h such that $c^h \neq 0$, satisfying

$$(2.1)\quad z(p,\beta p) + y(p,\beta p) + \sum_{h=1}^{H} c^h - a - B_1(p,\beta p)b - B_2(p,\beta p)b = 0$$

$$(2.2)\quad p'B_1(p,\beta p) + \beta p'B_2(p,\beta p) \leq 0,\ b \geq 0$$

$$(2.3)\quad p'B_1(p,\beta p)b + \beta p'B_2(p,\beta p)b = 0$$

$$(2.4)\quad \beta_h \leq \beta < 1 \text{ for } h = 1,\ldots H, \text{ with } \beta_h = \beta \text{ for every } h \text{ such that } c^h \neq 0.$$

$$(2.5)\quad Du^h(c^h) = \lambda^h \text{ for every } h \text{ such that } c^h \neq 0.$$

Notation is explained in the previous paper. It is shown there that a steady state always exists although it need not be the case that all perfect foresight equilibria asymptotically approach a steady state.

In order to consider equilibria which begin and remain close to a steady state, the initial conditions have to be consistent with the steady state, or very near to being so. One obvious requirement is that the preferences of the initial old be such that their demand, at the steady state prices, is the same as the demand of the old in all other periods. Specifically, one must have

$$z_0(p) = z(p,\beta p) - B_2(p,\beta p)b \tag{2.6}$$

where the right hand side represents the second period consumption by members of finite-lived generations $t \geq 1$, in excess of their second period endowments and the net output of production projects undertaken in the first period of their lives. This is not simply a matter of correct specification of the preferences of the initial

old. Their endowments must be increased to $\bar{z}_0 = \bar{z} + B_2(p,\beta p)b$ so that the total available goods at the beginning of the first period equals the total available goods at the beginning of each period in steady state, $\bar{y} + \bar{z} + a + B_2(p,\beta p)b$.

Nor is it possible, in general, to maintain the assumption made in the previous paper, that at the beginning of the first period no agents have accumulated debts; for it is generally true of steady states that $p'c^h \neq p'a^h$. This means that if the steady state prices prevail from period one onward -- that is, $p_t = \beta^{t-1}p$ for all $t \geq 1$ -- the consumption plan $c_t^h = c^h$ for all $t \geq 1$ would be optimal for consumer h among the set of consumption programs costing no more than it, but this set would not correspond to h's budget set, defined by

$$\sum_{t=1}^{\infty} p_t' c_t^h \leq \sum_{t=1}^{\infty} p_t' a^h.$$

Therefore the steady state cannot prevail from period one on, but can only be approached asymptotically. In fact, it is argued in the previous paper that infinite-lived

agents with $\beta_h < \beta$ consume nothing in the steady state. This plainly cannot be true from period one onward, as their endowments have positive value and any consumption plan involving some consumption is preferred to never consuming at all.

The solution is to allow the infinite-lived agents and the members of finite-lived generation zero to start out with net claims against each other. These claims represent contracts entered into before the period being modeled. Since there are no nominal quantities in the model,[2] these claims cannot be denominated in terms of nominal aggregates. Hence we assume that each agent's claim is a vector of goods. Let the vector of initial claims held by each infinite-lived consumer be denoted w^h, for h = 1,...H. Positive components indicate net claims against other agents and negative components indicate debts. The budget constraint of consumer h is then

$$\sum_{t=1}^{\infty} p_t' c_t^h \leq \sum_{t=1}^{\infty} p_t' a^h + p_t' w^h \tag{2.7}$$

We assume that the net claims of the infinite-lived consumers satisfy

$$\sum_{h=1}^{H} w^h - \bar{z}_0 \in Q_1 \tag{2.8}$$

$$-\sum_{h=1}^{H} w^h - \bar{y} - a \in Q_1 \tag{2.9}$$

where $\bar{z}_0$ is the aggregate endowment vector of the initial old, as defined above, and

$$\begin{aligned} Q_1 &\equiv \{q_1 | (q_1, 0) \in Q\} \\ &= \{q_1 | (q_1, q_2) \in Q \text{ for some } q_2\} \end{aligned}$$

where Q is the production cone. The upper bound (2.8) indicates that the net claims of the infinite-lived consumers on the initial old do not exceed what can be produced using the aggregate endowment vector of the initial old, so that it is possible for the claims to be paid. The lower bound (2.9) indicates that the net obligations of the infinite-lived consumers to the initial old do not exceed what can be produced using the

aggregate endowment vector of infinite-lived consumers plus that of the young of generation one, so that sufficient goods exist in period one for the debts to be paid off -- perhaps by incurring new debts to the members of generation one. Since the net initial claims of all agents must sum to zero, the total net initial claims of the members of generation zero must be $-\sum_{h=1}^{H} w^h$. Therefore (A.2) of the previous paper[3] must be modified to read:

$$p_1' z_0(p_1) = -\sum_{h=1}^{H} p_1' w^h. \tag{2.10}$$

This is consistent with (2.6) since

$$p' z_0(p) = p' z(p,\beta p) - p' B_2(p,\beta p) b$$

$$= \frac{1}{1-\beta}\left[p' z(p,\beta p) + p' y(p,\beta p) - p' B_1(p,\beta p) b - p' B_2(p,\beta p) b\right]$$

$$= \frac{1}{1-\beta}\left[p' a - \sum_{h=1}^{H} p' c^h\right]$$

$$= -\sum_{h=1}^{H} p' w^h.$$

With these modifications the definition of perfect foresight equilibrium is again:

Definition. A perfect foresight equilibrium of economy $(z_0, y, V^h, a^h, w^h, Q)$ is a price sequence $\{p_t\}_1^\infty$, an activity sequence $\{b_t\}_1^\infty$, and consumption sequences $\{c_t^h\}_1^\infty$, for h = 1,...H, such that for each h, $\{c_t^h\}_1^\infty$ maximizes V^h subject to (2.7),

$$(2.11)\quad z_0(p_1) + y(p_1,p_2) + \sum_{h=1}^{H} c_1^h - a - B_1(p_1,p_2)b_1 = 0,$$

$$(2.12)\quad z(p_{t-1},p_t) + y(p_t,p_{t+1}) + \sum_{h=1}^{H} c_t^h - a - B_1(p_t,p_{t+1})b_t - B_2(p_{t-1},p_t)b_{t-1} = 0 \text{ for } t \geq 2,$$

$$(2.13)\quad p_t'B_1(p_t,p_{t+1}) + p'_{t+1}B_2(p_t,p_{t+1}) \leq 0 \text{ and } b_t \geq 0 \text{ for all } t, \text{ and}$$

$$(2.14)\quad p_t'B_1(p_t,p_{t+1})b_t + p'_{t+1}B_2(p_t,p_{t+1})b_t = 0 \text{ for all } t.$$

The steady state (p,β,b,c^h) is an equilibrium from period one on if the initial claims are such that

$$(2.15)\quad p'c^h = p'a^h + (1-\beta)p'w^h$$

for all h. Note that it is always possible to satisfy (2.15) with claims satisfying (2.8) and (2.9) since the only restriction placed by (2.15) on $w \equiv \sum_{h=1}^{H} w^h$ is

$$(1-\beta)p'w = \sum_{h=1}^{H} p'c^h - p'a$$

so that

$$p'(w-\bar{z}_0) = \frac{1}{1-\beta}\, p'(B_1 b+B_2 b-y-z-(1-\beta)(\bar{z}+B_2 b))$$
$$= p'(-\bar{z}-z) < 0,$$

and

$$p'(-w-\bar{y}-a) = \frac{1}{1-\beta}\, p'(-B_1 b-B_2 b+y+z-(1-\beta)(\bar{y}+a))$$
$$= \frac{1}{1-\beta}\, p'(-\beta B_1 b-\beta B_2 b+\beta y+\beta z-(1-\beta)(\sum_{h=1}^{H} c^h+\bar{y}))$$
$$= p'(B_1 b-y-\sum_{h=1}^{H} c^h-\bar{y}) < p'B_1 b$$

Here, all functions are evaluated at $(p,\beta p)$. These inequalities, and the fact that

$$-\bar{y}-a-B_1 b << \bar{z}+B_2 b = \bar{z}_0$$

in any steady state, imply that a w exists satisfying

$$-\bar{y}-a-B_1 b \ll w \ll \bar{z}_0$$

This w satisfies (2.8) and (2.9) since $0 \in Q_1$ and $B_1 b \in Q_1$. The aggregate w can then be allocated among the individual w^h's by any scheme satisfying (2.15).

These initial claims cannot be incorporated into the specification of the endowments for the first period, as it may be necessary to have $a^h + w^h$ with some negative components in order to create initial conditions consistent with a given steady state.[4] Thus we cannot always invoke the generalization of the theorem of Wilson (1981), presented in the previous paper, to be sure equilibrium exists when there are initial claims. Indeed, it is clear that if initial debts are sufficiently large compared to an infinite-lived agent's subsequent endowment stream, an equilibrium may not exist.[5] We have not sought to characterize the amount of initial debt consistent with the existence of perfect foresight equilibrium. However, since we specify initial conditions so as to be consistent with a particular

steady state, we always know that one equilibrium exists, the steady state itself.

We employ the methods of dynamical systems theory to analyze local determinacy. The criteria of local determinacy is familiar from capital theory, where it is often referred to as "saddle-point stability." A steady state x^* of a dynamical system

$$\dot{x} = f(x)$$

has this property if the matrix of derivatives $Df(x^*)$ has exactly the same number of eigenvalues with negative real parts as there are "predetermined" variables in the vector x. That is, if the initial conditions fix initial values for n_1 out of the n elements of x, there must be exactly n_1 eigenvalues of $Df(x^*)$ with negative real parts.

For example, in the familiar case of a two-dimensional system with one predetermined variable (x_1), perfect determinacy requires that $Df(x^*)$ have one eigenvalue with negative real part and one with positive real part, so that x^* is a "saddle point" as shown in

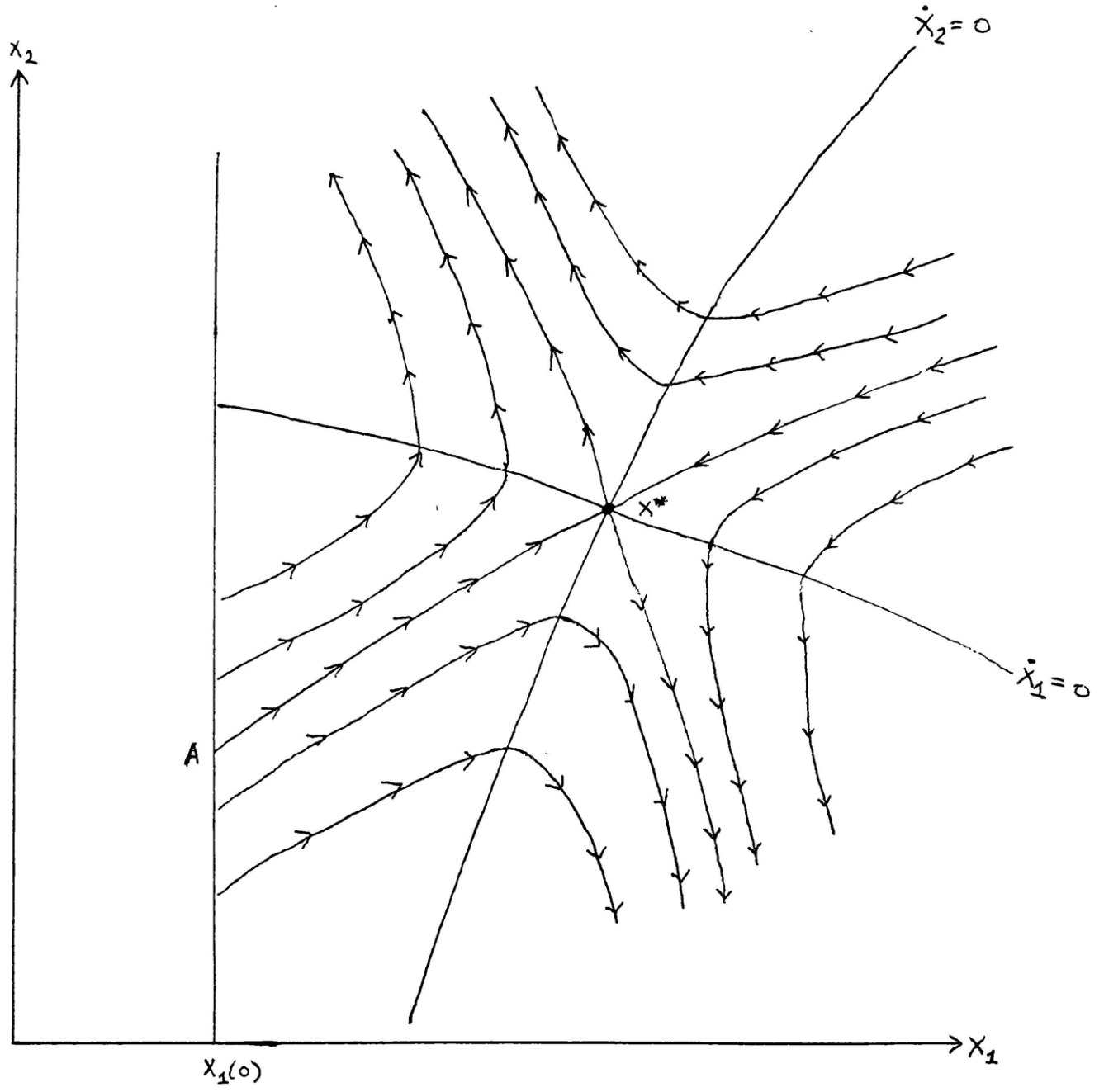


Figure 1. Perfect Determinacy.

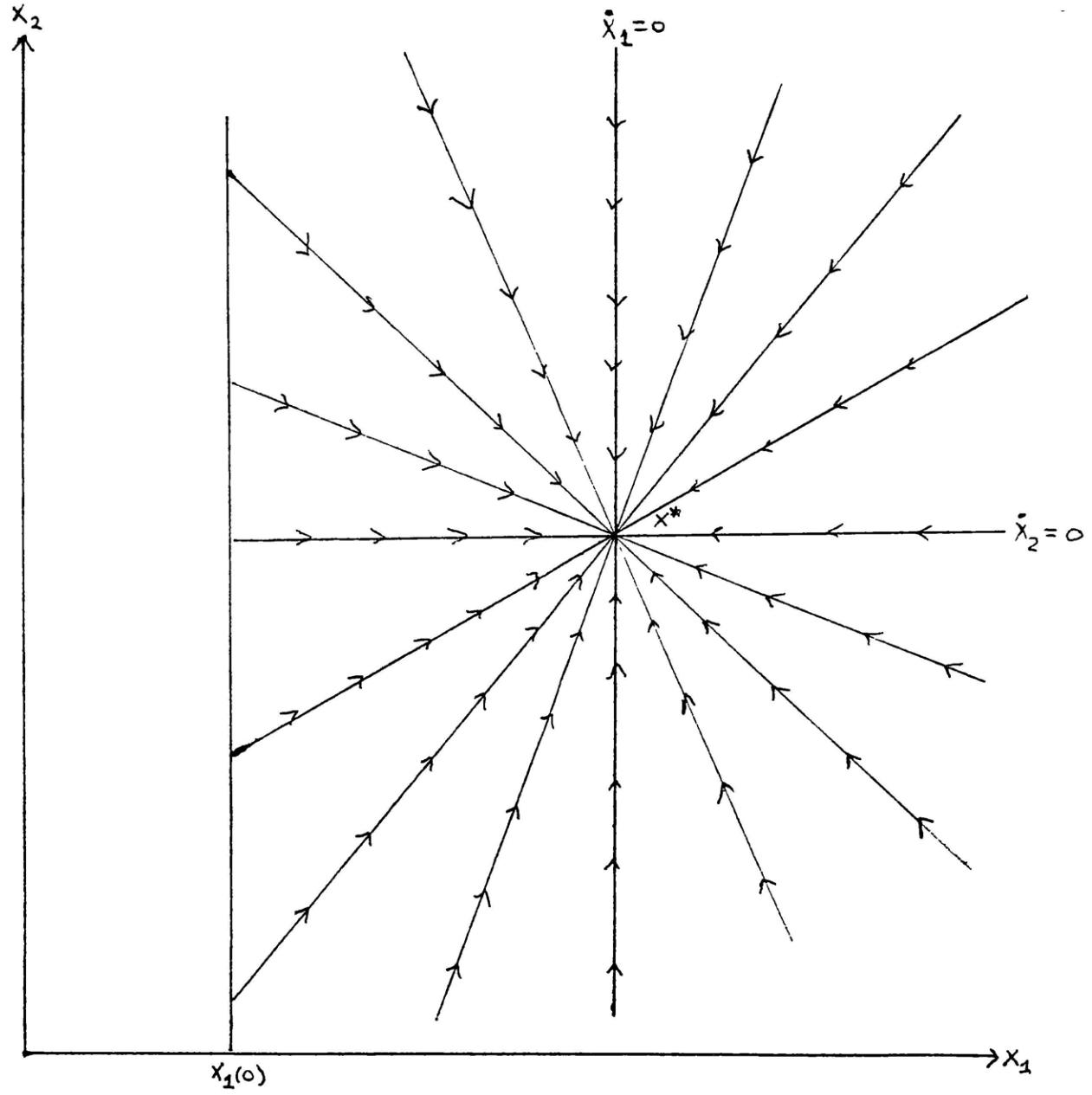


Figure 2. Indeterminacy.

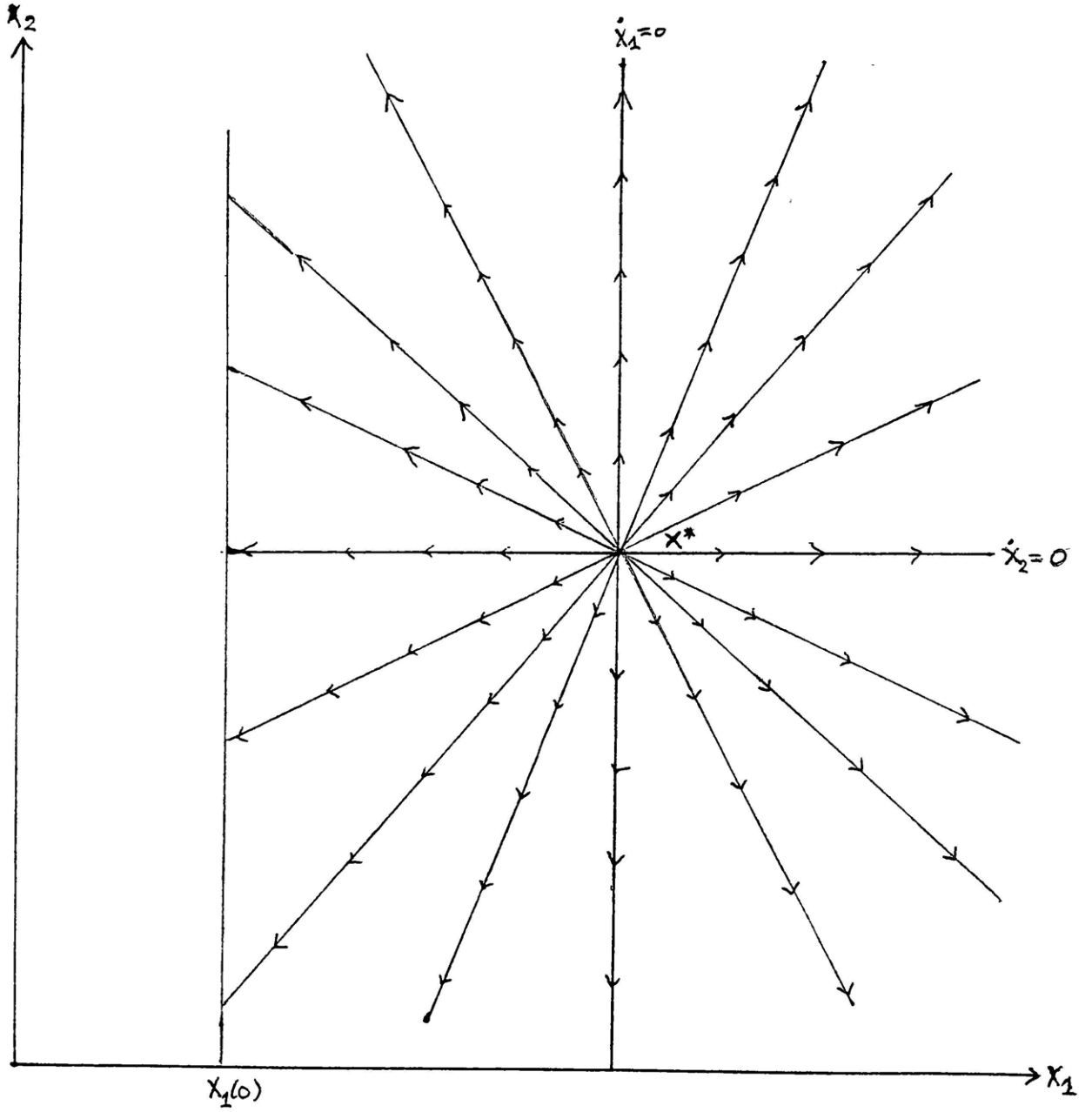


Figure 3. Lack of Structural Stability.

Figure 1. In that case, for a given specification $x_1(0)$ of the predetermined variable, there is a single value for x(0), point A in Figure 1, that extends to a dynamic path converging to the steady state. If both eigenvalues have negative real parts (and are distinct), x* would be a "stable focus" as shown in Figure 2. In that case, for a given specification $x_1(0)$ of the predetermined variable, $x_2(0)$ can take any of a continuum of values, and still extend to a dynamic path converging to x*. Thus the equilibrium is indeterminate. And if both eigenvalues have positive real parts (and are distinct), x* would be an "unstable focus" as shown in Figure 3. In that case no paths converge to x*, and so most specifications of $x_1(0)$ -- any specification other than $x_1(0) = x_1^*$ exactly -- are not consistent with any equilibrium path converging to the steady state.

In the case of more than two dimensions, the criterion for determinacy generalizes to that above. Let the solutions to the dynamical equations from t = 0 onward be parameterized by x(0), the possible values of which constitute an n-dimensional manifold S. If Df(x*) has n^s eigenvalues with negative real parts, then there

is an n^s-dimensional submanifold W of values x(0) such that

$$\lim_{t\to\infty} x(t) = x^*,$$

which set includes the value x^*. Initial conditions consistent with the steady state specify the n_1 predetermined elements of x(0). Determinacy occurs if elements of W consistent with these initial conditions are locally unique. Let points in S consistent with the initial conditions be called V, an $n-n_1$-dimensional submanifold. Then the condition for determinacy is

$$\dim (V \cap W) = 0,$$

which holds generically if

$$\dim V + \dim W \leq \dim S,$$

i.e., $n^s \leq n_1$. If $n^s > n_1$, there must be a continuum of

points in $V \cap W$, and hence a continuum of solutions to the dynamical equations, all consistent with the initial values of the n_1 predetermined variables and all converging asymptotically to the steady state x^*; the equilibrium is indeterminate. On the other hand, perfect determinacy also requires structural stability -- that there continues to be an element of $V \cap W$ nearby if the initial conditions are perturbed slightly. This cannot happen if $n^s < n_1$, so that perfect determinacy requires $n^s = n_1$, i.e.,

$$\dim V + \dim W = \dim S.$$

Similar methods may be applied to discrete time dynamical systems. A steady state x^* of a nonlinear difference equation

$$x_{t+1} = f(x_t)$$

is locally determinate if the matrix of derivatives $Df(x^*)$ has exactly the same number of eigenvalues with

modulus less than one as there are predetermined variables in the vector x. Again, equilibrium is indeterminate if the number of eigenvalues with modulus less than one is greater than this quantity, and equilibrium is not structurally stable if the number of eigenvalues with modulus less than one is smaller than this quantity.

Consider now the discrete time dynamical system defined by equations (2.12)-(2.14). Let the stable manifold W be the set of initial conditions that have an extension to a path that satisfies all of the equilibrium conditions except (2.11), and that converges asymptotically to the steady state. Let the initial conditions manifold V be the set of initial conditions that satisfy (2.11), and (2.13) and (2.14) for t=1. Also let the solution manifold S be the set of initial conditions that have an extension to a path that satisfies all of the equilibrium conditions except (2.11). Both V and W are submanifolds of S. They intersect transversally in S if the sum of their tangent spaces at each point in the intersection is the tangent space of S. Perfect determinacy occurs if V and W intersect transversally and

(2.16) $\dim V + \dim W = \dim S.$

If dim V + dim W exceeds dim S, there are not enough equilibrium conditions for even equilibria converging asymptotically to the steady state to be locally unique, so that equilibrium is indeterminate. The number of dimensions by which dim V + dim W exceeds dim S, which is $\dim(V \cap W)$, is called the dimension of indeterminacy. If (2.16) holds, equilibria converging asymptotically to the particular steady state are locally unique. If dim V + dim W is less than dim S, then the intersection between V and W (which by construction includes at least the steady state) is not transversal. This means that their intersection is not generic -- small perturbations of the initial conditions exist under which there would be no equilibria converging asymptotically to the steady state. We show that all three cases are possible.

We determine $\dim(V \cap W)$ by considering the tangent spaces $\tilde{V}$ and $\tilde{W}$ of V and W near the steady state, which is known to belong to $V \cap W$. These tangent spaces are the linear spaces of solutions to the linearizations of the corresponding equilibrium conditions around the steady

state values. The dimension of $\tilde{W}$, which equals the dimension of W, then corresponds to the number of eigenvalues of the linearized equilibrium conditions that have modulus less β. (See theorem 6.17 of Irwin, 1980.)

The linearization is straightforward in the case of steady states in which all infinite-lived agents consume, i.e., $c^h >> 0$ for all h. This is not, however, a particularly general situation, for it is not possible if the infinite-lived agents have different rates of time preference. If $c^h = 0$ for some agent, one cannot linearize around the steady state, strictly speaking. For that agent consumes a small amount on paths near the steady state, and converging to it asymptotically, so there are conditions of the form

$$(2.17) \quad \beta_h^{t-1} Du^h(c_t^h) = \lambda^h p_t,$$

for t = 1,2,..., among the equilibrium conditions. Equation (2.17) cannot be linearized around the steady state (i.e., around $c^h = 0$) because of (B.3). So what we consider is not the steady state and initial conditions

consistent with it, but an equilibrium very close to it in which all agents consume at least a small amount, and initial conditions consistent with equilibrium, which are close to initial conditions that would allow the steady state. Linearization of the equilibrium conditions around this equilibrium then allows us to determine the dimension of the manifold of equilibrium paths converging to the steady state.

But as it turns out, the agents who do not consume in steady state can be ignored in determining the dimension of indeterminacy of equilibrium near the steady state. That is, their endowment may be treated as land in the possession of some member of finite-lived generation zero, while their consumption may be ignored altogether. In order to insure this, we make the following additional assumption on the u^h's:

(B.4) Let $d^h(p,Y)$ be the solution to max $u^h(d^h)$ subject to $p'd^h \leq Y$, and let $\lambda^h(p,Y)$ be the Lagrange multipliers on the budget constraint in the problem. Then for any $p >> 0$,

$$\lim_{Y \to 0} \lambda^h(p,Y)[D^2u^h(d^h(p,Y))]^{-1} = 0$$

in all components.

This assumption is satisfied by most familiar utility functions that satisfy (B.1) - (B.3) such as the constant elasticity of substitution family. It implies that when conditions (2.17) are linearized around some near-steady-state equilibrium yielding

$$\beta_h^{t-1}[c_t^h - \bar{c}_t^h] = \bar{\lambda}^h[D^2u^h(\bar{c}_t^h)]^{-1}[(p_t - \bar{p}_t) + (\frac{\lambda^h - \bar{\lambda}^h}{\bar{\lambda}^h})\bar{p}_t]$$

the size of the deviations in c_t^h required in the case of a given perturbation of the price sequence are of much smaller order. Hence conditions (2.17) have no significant effect on the dynamics, and the same degree of indeterminacy is obtained if one replaces (2.17) by $c_t^h = 0$ for all t.

Thus one obtains a set of conditions that can be linearized around the steady state.

We therefore consider only those agents $h=1,\ldots J \leq H$ for whom $\beta_h = \beta_1$, and who consume in steady state. The vector a represents the sum of the endowments of these J agents, plus any land that exists, or the endowments of any other infinite-lived agents who do not consume asymptotically. First we present the analysis for the case $J = 0$, and then for the case $J \geq 1$.

III. Steady States in Which No Infinite-Lived Agents Consume

We assume that activities $j=1,\ldots,k\leq m$ are the only ones that yield zero profits in steady state, and that all of them are operated in steady state (i.e., $b_j > 0$ for $j=1,\ldots,k$). Therefore, at all solutions near steady state, activities $j>k$ yield negative profits and are not operated, while activities $j\leq k$ are operated and hence must yield exactly zero profits. It follows that we can ignore activities $j>k$, making $b>>0$ a k-vector, and B_1 and B_2 nxk matrices. We can then ignore condition (2.13).

We also adopt the conventions

(3.1a) $$D_i y(p,\beta p) = D_i y$$

(3.1b) $$D_i z(p,\beta p) = D_i z$$

(3.2a) $$B_j(p,\beta p) = B_j$$

(3.2b) $$D_j B_i(p,\beta p)b = K_{ij}$$

for i,j = 1,2. Then linearizing equilibrium conditions (2.12) and (2.14) around a steady state in which $p_t = \beta^{t-1} p$, $b_t = b$, $c_t^h = c^h$, with the substitution $c_t^h = 0$ even for paths near the steady state, as discussed above, yields:

(3.3)
$$[\beta D_1 z - \beta K_{21}](p_{t-1} - \beta^{t-2} p) + [\beta D_2 z + D_1 y - \beta K_{22} - K_{11}](p_t - \beta^{t-1} p)$$
$$+ [D_2 y - K_{12}](p_{t+1} - \beta^{t} p) - \beta^{t-1} B_1(b_t - b) - \beta^{t-1} B_2(b_{t-1} - b) = 0$$

(3.4) $$B_1'(p_t - \beta^{t-1} p) + B_2'(p_{t+1} - \beta^{t} p) = 0.$$

This is a set of n+k stationary linear difference equations for n+k state variables $(p_t - \beta^{t-1} p)$ and

$\beta^{t-1}(b_t - b)$.

The coefficients of (3.3) and (3.4) satisfy identities

$$D_1 y \cdot p + \beta D_2 y \cdot p = 0 \tag{3.5}$$

$$D_1 z \cdot p + \beta D_2 z \cdot p = 0 \tag{3.6}$$

$$p' D_1 y + \beta p' D_1 z = - y'(p, \beta p) \tag{3.7}$$

$$p' D_2 y + \beta p' D_2 z = - z'(p, \beta p) \tag{3.8}$$

$$K_{11} p + \beta K_{12} p = 0 \tag{3.9}$$

$$K_{21} p + \beta K_{22} p = 0 \tag{3.10}$$

$$p' K_{11} + \beta p' K_{21} = 0 \tag{3.11}$$

$$p' K_{12} + \beta p' K_{22} = 0 \tag{3.12}$$

Identities (3.5) and (3.6) follow from the homogeneity of y and z, and (3.9) and (3.10) follow from the homogeneity

of B_1 and B_2. Identities (3.7) and (3.8) follow from (A.2), while (3.11) and (3.12) follow from (3.26).

We also know from (2.2) and (2.3) that

$$p'B_1 + \beta p'B_2 = 0 \tag{3.13}$$

and from (C.4) that

$$p'B_1 \leq 0. \tag{3.14}$$

And finally, we know from the convexity of profit functions that

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

is a non-negative symmetric matrix. That is, its eigenvalues are all real and non-negative. Other than these relations, the assumptions we have made, and economic theory, place no restrictions on the coefficients of (3.3) and (3.4), so any conclusion that may be drawn about local determinacy must follow from the relations listed in this paragraph.

The system defined by (3.3) and (3.4) may be written as

$$\phi_t = G\,\phi_{t-1} \tag{3.15}$$

where

$$\phi_t \equiv \begin{bmatrix} \beta^{t-1}(b_t - b) \\ p_t - \beta^{t-1}p \\ p_{t+1} - \beta^{t}p \end{bmatrix}$$

and

$$G \equiv \begin{bmatrix} -B_1 & (\beta D_2 z + D_1 y - \beta K_{22} - K_{11}) & D_{2y} - K_{12} \\ 0 & I_n & 0 \\ 0 & B_1' & B_2' \end{bmatrix}^{-1} \begin{bmatrix} \beta B_2 & -(\beta D_1 z - \beta K_{21}) & 0 \\ 0 & 0 & I_n \\ 0 & 0 & 0 \end{bmatrix}$$

In writing this we make use of the following regularity assumptions.

$$\text{Det}\ (D_2 y - K_{12}) \neq 0. \tag{R.1}$$

(R.2) $$\text{Det } (B'_2(D_2y-K_{12})^{-1}B_1)\neq 0.$$

(R.1) is true in general since the conditions (3.5)-(3.14) do not imply otherwise, and if no single-period production activities are in use,[6] (R.2) is also true in general. These guarantee that the matrix on the left above is inevitable. We adopt the shorthand notation

$$\begin{aligned} e &= D_2y-K_{12} \\ f &= \beta D_2z+D_1y-\beta K_{22}-K_{11} \\ g &= \beta D_1z-\beta K_{21} \\ u &= B_1 \\ v &= B_2 \end{aligned}$$

The matrix G is then given by

(3.16) $$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ 0 & 0 & I_n \\ G_{31} & G_{32} & G_{33} \end{bmatrix}$$

where

$$
\begin{aligned}
G_{11} &= -\beta(v'e^{-1}u)^{-1}\ v'e^{-1}v \\
G_{12} &= \ (v'e^{-1}u)^{-1}\ v'e^{-1}g \\
G_{13} &= -(v'e^{-1}u)^{-1}\ (u'-v'e^{-1}f) \\
G_{31} &= \beta[\,e^{-1}v-e^{-1}u(v'e^{-1}u)^{-1}v'e^{-1}v] \\
G_{32} &= -e^{-1}g+e^{-1}u(v'e^{-1}u)^{-1}v'e^{-1}g \\
G_{33} &= -e^{-1}f-e^{-1}u(v'e^{-1}u)^{-1}(u'-v'e^{-1}f)
\end{aligned}
$$

The set of solutions to the equations (3.15) form a 2n - dimensional linear space, rather than a 2n+k- dimensional linear space, because the k equations (3.4) are restrictions that each vector ϕ_t must satisfy, rather than dynamic equations relating ϕ_t and ϕ_{t+1}. This means that G has k zero eigenvalues, corresponding to left zero eigenvectors $(0,B_1',B_2')$. The 2n right eigenvectors corresponding to the 2n nonzero eigenvalues span a 2n-dimensional linear space of solutions $\tilde{S}$. We are interested in finding the subspace of solutions that converge asymptotically to the steady state.

Our identities guarantee that one eigenvalue of G is β, with the corresponding right eigenvector being

$$
(3.17) \qquad G \begin{bmatrix} 0 \\ p \\ \beta p \end{bmatrix} = \beta \begin{bmatrix} 0 \\ p \\ \beta p \end{bmatrix}.
$$

This one-dimensional space of solutions corresponds to a uniform rescaling of all prices in all periods, which has no real significance. In the case that a=0 -- that is, neither land nor infinite-lived consumers exist -- G has 1 as an eigenvalue, because of the relation

$$(3.18)\quad [\beta p'B_2 \quad -\beta p'(D_1 z - K_{21}) \quad p'(D_2 y - K_{12}) + b'B_2'] G$$
$$= [\beta p'B_2 \quad -\beta p'(D_1 z - K_{21}) \quad p'(D_2 y - K_{12}) + b'B_2' + a']$$

However, when $a \neq 0$, this is no longer an eigenvector, and the relations (3.5)-(3.14) do not determine any other eigenvalues. We make the following regularity assumption, which is generically valid:

(R.3) G has 2n non-zero eigenvalues; these eigenvalues have the same modulus if and only if they are complex conjugates.

In the case that $a \neq 0$, we also assume:

(R.4) G has no eigenvalue equal to one.

Note that (R.4) can only be true if $a \neq 0$, because of

(3.18). Let n^s be the number of eigenvalues of G of modulus less than β. Then, if we ignore the uniform price renormalizations, $\tilde{S}$ contains an n^s- dimensional subspace, $\tilde{W}$, of equilibria converging asymptotically to the steady state and a $2n-n^s-1$ dimensional subspace of equilibria diverging from it. If $a=0$, then G has an eigenvalue equal to one and an eigenvalue equal to β, but no others are determined by our general assumptions. Therefore, in the case of a non-monetary equilibrium, n^s can take any value between $0 \leq n^s \leq 2n-1$ if $\beta > 1$ and $0 \leq n^s \leq 2n-2$ if $\beta < 1$, while in the case of a monetary equilibrium,[7] with $\beta = 1$, n^s can take any value between $0 \leq n^s \leq 2n-1$. This agrees with the results of Kehoe and Levine (1983) for the case of pure exchange economies. If $a>0$, then one eigenvalue must be β, but no others are determined by our general assumptions. Therefore n^s can take any value between $0 \leq n^s \leq 2n-1$.

Elements of $\tilde{S}$ may be identified with the values of ϕ_1 for each solution. The initial conditions manifold V consists of the solutions which satisfy

$$(3.19) \quad z_0(q,p_1) + y(p_1,p_2) - a - B_1(p_1,p_2)b_1 = 0$$

where q represents the vector of land prices for land traded before its period one yield, which in equilibrium must be given by

$$q = \sum_{t=1}^{\infty} p_t \tag{3.20}$$

It is assumed that the members of generation zero hold the initial claims to the vector of land a, as well as the period one endowment of $\bar{z} + B_2 b$ (as discussed in section II). Because the endowment due to land ownership extends over all periods while the agents consume only in period one, it is convenient to define z_0 as their period one demand in excess of the endowment $\bar{z} + B_2 b$, but not in excess of their land endowment. Therefore $z_0(q, p_1)$ in (3.19) satisfies the identity

$$p_1' z_0(q, p_1) \equiv q'a \tag{3.21}$$

In accordance with (2.6), $z_0(q, p_1)$ must also satisfy

$$z_0\left(\frac{p}{1-\beta}, p\right) = z(p, \beta p) - B_2 b. \tag{3.22}$$

Linearizing (3.19) around the steady state values $p_1 = p$, $p_2 = \beta p$, $b_1 = b$, $q = \frac{p}{1-\beta}$ yields

$$(3.23) \qquad D_1 z_0\left(q-\frac{p}{1-\beta}\right) + \left[D_2 z_0 + D_1 y - K_{11}\right](p_1-p) + \left[D_2 y - K_{12}\right](p_2-\beta p) - B_1(b_1-b) = 0.$$

The coefficients $D_i z_0$ satisfy identities

$$(3.24) \qquad D_1 z_0 \cdot \left(\frac{p}{1-\beta}\right) + D_2 z_0 \cdot p = 0$$

$$(3.25) \qquad p' D_1 z_0 = a'$$

$$(3.26) \qquad p' D_2 z_0 = - z_0'\left(\frac{p}{1-\beta}, p\right)$$

(3.24) follows from the homogeneity of the excess demand function, while (3.25) and (3.26) result from differentiation of (3.21). For any solution of (3.15), $\left(q-\frac{p}{1-\beta}\right)$ is not independent of ϕ_1, but must satisfy

$$\begin{aligned}\left(q-\frac{p}{1-\beta}\right) &= \sum_{t=1}^{\infty}\left(p_t-\beta^{t-1}p\right)\\ &= (p_1-p) + (0\ 0\ I_n)\sum_{t=1}^{\infty}\phi_t\\ &= (p_1-p) + (0\ 0\ I_n)\sum_{t=1}^{\infty}G^{t-1}\phi_1\\ &= \left[(0\ I_n\ 0) + (0\ 0\ I_n)(I_{2n+k}-G)^{-1}\right]\phi_1\end{aligned}$$

where $(I_{2n+k}-G)$ has an inverse because of (R.4). Therefore (3.23) may be rewritten

$$\begin{aligned}(3.27)\qquad &\left[(-B_1\ D_1z_0+D_2z_0+D_1y-K_{11}\ D_2y-K_{12})\right.\\ &\left.+ (0\ 0\ D_1z_0)(I_{2n+k}-G)^{-1}\right]\phi_1 = 0\end{aligned}$$

This set of linear restrictions characterizes the tangent space $\tilde{V}$ of the initial conditions manifold.

Our assumptions require that the linear restrictions (3.27) have two properties. First of all,

$$\begin{bmatrix} 0 \\ p \\ \beta p \end{bmatrix}$$

must be a solution. This follows from (3.5), (3.9), (3.17), and (3.24). And second, (3.27) comprises only

n-1 linearly independent restrictions. To see this, note that (3.18) implies

$$[\beta p'B_2 \; -\beta p'(D_1 z - K_{21}) \quad p'(D_2 y - K_{12}) + b'B_2'] = [0 \; 0 \; -a'](I_{2n+k} - G)^{-1}.$$

Using (3.25), we obtain

$$[\beta p'B_2 \; -\beta p'(D_1 z - K_{21}) \quad p'(D_2 y - K_{12}) + b'B_2'] = [0 \; 0 \; -p'D_1 z_0](I_{2n+k} - G)^{-1}.$$

Then the left side of (3.27), premultiplied by p', becomes

$$[-(p'B_1 + \beta p'B_2) \quad p'(D_1 z_0 + D_2 z_0 + D_1 y - K_{11} + \beta D_1 z - \beta K_{21}) \quad -b'B_2']\phi_1$$

Using (3.7), (3.11), (3.13), (3.25) and (3.26), this expression is equal to

$$[0 \quad a' - z_0' - y' \quad -b'B_2']\phi_1 = [0 \quad -b'B_1' \quad -b'B_2']\phi_1 = 0$$

Where (2.11) is used in the first equality and (3.4) in the second. Thus this linear combination of the

restrictions is guaranteed to equal zero without imposing (3.27). Hence $\tilde{V}$ is an n+1 dimensional linear space, of which one dimension corresponds to uniform price renormalizations.

If $\tilde{V}$ and $\tilde{W}$ intersect transversally, as they do generically, the dimension of indeterminacy is $n^s+(n+1)-2n=n^s-n+1$, which may be as large as n. On the other hand, if $n^s = n-1$, the steady state equilibrium is exactly determinate. If $n^s<n-1$ -- and n^s may be as low as zero -- the equilibrium is not structurally stable. That is, under certain small perturbations of the initial conditions -- say, of the distribution of endowments among members of generation zero -- there ceases to be an equilibrium near the steady state.

IV. Steady States with Infinite-Lived Comsuners

Let there be $J > 0$ infinite-lived consumers, each of whose time discount factor is $\beta_h = \beta$, the steady state discount factor for future prices, and each of whom has steady state consumption $c^h >> 0$. In this case the linearized equilibrium conditions (3.3) and (3.4) become

$$(4.1)\qquad [\beta D_1 z - \beta K_{21}](p_{t-1} - \beta^{t-2}p) + [\beta D_2 z + D_1 y - \beta K_{22} - K_{11}](p_t - \beta^{t-1}p)$$

$$+ [D_2 y - K_{12}](p_{t+1} - \beta^{t}p) + \beta^{t-1} \sum_{h=1}^{J} \sum_{s=1}^{\infty} \left[\frac{\partial c_t^h}{\partial p_s}\right](p_s - \beta^{s-1}p)$$

$$- \beta^{t-1} B_1(b_t - b) - \beta^{t-1} B_2(b_{t-1} - b) = 0.$$

$$(4.2)\qquad B_1'(p_t - \beta^{t-1}p) + B_2'(p_{t+1} - \beta^{t}p) = 0.$$

Because each $c_t^h(p_1, p_2, \ldots)$ is a function of the entire price sequence, instead of only a finite number of prices, it might seem that this system of linear equations cannot be analyzed by the methods used for finite-order linear difference equations, illustrated in the preceding section. However, the additive separability and the stationarity of the utility functions V^h causes the infinite matrix $\left[\frac{\partial c_t^h}{\partial p_s}\right]$ to take the simple form

$$\left[\frac{\partial c_t^h}{\partial p_s}\right] = \delta_{st}\lambda^h \left[V_{tt}^h\right]^{-1} - \frac{\lambda^h\left[V_{tt}^h\right]^{-1} p_t p_s'\left[V_{ss}^h\right]^{-1}}{\sum_{v=1}^{\infty} p_v'\left[V_{vv}^h\right]^{-1} p_v}$$

$$- \frac{\left[V_{tt}^h\right]^{-1} p_t (c_s^h - a^h - \delta_{s1} w^h)'}{\sum_{v=1}^{\infty} p_v'\left[V_{vv}^h\right]^{-1} p_v}$$

where $\left[V_{tt}^h\right]$ denotes the nxn matrix $\frac{\partial^2 V^h}{\partial c_t^{h2}}$, δ_{st} is the Kronecker delta, and λ^h is the Lagrange multiplier in (2.5). The first two terms are a Slutsky matrix of substitution effects, while the last term is the income effect. Evaluated at a steady state where $p_t = \beta^{t-1} p$, $c_t^h = c^h$, $b_t = b$ for all t, one has

$$\left[V_{tt}^h\right] = \beta^{t-1} D^2 u^h(c^h)$$

so that

$$\left[\frac{\partial c_t^h}{\partial p_s}\right] = \delta_{st}\lambda^h\beta^{1-t}[D^2u^h]^{-1} - \lambda^h(1-\beta)\frac{[D^2u^h]pp'[D^2u^h]^{-1}}{p'[D^2u^h]^{-1}p}$$

$$- (1-\beta)\frac{[D^2u^h]^{-1}p(c^h - a^h - \delta_{s1}w^h)'}{p'[D^2u^h]^{-1}p}$$

Therefore one can write

$$\sum_{h=1}^{J}\left[\frac{\partial c_t^h}{\partial p_s}\right] = \delta_{st}\beta^{1-t}A - E + \delta_{s1}D$$

where the definitions of matrices A,E and D are obvious from the preceding line. Note that these matrices satisfy the identities

$$Ep = (1-\beta)(A + D)p \tag{4.3}$$

$$p'E = (1-\beta)p'A + (1-\beta)\left[\sum_{h=1}^{J}(c^{h'} - a^{h'})\right] \tag{4.4}$$

where the first follows from the homogeneity of the infinite-lived consumers' demand functions, and the second from differentiating their budget constraints.

Equation (4.1) now takes the form

$$(4.5)\qquad [\beta D_1 z - \beta K_{21}](p_{t-1} - \beta^{t-2}p) + [\beta D_2 z + D_1 y - \beta K_{22} - K_{11}](p_t - \beta^{t-1}p)$$
$$+ [D_2 y - K_{12}](p_{t+1} - \beta^t p) + A(p_t - \beta^{t-1}p) - \beta^{t-1} E \sum_{s=1}^{\infty} (p_s - \beta^{s-1}p)$$
$$+ \beta^{t-1} D(p_1 - p) - \beta^{t-1} B_1(b_t - b)$$
$$- \beta^{t-1} B_2(b_{t-1} - b) = 0.$$

Subtracting β times this equation for period t from the equation for period t + 1 yields

$$(4.6)\qquad [-\beta^2 D_1 z + \beta^2 K_{21}](p_{t-1} - \beta^{t-2}p) +$$
$$[\beta D_1 z - \beta K_{21} - \beta^2 D_2 z - \beta D_1 y$$
$$+ \beta^2 K_{22} + \beta K_{11} - \beta A](p_t - \beta^{t-1}p) +$$
$$[\beta D_2 z + D_1 y - \beta K_{22} - K_{11} + A$$
$$- \beta D_2 y + \beta K_{12}](p_{t+1} - \beta^t p) +$$
$$[D_2 y - K_{12}](p_{t+2} - \beta^{t+1}p) - \beta^t B_1(b_{t+1} - b)$$
$$- \beta^t (B_2 - B_1)(b_t - b) + \beta^t B_2(b_{t-1} - b) = 0.$$

This is at last a finite-order stationary linear difference equation. However, because of the first differencing operation, not all solutions to (4.6) are solutions of (4.5). Once the solutions of (4.6) have been identified, it is necessary to eliminate those that do not satisfy (4.5). It suffices for this to impose (4.5) for $t = 2$, since this equation plus (4.6) for all $t \geq 2$ is equivalent to (4.5) for all $t > 2$. Hence our method of analysis is:

(i) identify the linear space of solutions to (4.6);

(ii) impose (4.5) for $t = 2$ (the period two conditions) to identify subspace $\tilde{S}$;

(iii) identify the subspace $\tilde{W}$ of $\tilde{S}$ which converges asymptotically to the steady state; and finally

(iv) consider the intersection of

$\tilde{W}$ with $\tilde{V}$, the subspace of $\tilde{S}$ consistent with the initial conditions.

Equation (4.6) may be written as

$$\phi_t = F\,\phi_{t-1} \tag{4.7}$$

where now

$$\phi_t \equiv \begin{bmatrix} \beta^{t-1}(b_t - b) \\ \beta^{t}(b_{t+1} - b) \\ p_t - \beta^{t-1}p \\ p_{t+1} - \beta^{t}p \\ p_{t+2} - \beta^{t+1}p \end{bmatrix}$$

and

$$F = \begin{bmatrix} \beta(B_1 - B_2) & -B_1 & F_2 & F_1 & D_2 y - K_{12} \\ 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & B_1' & B_2' \\ I_k & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \cdot$$

$$\begin{bmatrix} -\beta^2 B_2 & 0 & -\beta^2(D_1 z - K_{21}) & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n \\ 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I_k & 0 & 0 & 0 \end{bmatrix}$$

using as shorthand

$$F_1 = \beta D_2 z + D_1 y - \beta D_2 y - \beta K_{22} - K_{11} + \beta K_{12} + A.$$

$$F_2 = \beta D_1 z - \beta^2 D_2 z - \beta D_1 y - \beta K_{21} + \beta^2 K_{22} + \beta K_{11} - \beta A.$$

It is easily verified that (R.1) and (R.2) suffice for the matrix on the left to be invertible. The $(3n + 2k) \times (3n + 2k)$ matrix F has 2k zero eigenvalue, corresponding to the left zero eigenvectors[8]

$$(0 \quad 0 \quad 0 \quad B_1' \quad B_2')F = (0 \quad 0 \quad 0 \quad 0 \quad 0)$$

$$(0 \quad 0 \quad B_1' \quad B_2' \quad 0)F^2 = (0 \quad 0 \quad 0 \quad B_1' \quad B_2')F$$

$$= (0 \quad 0 \quad 0 \quad 0 \quad 0).$$

The remaining $3n$ eigenvalues are

(i) $n-k$ eigenvalues β corresponding to the right eigenvectors

$$F \begin{bmatrix} 0 \\ 0 \\ r \\ \beta^2 r \\ \beta\, r \end{bmatrix} = \beta \begin{bmatrix} 0 \\ 0 \\ r \\ \beta^2 r \\ \beta\, r \end{bmatrix}$$

where r takes any value in the $n-k$-dimensional linear space of n-vectors satisfying

$(B_1' + \beta B_2')r = 0$;

(ii) k more eigenvalues β, corresponding to the right eigenvectors

$$F \begin{bmatrix} s \\ \beta s \\ 0 \\ 0 \\ 0 \end{bmatrix} = \beta \begin{bmatrix} s \\ \beta s \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where s is any k-vector;

(iii) the 2n non-zero eigenvalues of $\hat{G}$, μ_j for $j=1,\ldots 2n$, where $\hat{G}$ is defined by (3.16) with the substitution

$$f = \beta D_2 z + D_1 y - \beta K_{22} - K_{11} + A$$

for our previous notation; these eigenvalues correspond to the right eigenvectors

(4.8)
$$F \begin{bmatrix} u_j \\ \mu_j u_j \\ v_j \\ \mu_j v_j \\ \mu_j^2 v_j \end{bmatrix} = \mu_j \begin{bmatrix} u_j \\ \mu_j u_j \\ v_j \\ \mu_j v_j \\ \mu^2 v_j \end{bmatrix}$$

where the eigenvectors of $\hat{G}$ are

$$\hat{G}\begin{bmatrix} u_j \\ v_j \\ \mu_j v_j \end{bmatrix} = \mu_j \begin{bmatrix} u_j \\ v_j \\ \mu_j v_j \end{bmatrix}$$

This exahusts the eigenvalues of F. Note that β need not be among the eigenvalues of $\hat{G}$, for we now have

$$\hat{G}\begin{bmatrix} 0 \\ p \\ \beta p \end{bmatrix} = \begin{bmatrix} \beta[(B_2'(D_2y - K_{12})^{-1}B_1)^{-1}B_2'(D_2y - K_{12})^{-1}Ap] \\ \beta p \\ \beta^2 p - \beta[D_2y - K_{12})^{-1} - (D_2y - K_{12})^{-1}B_1 \cdot \\ \quad (B_2'(D_2y - K_{12})^{-1}B_1)^{-1}B_2'(D_2y - K_{12})^{-1}]Ap \end{bmatrix} \tag{4.9}$$

Thus our general assumptions determine none of the eigenvalues of $\hat{G}$. We make the following regularity assumptions which are generically valid.

(R.5) $\hat{G}$ has 2n non-zero eigenvalues; these eigenvalues have the same modulus only if they are complex conjugates.

(R.6) $\hat{G}$ has no eigenvalue equal to one, and none with modulus exactly equal to β.

Then paralleling the notation of the previous section, we let n^s be the number of eigenvalues of $\hat{G}$ with modulus less than β, which may range over $0 \leq n^s \leq 2n$.

The solutions to (4.7) form a 3n-dimensional linear space, rather than a 3n + 2k-dimensional linear space, because equations (4.2) impose 2k restrictions that each vector ϕ_t must satisfy. These solutions may be identified with the values of ϕ_1 for each one. The 3n-dimensional space may be parameterized by

$$(4.10)\quad \begin{bmatrix} b_1 - b \\ \beta(b_2 - b) \\ p_1 - p \\ p_2 - \beta p \\ p_3 - \beta^2 p \end{bmatrix} = \sum_{j=1}^{2n} d_j \begin{bmatrix} u_j \\ \mu_j u_j \\ v_j \\ \mu_j v_j \\ \mu_j^2 v_j \end{bmatrix} + \begin{bmatrix} s \\ \beta s \\ r \\ \beta r \\ \beta^2 r \end{bmatrix}$$

where as discussed above, the coefficients r must satisfy

$$(4.11)\quad (B_1' + \beta B_2')r = 0$$

Of the 3n dimensions, there is an n^s- dimensional subspace of solutions converging asymptotically to the steady state, a $2n-n^s$-dimensional subspace diverging from it, and an n-

dimensional subspace of solutions which (in the linear approximation) neither converge nor diverge.[9]

The period two conditions are then

$$[\beta D_1 z - \beta K_{21} + \beta D](p_1 - p) + [\beta D_2 z + D_1 y - \beta K_{22} - K_{11} + A](p_2 - \beta p) +$$

$$[D_2 y - K_{12}](p_3 - \beta^2 p) - \beta E \sum_{s=1}^{\infty} (p_s - \beta^{s-1} p) - \beta B_1 (b_2 - b) - \beta B_2 (b_1 - b) = 0.$$

In terms of the notation in (4.10), these additional restrictions are[10]

$$(4.12) \qquad [\beta D_2 y + \beta D_2 z + D_1 y + D_1 z - \beta K_{12} - \beta K_{22} - K_{11} - K_{21} + A + D - \frac{1}{1-\beta} E] r$$

$$[B_1 + B_2]\, s = \sum_{j=1}^{2n} d_j \left[\frac{1}{1-\mu_j} E v_j - D v_j\right].$$

This is a set of n linear restrictions on the coefficients that are, in general, linearly independent. It follows from (3.5), (3.6), (3.9), (3.10) and (4.3) that

$$\begin{bmatrix} 0 \\ 0 \\ \\ p \\ \beta p \\ \beta^2 p \end{bmatrix}$$

is one solution, so that the subspace of uniform price renormalizations belongs to $\tilde{S}$. Let us restrict our attention to the n^s + n-dimensional space of solutions to (4.7) spanned by eigenvectors with eigenvalues of modulus β or less. Since conditions (4.12) are in general linearly independent of (4.7), and there is no relation linking (4.12) and particular eigenvectors of $\hat{G}$, in general the linear space $\tilde{W}$ of solutions to (4.7) and (4.12) that do not diverge from the steady state -- at least in the linearized system -- is of dimension $\max(n^s,1)$ and includes the subspace of uniform price renormalizations.

Now it follows from the definitions of D and E that all columns of both vectors are linear combinations of the J vectors $[D^2 u^h]^{-1} p$ for h = 1,...,J. We make the following generically valid assumption.

(R.7) $\quad \text{Rank}[D\ E] = m \equiv \min[n, J].$

Then (4.12) requires that the n-vector

$$[\beta D_2 y + \beta D_2 z + D_1 y + D_1 z - \beta K_{12} - \beta K_{22} - K_{11} - K_{21} + A]r - [B_1 + B_2]s$$

lie in a particular m-dimensional subspace. If J=1, this space consists of multiples of a single vector, Ap. In general, it is an m-dimensional linear space, containing the vector Ap. We also assume the following generically valid regularity condition.

(R.8) $$\text{Det} \begin{bmatrix} \beta D_2 y + \beta D_2 z + D_1 y + D_1 z - \beta K_{12} - BK_{22} - K_{11} - K_{21} + A & -B_1 - B_2 \\ B_1' + \beta B_2' & 0 \end{bmatrix} \neq 0$$

This condition amounts to an assumption that the linearized conditions for a steady state yield a locally determinate solution.[11] If it holds, (4.11) and (4.12) imply that (r,s) must belong to an m-dimensional linear

space, which contains (p,0). This solution corresponds to a uniform price renormalization, with no economic significance. If m=1, this is the only restriction, and the period two restrictions become

(i) s=0 (k restrictions),

(ii) r is proportional to p (n-k-1 restrictions, given (4.11)), and

(iii) one restriction on the d_j following from (4.12),

$$\sum_{j=1}^{2n} d_j \left[\frac{1}{1-\mu_j}(p'A + c' - a') - w'\right]v_j = 0$$

In general, there are (n-m) restrictions on (r,s), and m restrictions involving the d_j. It is now evident what the significance of non-zero values of (r,s) is. The previous paper shows that when J > 1, steady states are generically not locally unique -- rather, one has an m-dimensional continuum of steady states -- one dimension of uniform price renormalizations and m-1 dimensions of relative price variations corresponding to different

choices of weights σ_j in (I.5.6). We see now that the n^s-dimensional linear space $\tilde{W}$ includes an m-dimensional subspace for which $(r,s) \neq 0$. These paths do not converge asymptotically to the steady state under consideration, strictly speaking, but rather to steady states very close to it, other members of the m-dimensional continuum of steady states (m-1 dimensional, if one ignores renormalizations).

The initial conditions manifold V is defined by the condition

$$(4.13) \qquad z_o(q,p_1) + y(p_1,p_2) + \sum_{h=1}^{J} c_1^h \ (p_1,p_2,\ldots) - a - B_1(p_1,p_2)b = 0$$

where as before q represents the vector of land prices. The members of finite-lived generation zero are assumed to hold claims to a vector of land

$$a - \sum_{h=1}^{J} a^h$$

as well as to a period one endowment of $\bar{z} + B_2 b$.

Therefore $z_o(q,p_1)$ in (4.13) satisfies an identity

$$(4.14) \qquad p_1' z_o(q,p_1) \equiv q'[a - \sum_{h=1}^{J} a^h] - \sum_{h=1}^{J} p_1' w^h .$$

In order for the steady state to be consistent with the initial conditions, it must also satisfy

$$(4.15) \qquad z_o(\frac{p}{1-\beta}, p) = z(p,\beta p) - B_2 b$$

Linearizing (4.13) around the steady yields

$$(4.16) \quad D_1 z_o(q - \frac{p}{1-\beta}) + [D_2 z_o + D_1 y - K_{11} + A + D](p_1 - p) +$$
$$[D_2 y - K_{21}](p_2 - \beta p) - B_1(b_1 - b) - E \sum_{s=1}^{\infty} (p_s - \beta^{s-1} p) = 0$$

where the coefficients $D_i z_o$ satisfy identities

$$(4.17) \qquad D_1 z_o \frac{p}{1-\beta} + D_2 z_o p = 0$$

$$(4.18) \qquad p' D_1 z_o = a' - \sum_{h=1}^{J} a^{h'}$$

$$(4.19) \qquad p'D_2z_o = - z_o'\left(\frac{p}{1-\beta},\ p\right) - \sum_{h=1}^{J} w^{h'}$$

As before, q is given by (3.20), so (4.16) may be rewritten

$$(4.20) \quad [-B_1 \quad 0 \quad D_2z_o+D_1y - K_{11}+A+D \quad D_2y-K_{12} \quad 0]\phi_1 =$$

$$[0 \quad 0 \quad E-D_1z_o \quad 0 \quad 0][I_{3n+2k}-F]^{-1}\phi_1$$

where (R.6) is used to guarantee the invertibility of $[I_{3n+2k} - F]$. This set of linear restrictions characterizes the tangent space $\tilde{V}$ of the initial conditions manifold.

Again we find that $\tilde{V}$ contains the subspace of uniform price renormalizations, i.e., that

$$\begin{bmatrix} 0 \\ 0 \\ p \\ \beta p \\ \beta^2 p \end{bmatrix}$$

is one solution to (4.20). This follows from (3.5), (3.9), (4.3) and (4.17). We also find that (4.20)

comprises only n-1 linearly independent conditions. For the definition of $\hat{G}$ implies that

$$[\beta p'B_2 \quad -\beta p'(D_1 z - K_{21}) \quad p'(D_2 y - K_{12}) + b'B_2'](I_{2n+k} - \hat{G})$$
$$= [0 \quad 0 \quad \frac{p'E}{1-\beta} - (a' - \sum_{h=1}^{J} a^{h'})]$$

so that

$$[\beta p'B_2 \quad -\beta p'(D_1 z - K_{21}) \quad p'(D_2 y - K_{12}) + b'B_2']$$
$$= [0 \quad 0 \quad \frac{p'E}{1-\beta} - (a' - \sum_{h=1}^{J} a^{h'})](I_{2n+k} - \hat{G})^{-1}$$

Multiplying this on the right by

$$\sum_{j=1}^{2n} d_j \begin{bmatrix} u_j \\ v_j \\ \mu_j v_j \end{bmatrix}$$

yields

$$(4.21)\quad \sum_{j=1}^{2n} \left(d_j[\mu_j p'(D_2 y - K_{12}) + \mu_j b' B_2' - \beta p'(D_1 z - K_{21})]v_j \right.$$

$$\left. + d_j \beta p' B_2 u_j \right) = \sum_{j=1}^{2n} d_j\left[\frac{\mu_j}{1-\mu_j}\left(\frac{p'E}{1-\beta} - \left(a' - \sum_{h=1}^{J} a^h{}'\right)\right)\right] v_j$$

Using the parameterization (4.10), (4.20) multiplied on the left by p' becomes

$$\sum_{j=1}^{2n} d_j[\mu_j p'(D_2 y - K_{12}) + p'(D_2 z_o + D_1 y$$

$$- K_{11} + A + D)]v_j - \sum_{j=1}^{2n} d_j p' B_1 u_j$$

$$+ [\beta p'(D_2 y - K_{12}) + p'(D_2 z_o + D_1 y - K_{11} + A + D)]r$$

$$- p' B_1 s = \sum_{j=1}^{2n} d_j\left[\frac{1}{1-\mu_j} p'(E - D_1 z_o)\right] v_j$$

$$+ \left[\frac{1}{1-\beta} p'(E - D_1 z_o)\right] r.$$

Using (3.7), (3.11), (3.15), (4.4), (4.13), (4.18), and (4.19), this becomes

$$\sum_{j=1}^{2n} d_j[\mu_j p'(D_2 y - K_{12}) - \beta p'(D_1 z - K_{21})$$

$$+ \frac{p'E}{1-\beta} - \frac{\beta}{1-\beta} p'D + \frac{\mu_j}{1-\mu_j}(a' - \sum_{h=1}^{J} a^{h'})$$

$$- \frac{1}{1-\mu_j} p'E - b'B_1']v_j + \sum_{j=1}^{2n} d_j \beta p'B_2 u_j =$$

$$- [\beta p'(D_2 y - K_{12}) - \beta p'(D_1 z - K_{21}) - \frac{\beta}{1-\beta} p'D$$

$$+ \frac{\beta}{1-\beta}(a' - \sum_{h=1}^{J} a^{h'}) - b'B_1']r - \frac{\beta}{1-\beta}[p'B_1 + p'B_2]s$$

which, substituting (4.21), and using the fact that $B_1'v_j + \mu_j B_2'v_j = 0$ for all j, becomes

$$(4.22) \qquad \sum_{j=1}^{2n} d_j[\frac{1}{1-\mu_j}(\frac{\beta}{1-\beta})p'E - \frac{\beta}{1-\beta}p'D]v_j =$$

$$- [\beta p'(D_2 y - K_{12}) - \beta p'(D_1 z - K_{21}) - \frac{\beta}{1-\beta}p'D +$$

$$\frac{\beta}{1-\beta}(a' - \sum_{h=1}^{J} a^{h'}) - b'B_1']r - \frac{\beta}{1-\beta}[p'B_1 + p'B_2]s$$

But (4.22) must be true for any solution to (4.5), for equation (4.12), multiplied on the left by $\frac{\beta}{1-\beta}p'$, yields (4.22). Therefore (4.20) comprises only n-1 linearly independent restrictions, and so $\tilde{V}$ is a linear space of dimension n+1, of which one dimension corresponds to uniform price renormalizations. Hence

$$\dim(\tilde{V} \cap \tilde{W}) = \max(n^s - n + 1, 1)$$

of which one dimension corresponds to uniform price renormalizations. If we ignore this dimension, the dimension of indeterminacy is $n^s - n$ which may be as large as n. If $n^s=n$, the steady state is exactly determinate, and if $n^s<n$ --it may be as low as zero--the equilibrium is not structurally stable.

V. Conditions for Determinacy of Equilibrium

We have shown that the local determinacy of a steady state equilibrium depends on n^s, the number of non-zero eigenvalues of modulus less than β of the matrix $\hat{G}$. (In the case that no infinite-lived agents consume in steady state, it is the matrix G, but G is simply $\hat{G}$ with A=0.) The eigenvalues of $\hat{G}$ are the solutions to the characteristic equation

$$\text{Det}(\hat{G}-\mu I_{2n+k}) = 0$$

Similarity transformations show that the 2n non-zero solutions to this equation are the 2n solutions to

$$(5.1) \qquad \text{Det} \begin{bmatrix} e\mu^2+f\mu+g & \mu u+\beta v \\ \mu v'+u' & 0 \end{bmatrix} = 0,$$

recalling the shorthand

$$
\begin{aligned}
e &= D_2 y - K_{12} \\
f &= \beta D_2 z + D_1 y - \beta K_{22} - K_{11} + A \\
g &= \beta D_1 z - \beta K_{21} \\
u &= B_1 \\
v &= B_2
\end{aligned}
$$

As noted previously, when there exist infinite-lived agents who consume in steady state, n^s may be anything between zero and 2n. When land exists, but no infinite-lived agents who consume in steady state, there must be one β eigenvalue, but the number less than β can be anything between zero and 2n-1. When neither land nor infinite-lived agents exist, there must be one β eigenvalue and an eigenvalue equal to one, but the remaining 2n-2 can take any values. The second case corresponds to A=0 and $a \neq 0$, while the latter case corresponds to A=0 and a=0. Now if the population of infinite-lived agents who consume at steady state is small, or they consume very little, the elements of A are small. For A is the sum over such agents of $\lambda^h [D^2 u^h]^{-1}$, and because of (B.4), this sum is small if either the number of agents summed over is small enough or steady state consumption is small enough.

It is evident from (5.1) that the eigenvalues vary continuously as A is varied, keeping the other terms constant, and therefore whenever A is sufficiently small, there is an eigenvalue near β. It is not possible, however, to say in general whether this eigenvalue is slightly greater than or slightly less than β in modulus,[12] so merely knowing that the infinite-lived consumers are small does not suffice to place any restrictions on n^s. This consideration does serve, though, to indicate that the mere existence of infinite-lived agents who consume in steady state need have no great effect on the local determinacy of equilibrium. If a steady state in which no infinite-lived agents consume is perturbed by adding a small population of infinite-lived agents, with a rate of time preference close to the rate of return prevailing in the previous steady state, that consume a small amount in the new steady state, then n^s either remains the same or is increased by one (because the former β eigenvalue may become slightly less than β), and so the dimension of indeterminacy either remains the same or is decreased by one (because it is n^s-n+1 in the former case and n^s-n in the latter).

Similarly, if the population of infinite-lived agents who consume in steady state is small and the total amount of land (or endowments of infinite-lived agents) is small, then both A and a are small in all elements, and by a similar continuity argument, there must be an eigenvalue near β, and another near one. As above, it is not possible in general to say whether the eigenvalue near β is greater or less than β in modulus, but if β is not near one, there must be at least one eigenvalue greater than β in modulus (the eigenvalue near one) and so n^s can be at most 2n-1 in such a case, and the maximum possible dimension of indeterminacy is n-1. As with infinite-lived consumers, it is evident that the mere presence of infinite-lived assets (or the infinite-horizon endowment stream of an agent who does not consume at steady state) need have no great effect on the local determinacy of equilibrium. If a steady state without any such assets is perturbed by adding a small amount of land, the new steady state has the same value of n^s, and hence the same dimension of indeterminacy (n^s-n+1) as the previous steady state.

The special form of the matrix in (5.1) allows

further generalizations to be made. If k=n -- the number of production activities operating in steady state is as large as the number of goods in each period -- then the solutions to (5.1) are the n solutions to

$$\text{Det}(\mu B_1 + \beta B_2) = 0$$

and the n solutions to

$$\text{Det}(\mu B_2' + B_1') = 0$$

or the n eigenvalues of $-\beta B_1^{-1} B_2$ and the n eigenvalues of $-B_2^{-1} B_1$. These two sets of eigenvalues are related to each other in such a way that if μ belongs to the first set, then $\frac{\beta}{\mu}$ belongs to the second. This implies a pseudo-splitting of the eigenvalues of $\hat{G}$. That is, there are exactly n less than $\sqrt{\beta}$ in modulus and n greater than $\sqrt{\beta}$ in modulus (assuming none are exactly $\sqrt{\beta}$ in modulus). If either land or infinite-lived consumers exist, $\beta<1$, so $\beta<\sqrt{\beta}$ and therefore $n^s<n$. If no infinite-lived agents consume in steady state, one eigenvalue equals β, so $n^s<n-1$. Therefore indeterminacy is impossible in this

case; the equilibrium is either perfectly determinate, or not structurally stable. However, no approximation of the conditions k=n is possible; no restrictions on n^s follow from k nearly as large as n.

There are, however, conditions that guarantee pseudo-splitting without requiring that k=n. If f is a symmetric matrix and $g=\beta e'$, the matrix in (5.1) has the property that

$$\begin{aligned}
&\text{Det}\begin{bmatrix} \mu^2 e+\mu f+g & \mu u+\beta v \\ \mu v'+u' & 0 \end{bmatrix} \\
&= \left(\frac{\mu}{\beta}^2\right)^n \text{Det}\begin{bmatrix} g'+\left(\frac{\beta}{\mu}\right) f'+\left(\frac{\beta}{\mu}\right)^2 e' & u+\left(\frac{\beta}{\mu}\right) v \\ \beta v'+\left(\frac{\beta}{\mu}\right) u' & 0 \end{bmatrix} \\
&= \left(\frac{\mu}{\beta}^2\right)^n \text{Det}\begin{bmatrix} \left(\frac{\beta}{\mu}\right)^2 e+\left(\frac{\beta}{\mu}\right) f+g & \left(\frac{\beta}{\mu}\right) u+\beta v \\ \left(\frac{\beta}{\mu}\right) v'+u' & 0 \end{bmatrix}
\end{aligned}$$

so that whenever μ is an eigenvalue, $\frac{\beta}{\mu}$ is also one. The conditions for this to occur are

$$(5.2)\quad \beta D_2 z+D_1 y-\beta K_{22}-K_{11}+A=\beta D_2 z'+D_1 y'-\beta K_{22}'-K_{11}'+A'$$

(5.3) $D_1 z - K_{21} = D_2 y' - K_{12}'$

From their definitions, A and

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

must be symmetric matrices; hence for (5.2) and (5.3) to be satisfied, one needs only for

$$\begin{bmatrix} D_1 y & D_2 y \\ D_1 z & D_2 z \end{bmatrix}$$

to be a symmetric matrix as well. These demand derivatives may be decomposed into income effects and substitution effects in the usual manner, and, as always, the matrix of substitution effects is symmetric. Therefore (5.2) and (5.3) are satisfied if there are no income effects. This occurs if all finite-lived consumers in steady state consume exactly their endowment. The conditions hold approximately if the consumption pattern of finite-lived consumers in steady

state is approximatly the same as their endowments, or if there is a great degree of elasticity of substitution between goods in the preferences of the finite-lived consumers.

But even if the matrix of demand derivatives is not roughly symmetric, conditions (5.2) and (5.3) hold approximately if the A and K_{ij} terms, which necessarily have the desired symmetry properties, are much larger than the demand derivatives in e,f, and g. This happens if there is a great degree of substitutability in production -- K_{ij} terms are large -- or the infinite-lived consumers are large in the sense discussed above -- A is large. In the latter case, (5.2) holds approximately, but (5.3) need not be close to being true. However, because in this case the f terms are large compared to all terms in e and g, pseudo-splitting still occurs. For if the elements of A are large compared to the elements of $D_i y$, $D_i z$, and K_{ij}, the degree-2n polynomial equation (5.1) becomes approximately

$$\left(\text{small coefficients on } \mu^{2n} \text{ through } \mu^{n+k+1}\right)$$

$$+ \mu^{n-k} \operatorname{Det}\begin{bmatrix} A & \mu u+\beta v \\ \mu v'+u' & 0 \end{bmatrix}$$

$$+ \left(\text{small coefficients on } \mu^{n-k-1} \text{ through } 1\right) = 0$$

Its solutions are approximately the following:

(i) n-k solutions very large, of order $|A|^{1/n}$

(ii) n-k solutions very small, of order $|A|^{-1/n}$, and

(iii) 2k solutions approximately equal to the 2k solutions to

$$\text{Det} \begin{bmatrix} A & \mu u + \beta v \\ \mu v' + u' & 0 \end{bmatrix} = 0$$

The third set of solutions has the pseudo-splitting property, by the same reasoning as above, since A is symmetric; that is, there are k greater than $\sqrt{\beta}$ and k less than $\sqrt{\beta}$. The first set of solutions must be greater than $\sqrt{\beta}$, for $|A|$ large enough, and the second set must be less

than $\sqrt{\beta}$, for $|A|$ large enough. Hence, large $|A|$ suffices to guarantee pseudo-splitting.

These results are summarized in the following proposition.

Proposition 5.1: Indeterminacy is impossible (either the equilibrium is perfectly determinate, or is not structurally stable) in any of the following cases:

(i) if the infinite-lived consumers are sufficiently more numerous or consume sufficiently more than do the finite-lived consumers;

(ii) if the production technology allows a sufficient amount of substitution;

(iii) if the number of independent production activities in use is equal to the number of goods in each period; or

(iv) if in the excess demands of the finite-lived

consumers, substitution effects are sufficiently large compared to income effects.

It follows that for indeterminacy to be a problem, the economy must not be close to any of these extreme cases -- the infinite-lived consumers must be a sufficiently small fraction of the economy, the production technology must be sufficiently inflexible, the tastes of the finite-lived consumers must be sufficiently inflexible, and the number of independent production activities operated in equilibrium must be less than the number of goods in each period.

This result seems to exhaust the conclusions about the conditions for determinacy that may be drawn without restricting one's attention to special functional forms. The illustrative example given in the Appendix shows that indeterminacy is possible in the presence of production and infinite-lived consumers, and may allow the reader some sense of the size of parameters required for the Proposition not to apply.

APPENDIX

To illustrate the claim made in the text that n^s can range between zero and 2n, we here present examples for the case n = 2 and k = 1. In each of these examples, there is a single infinite-lived consumer with utility function

$$V = \sum_{t=1}^{\infty} (0.9)^{t-1}(\log c_{1t} + \alpha \log c_{2t})$$

who consumes in the steady state. This implies $\beta = .9$ in each example. There is a single production activity in use in the steady state, with coefficients B_1 and B_2. In each example, quantities and activities have been normalized so that in the steady state

$$p_1 = p_2 = 1, \; b = 1$$

The finite-lived consumers are specified only by giving the values of y and z and their derivatives at $(p, \beta p)$. Because the values given satisfy (3.5) - (3.8), it follows from a theorem of Debreu (1974) that there exists a set of no more than four utility-maximizing consumers whose aggregate excess demands and their derivatives would have the specified values at $(p, \beta p)$.

Example 1: Suppose that in the steady state y = (-890, -210), z = (877.78, 344.44), c = (5,300), the infinite-lived consumer is specified by a = (72.78, 340.00), α = 60, the production activity is B_1 = (-100, -30), B_2 = (20, 124.44), and the Jacobian matrix

$$\begin{bmatrix} D_1 y & D_2 y \\ D_1 z & D_2 z \end{bmatrix}$$

has the value

$$\begin{bmatrix} -180 & 90 & 200 & -100 \\ 270 & -180 & -300 & 200 \\ 222.22 & 222.22 & -22.22 & -471.60 \\ 666.67 & 111.11 & -841.98 & -22.22 \end{bmatrix}$$

At such a steady state the eigenvalues of G are:

$$\lambda_{1,2} = .3637 \pm .6973i \quad \text{(modulus .7865)}$$
$$\lambda_3 = .6574$$
$$\lambda_4 = -.8523$$

Therefore $n^s = 4$, and the dimension of indeterminacy is two.

Example 2: Suppose that in the steady state $y = (-630, 60)$, $z = (567.78, 65.55)$, $c = (5,25)$, the infinite-lived consumer is specified by $a = (22.78, 56.11)$, $\alpha = 5$, the production activity is $B_1 = (-100, -30)$, $B_2 = (20, 124.44)$, and the Jacobian matrix has the value

$$\begin{bmatrix} -180 & 90 & 200 & 100 \\ -90 & -270 & 100 & 300 \\ 333.33 & 222.22 & -244.44 & -372.84 \\ 666.67 & 111.11 & -719.76 & -144.44 \end{bmatrix}$$

At such a steady state, the eigenvalues of G are:

$$\lambda_{1,2} = .0424 \pm .8804i \text{ (modulus .8814)}$$
$$\lambda_3 = .8488$$
$$\lambda_4 = 49.42$$

Therefore $n^s = 3$, and the dimension of indeterminacy is one.

Example 3: Suppose that in the steady state $y = (-.2, -.2)$, $z = (.22, .22)$, $c = (5, 5)$, the infinite-lived consumer is specified by $a = (6.02, 3.91)$, $\alpha = 1$, the production is $B_1 = (-2, 1)$, $B_2 = (1, .11)$, and the Jacobian matrix has the value

$$\begin{bmatrix} -.9 & -.9 & 1 & 1 \\ -.9 & -.9 & 1 & 1 \\ 1.11 & 1.11 & -.11 & -2.36 \\ 1.11 & 1.11 & -2.36 & -.11 \end{bmatrix}$$

At such a steady state the eigenvalues of G are:

$$\lambda_1 = .1777$$
$$\lambda_2 = .3326$$
$$\lambda_3 = 2.7759$$
$$\lambda_4 = 5.4870$$

Therefore $n^s = 2$, and the equilibrium is perfectly determinate.

Example 4: Suppose that in the steady state $y = (160, 150)$ $z = (-156.67, -187.78)$, $c = (5,200)$, the infinite-lived consumer is specified by $a = (128.33, 23.33)$, $\alpha = 40$, the production activity is $B_1 = (-140, -30)$, $B_2 = (20, 168.89)$, and the Jacobian matrix has the value

$$\begin{bmatrix} -180 & -360 & 200 & 400 \\ -180 & -90 & 200 & 100 \\ 111.11 & 111.11 & -22.22 & -224.69 \\ 111.11 & 222.22 & -248.14 & -122.22 \end{bmatrix}$$

At such a steady state, the eigenvalues of G are:

$$\lambda_{1,2} = .1878 \pm .9324i \quad \text{(modulus .9511)}$$
$$\lambda_3 = 1.4064$$
$$\lambda_4 = .2613$$

Therefore $n^s = 1$ and the equilibrium is not structurally stable.

Example 5: Suppose that in the steady state $y = (60, 730)$, $z = (-156.67, -721.11)$, $c = (5,300)$, the infinite-lived consumer is specified by $a = (28.33, 170)$, $\alpha = 60$, the production activity is $B_1 = (-140, -30)$, $B_2 = (20, 168.89)$, and the Jacobian matrix has the value

$$\begin{bmatrix} -180 & -540 & 200 & 600 \\ -180 & -90 & 200 & 100 \\ 222.22 & -333.33 & -22.22 & 145.68 \\ 111.11 & 222.22 & -248.14 & -122.22 \end{bmatrix}$$

At such a steady state the eigenvalues of G are:

$$\lambda_{1,2} = .6158 \pm 1.1134i \quad \text{(modulus 1.2723)}$$
$$\lambda_3 = -.9814$$
$$\lambda_4 = 1.2846$$

Therefore $n^s = 0$, and again the equilibrium is not structurally stable.

FOOTNOTES

[1] This possibility was first shown in examples constructed by Calvo (1978) and Hahn (1982, pp. 107-110).

[2] It is shown in the previous paper that monetary equilibria are possible only if neither land nor infinite-lived consumers are present. Therefore we concern ourselves only with non-monetary equilibria in this paper.

[3] Hereafter, a reference to (A.2) means assumption (A.2) of the previous paper, and a reference to (I.5.3) means equation (5.3) of the previous paper.

[4] For example, consider a one-good economy with one infinite-lived consumer and one finite-lived consumer each generation, described by

$$y(p_t, p_{t+1}) = \frac{p_{t+1}}{p_t} - 1$$

$$z(p_t, p_{t+1}) = \frac{p_t}{p_{t+1}} - 1$$

$$u^1(c) = \log c$$

$$\beta^1 = \frac{3}{5}, \qquad a^1 = \frac{1}{2}$$

There is no production. The unique steady state involves price decrease at a rate $\beta = \frac{3}{5}$, steady-state consumption by the infinite-lived consumer each period of $c^1 = \frac{7}{30}$,

and a debt each period on his part of $\frac{2}{3}$. That is, each period the infinite-lived consumer must borrow $\frac{2}{5}$ units of the consumption good form the current young, in addition to his endowment of $\frac{1}{2}$ unit, in order to repay his debt of $\frac{2}{3}$ units to the current old and consume $\frac{7}{30}$ units. He will then have to repay the next generation's old $\frac{2}{3}$ units for the $\frac{2}{5}$ they lent him the period before, because prices will have fallen to $\frac{3}{5}$ their level in the previous period; the pattern is repeated forever. In this example, the amount of the infinite lived consumer's debt each period exceeds his endowment, although it still satisfies (2.9).

[5] For example, consider the economy of footnote 4, with an initial claim of $-\frac{3}{2} < w^1 < -1$. Then (2.10) implies $z_0(p_1) = -w^1$, and (2.11) becomes

$$-w^1 + \frac{p_2}{p_1} - 1 + c_1 - \frac{1}{2} = 0$$

Then $c_1 \geq 0$ implies $\frac{p_2}{p_1} < \frac{1}{2}$. But for any $t \geq 2$, the period t equilibrium condition

$$\frac{p_{t-1}}{p_t} + \frac{p_{t+1}}{p_t} + c_t = \frac{5}{2}$$

implies that if $\frac{p_t}{p_{t-1}} < \frac{1}{2}$, $\frac{p_{t+1}}{p_t} < \frac{p_t}{p_{t+1}}$. The rate of

price decrease must therefore become greater and greater. On the other hand, there can be no solution for non-negative p_{t+1} if $\frac{p_t}{p_{t+1}} < \frac{2}{5}$. Therefore, this must not occur for any t. But $\frac{p_t}{p_{t-1}} \geq \frac{2}{5}$ implies $\frac{p_{t-1}}{p_{t-2}} \geq \frac{10}{21}$, which implies $\frac{p_{t-2}}{p_{t-3}} \geq \frac{42}{85}$, and so on in a sequence converging to $\frac{1}{2}$. Hence however much smaller than $\frac{1}{2}$ the value of $\frac{p_2}{p_1}$ is, a contradiction is reached in a finite number of periods. Therefore no equilibrium is possible, even though the initial claim satisfies conditions (2.8) and (2.9).

[6] If a single-period production activity is in use in steady state, a column of B_2 consists entirely of zeroes, so that (R.2) cannot hold. However, in such a case it is possible to eliminate one good and one activity from the dynamical equations for each single-period activity in use, so that there are effectively $n-k_1$ goods and $k-k_1$ activities in use, where k_1 is the number of single-period production activities in use. Then the analysis presented in the text carries through,

with the conclusion that there may be as many as $n-k_1$ dimensions of indeterminacy.

[7] It is shown in the previous paper that monetary equilibria are possible in the case that $a=0$, but not if $a \neq 0$.

[8] Note that the k vectors $(0\ 0\ B_1'\ B_2'\ 0)$ are not themselves eigenvectors of F, but are zero eigenvectors of F^2. In the case of repeated eigenvalues of a non-symmetric matrix, there need not be as many eigenvectors as there are repetitions of a given eigenvalue -- i.e., the eigenvectors need not span the domain space of the linear operator. If F^2 has at least 2k zero eigenvalues (because it has at least 2k zero eigenvectors), then F must have that many as well, since for each eigenvalue λ of F there must be a corresponding eigenvalue λ^2 of F^2.

[9] The linearized analysis therefore does not suffice to tell us whether the corresponding solutions to the exact equations converge, diverge, or neither. Fortunately, as discussed below, an economic interpretation of the β eigenvalues exists that allows us to resolve this uncertainty without having to consider the non-linear dynamics.

[10] The derivation here assumes $\mu_j < 1$ for each eigenvalue of $\hat{G}$. It is of no importance whether this is true, as we shall restrict out attention to the subspace of (4.10) spanned by those eigenvectors whose eigenvalues are less

than β in modulus. Conditions (4.12) are well-defined even where not correct, because of (R.6).

11 The linearized versions of (2.1), (2.2), and (2.3) are

$$[D_1 z+\beta D_2 y+D_1 y+\beta D_2 y-K_{21}-\beta K_{22}-K_{11}-\beta K_{12}](p-\bar{p})$$

$$-[B_1+B_2](b-\bar{b}) + \sum_{h=1}^{J}(c^h-\bar{c}^h) = 0$$

$$[B_1' + \beta B_2'](p-\bar{p}) = 0$$

$$[D^2 u^h](c^h-\bar{c}^h) = \bar{\lambda}^h (p-\bar{p}) + \bar{p}(\lambda^h-\bar{\lambda}^h)$$

where we neglect the possibility of variation in β because of (2.4). Using the third equation to substitute for $(c^h-\bar{c}^h)$, the other two become

$$\begin{bmatrix} \beta D_2 y+\beta D_2 z+D_1 y+D_1 z-\beta K_{12}-\beta K_{22}-K_{11}-K_{21}+A & -B_1-B_2 \\ B_1' + \beta B_2' & 0 \end{bmatrix} \begin{bmatrix} p-\bar{p} \\ b-\bar{b} \end{bmatrix}$$

$$= \begin{bmatrix} -\sum_{h=1}^{J}[D^2 u^h]^{-1}\bar{p}(\lambda^h-\bar{\lambda}^h) \\ 0 \end{bmatrix}$$

If (R.8) holds, there is an m-dimensional linear space of solutions, which is swept out by varying the $(\lambda^h-\bar{\lambda}^h)$, for $h=1,2,\ldots,J$, which may be freely chosen. One solution is always

$$p-\bar{p} = \bar{p},\ b-\bar{b} = \bar{b},\ \lambda^h-\bar{\lambda}^h = -\bar{\lambda}^h$$

which corresponds to a uniform price renormalization of no economic significance. The remaining m-1-dimensional space of solutions corresponds to variations in the weights σ_j of (I.5.6). If (R.8) does not hold, there will be additional solutions, one dimension for each dimension by which the rank of the matrix in (R.8) falls below n+k.

[12] For example, in the case of a single good and no production, if we let the excess demand functions of the finite-lived population be written $z(\frac{p_{t+1}}{p_t})$ and $y(\frac{p_{t+1}}{p_t}) = -\frac{p_{t+1}}{p_t} z (\frac{p_{t+1}}{p_t})$, then the eigenvalue near β is approximately (for small $|A|$)

$$\beta - \frac{|A|}{c'(\beta)}$$

where $c(\beta) \equiv a-(1-\beta)z(\beta)$, as in the previous paper. Thus in Figure 1 of the previous paper, points A, C, and E represent steady states such that if a small population of infinite-lived agents were added with a rate of time preference close to the rate of return prevailing without them, and such that they would consume a small amount in the resulting new steady-state (near the old one), the new steady state would have an eigenvalue slightly <u>less</u> than β. On the other hand, points B and D represent steady states such that under a similar addition of a small population of infinite-lived agents, the new steady state would have an eigenvalue slightly <u>greater</u> than β. So both cases can occur.

[13] It is not, generically, possible for k to be greater than n, for in that case equations (3.13) could not, in general, all be satisfied.

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TRANSACTION COSTS, LIQUIDITY, AND OPTIMAL INFLATION

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This essay presents an explicit model of a "precautionary" demand for money balances,[1] and considers optimal monetary policy in the context of such a model. The most familiar models of liquidity preference illustrate Keynes' "transactions-motive" or "speculative-motive." For example, in the type of model originated by Baumol (1952) and Tobin (1956), it is assumed that the liquid asset (money) must be used at a certain regular rate for purchases, although other assets exist yielding a higher return. The higher-yield asset is less liquid, in the sense that a fixed brokerage cost is incurred whenever any of the asset is sold to obtain money with which to purchase consumption goods. Because sale of the asset whenever one wished to consume would be too costly, asset sales occur less frequently and money balances are held to allow consumption in the interim. Uncertainty plays no role in a "transactions" demand model of this sort at all. On the contrary, in the portfolio choice model originated by Tobin (1958), less liquid assets are those with greater uncetainty in their rate of return. All assets are traded in frictionless markets, but because of risk aversion on the part of savers, less risky assets will be held in

equilibrium even if they yield a somewhat lower expected return.

In the sort of model described here, by contrast, the demand for liquidity depends crucially upon the conjunction of uncertainty and market imperfection. Following Hicks (1974, pp. 41-42), liquidity is considered to be a matter of costly access to a market:[2]

> A firm which acquires a non-marketable asset - say a new factory, designed and equipped for its own particular purpose - has committed itself to a course of action, extending over a considerable time, with a fairly narrow band of subsequent choices attached to it. It has 'given hostages to fortune.' The acquisition of an easily marketable asset, on the other hand, can easily be revoked. There is not the same diminution of flexibility; the firm is in a position that is almost as flexible, after the acquisition, as before it. That, I suggest, is precisely what we mean by saying that the marketable asset possesses <u>liquidity</u>.

Liquidity in this sense matters if agents lack the information needed to be certain that the course of action to which they would commit themselves in the case of an illiquid investment project is desirable. Following Goldman (1974,1978), the model assumes agents to be uncertain about their future consumption preferences - specifically, about the time at which they wish to

consume[3] - although one might equivalently suppose that access to the consumption goods market is stochastic,[4] or that random events specific to individuals affect the benefit they can obtain from expenditures in a given period - e.g., that an accident can befall someone which will result in a permanent disability unless medical expenses occur immediately following the accident. At the time that an agent considers investment in productive assets, he does not yet know whether he will need to consume before the production process is completed. The available production opportunities have a structure such that intermediate goods produced in the process (here called "capital") are less liquid than is the final product, in the sense that transaction costs are incurred by an investor who wishes to sell his interest in a project before fruition, but not by one who waits and sells the final product.[5] The agent's alternative is to hold cash balances, which may be costlessly exchanged for consumption goods at any time.

The point of the exercise is not simply to illustrate another possible source of demand for liquidity. Rather, it is to demonstrate the possibility of welfare gains from a moderately inflationary monetary policy. In a celebrated

article, Friedman (1969) has argued that the optimal monetary policy would be one resulting in steady deflation,[6] at a rate such that holding cash balances would yield as high a return as is available on other riskless assets. Given that cash balances provide liquidity services, he argues, in order for them to be held at an efficient level, the cost of holding cash balances should be equal to the cost of supplying them. Since the cost of printing money is negligible, the opportunity cost of holding cash balances (instead of other riskless assets) should also be negligible. Inflation would have the character of a distorting tax, resulting in a dead-weight loss measured (with the usual caveats) by a triangle under a money demand curve.[7] Formal models of a transaction demand for money have supported this conclusion.[8] It is therefore commonly argued that if the government had no need of revenue (or could raise it through non-distorting taxes), any inflation must reduce welfare, although it is admitted that if the government needs a certain amount of revenue and has only distorting taxes of its disposal, then the (second-best) optimal tax policy could involve some use of the "inflation tax".

The present essay, however, illustrates that a moderate rate of inflation can be desirable, even when monetary policy is not used to generate revenue, all the new money printed being distributed to consumers in lump-sum transfers. Hahn (1971) has commented on the Friedman argument:

> The necessary conditions for Pareto-efficiency in a world of uncertainty with intertemporal choice will in general be fulfilled in a market economy only if money plays no role. There are therefore no grounds for supposing that the Friedman rule is either necessary or sufficient for Pareto-efficiency since it is of the essence of an explanation for the existence of money that other conventional necessary conditions are violated.

The present essay illustrates this possibility. The very problem that creates the demand for cash balances in this model - transaction costs incurred if capital must be sold, combined with uncertainty about when one will wish to consume - also vitiates the usual analysis of perfectly anticipated inflation as simply a distorting tax. Agents are unable to shift resources between early consumption and late consumption at the rate that would be possible if the supply of consumption goods in the economy were the only constraint, because of the absence of costless financial

intermediation. Inflationary monetary policy, by effectively taxing cash balances and distributing the proceeds as lump-sum transfers, redistributes income from late consumers to early consumers, and thus provides insurance against liquidity difficulties (the need for early consumption) that consumers were unable to arrange for themselves. This redistribution benefits all consumers, considered ex ante, before they know whether they will be early consumers or not. At moderate levels of inflation, the welfare gain due to this redistribution (which will be proportional to the rate of inflation) will outweigh the losses due to the distorting tax aspect (which, as a "triangle", is only a second-order effect).

This result illustrates the importance to macroeconomic theory of considering models which not only allow for market frictions, but which allow for uncertainty which places different agents in different circumstances, so that government interventions which differently affect agents in different circumstances can serve an insurance function.[9] Such models allow for the possibility of government intervention which improves upon the laissez-faire allocation of resources, without assuming that the central

authority possesses any information about the circumstances of particular individuals.

1. The Model

The sort of liquidity problem discussed above can be formally analyzed using the following simple model. Suppose that all economic activity occurs in discrete periods, and that each agent lives for three consecutive periods (numbered 0, 1, 2). The lifespans of agents overlap, so that in each period there are an equal number of agents in the zeroth period, the first period, and the second period of life. There is one consumption good, which is not storable, but which may be invested (used in the production of capital). For each unit of the consumption good invested in one period, one unit of capital (some intermediate good) is produced the following period. Capital has no consumption value, but is used to produce 1+R units of the consumption good the following period (two periods after the initial outlay of consumption good). Here $R > 0$ is given by the technology and is non-stochastic. The capital is completely used up by the production process. In addition to the good produced using

capital, each agent receives an endowment of w units of the consumption good during the zeroth period of his life; there are no subsequent endowments.

Consumption uncertainty is of the following sort. In the first period of life, each agent learns whether he is an early consumer or a late consumer. An early consumer can consume only in the first period of his life, a late consumer only in the second. A fraction λ of the agents in each generation turn out to be early consumers. In period zero, an agent knows only that he has the probability λ of being an early consumer. I assume that each agent is an expected-utility maximizer[10] and that all make their zeroth period investment decisions so as to maximize the same function[11]

$$W(c_1, c_2) = \lambda\left[\frac{c_1^{1-\gamma}}{1-\gamma}\right] + (1-\lambda)\beta\left[\frac{c_2^{1-\gamma}}{1-\gamma}\right] \qquad (1)$$

where γ is the coefficient of relative risk aversion, $\beta < 1$ is interpreted as a discount factor, c_1 represents the agent's consumption in the first period should he turn out to be an early consumer, and c_2 represents his consumption in the second period should he turn out to be a late

consumer.

These assumptions, except for the overlapping generations, are identical to those made by Diamond and Dybvig (1981). But whereas they assume that an early consumer can get one unit of consumption by liquidating his capital, I assume that this amount is determined by supply and demand. Transaction costs are represented by a fraction τ ($0 < \tau < 1$) of the capital that disappears in transit between buyer and seller. While a buyer pays price q per unit of capital that he receives, a seller gets only $q(1-\tau)$ per unit of capital that he sells. (The consumption good is taken as numeraire.) I assume that τ is given by technology, but that q is determined in the market.

In addition to investing in the production of capital or buying already existing capital, agents can store wealth by holding money balances. Money is assumed to be intrinsically valueless, held only because other agents will wish to hold it in the following period. Money can be exchanged for the consumption good at any time in a competitive market without incurring any transaction costs. A monetary authority creates new money (or destroys it) so as to maintain a steady rate of inflation π (which if negative indicates deflation). The new money is given in

each period to the agents in period zero of their lives; each agent receives real lump sum transfers T in period zero and none subsequently. (If $\pi < 0$, T is negative as well, meaning that period zero agents must sell some of their endowment for money with which to pay their tax liability; the money thus collected is burned by the monetary authority.) I here consider only steady states, so that π and T are the same for all generations.

Let the amount of consumption good invested by agents in period zero be n_1, and in period one (by agents who have found they are late consumers) be n_2. Let the amount of capital purchased by agents in period zero be k_1, and the amount held by late consumers in period one (whether purchased then or the result of investment in period zero) be k_2. Let the real money balances held by agents in period zero be m_1, and those held by late consumers in period one be m_2. Then the budget constraint for agents in period zero is

$$m_1 + n_1 + qk_1 \leq w + T \qquad (2)$$

and that for late consumers in period one is

$$m_2 + n_2 + q(k_2-n_1) \leq \frac{m_1}{1+\pi} + k_1(1+R) \qquad (3a)$$

if $k_2 \geq n_1$, and

$$m_2 + n_2 + q(1-\tau)(k_2-n_1) \leq \frac{m_1}{1+\pi} + k_1(1+R) \qquad (3b)$$

if $k_2 \leq n_1$. Consumption by early and late consumers respectively will be

$$c_1 = \frac{m_1}{1+\pi} + n_1 q(1-\tau) + k_1(1+R) \qquad (4a)$$

$$c_2 = \frac{m_2}{1+\pi} + n_2 q(1-\tau) + k_2(1+R) \qquad (4b)$$

The agent's choice problem, then, is to choose m_i, n_i, k_i $(i=1,2)$ to maximize (1) subject to constraints (2), (3) and (4), along with the constraint that all six quantities be non-negative, taking w, T, π, q, τ, and R as given independent of his actions.

For an economy, completely described by (w,π,τ,R), a steady state <u>equilibrium</u>[12] is a pair (T,q) such that the

solutions to the agents' choice problems taking these values as parametric satisfy

$$Z \equiv k_1 + (1-\lambda)(k_2-n_1) - \lambda(1-\tau)n_1 - (1-\lambda)(1-\tau)n_2 = 0 \qquad (5a)$$

if $k_2 > n_1$,

$$Z \equiv k_1 + (1-\lambda)(1-\tau)(k_2-n_1) - \lambda(1-\tau)n_1 - (1-\lambda)(1-\tau)n_2 = 0 \qquad (5b)$$

if $k_2 \leq n_1$, and

$$M \equiv m_1 + (1-\lambda)m_2 = \left(\frac{1+\pi}{\pi}\right)T \qquad (6)$$

Equation five says that the excess demand for capital is zero (recall that in each generation only fraction λ of the population sells its n_1 units of capital in period one, and only fraction $1-\lambda$ of the population buys additional capital (k_2-n_1) then and sells its n_2 units of capital in period two). Equation six is the relation between the amount of

real transfers and the rate of inflation that must hold for the money market to clear, where M denotes total demand for real balances.[13]

2. Steady State Equilibrium

Possible steady state equilibria are of three types. A "Type A" equilibrium is one in which only money is held. Then (3) and (4) reduce to

$$c_1 = \frac{m_1}{1+\pi} = \frac{w+T}{1+\pi}$$

$$c_2 = \frac{m_2}{1+\pi} = \frac{w+T}{(1+\pi)^2}$$

Using (6) to solve for the equlibrium level of transfers

$$T = \frac{(2+\pi-\lambda)\pi}{1+\lambda\pi} w$$

yields an equilibrium consumption pattern

$$c_1 = \frac{1+\pi}{1+\lambda\pi} w \qquad (7a)$$

$$c_2 = \frac{1}{1+\lambda\pi} w \qquad (7b)$$

A "Type B" equilibrium is one in which both money and capital are held. This means that the rate of return on money balances held for one period, $\frac{1}{1+\pi}$, must exactly equal the rate of return on purchasing capital and selling the final product the following period, $\frac{1+R}{q}$, since money and the consumption good are freely convertible. Hence for any such equilibrium,

$$q = (1+R)(1+\pi) \qquad (8)$$

It is shown in Appendix 1 that in such an equilibrium, no new investment n_2 is ever undertaken in period one, and that late consumers never sell any capital in period one, so that $n_2 = 0$, $k_2 \geq n_1$, and (3a) holds with equality. Under these circumstances (2) and (4) reduce to

$$c_1 = \frac{(W+T-n_1)}{1+\pi} + n_1(1+R)(1+\pi)(1-\tau) \qquad (9a)$$

$$c_2 = \frac{(W+T-n_1)}{(1+\pi)^2} + n_1(1+R) \qquad (9b)$$

It is shown in Appendix 1 that Type B equilibrium is possible only if

$$(1+R) > \frac{1}{(1+\pi)^2} > (1+R)(1+T) \qquad (10)$$

so that (1) is maximized by a choice of n_1 such that

$$\left(\frac{c_2}{c_1}\right)^{\gamma} = \left(\frac{1-\lambda}{\lambda}\right)\beta \frac{\left[(1+R) - \frac{1}{(1+\pi)^2}\right]}{\left[\frac{1}{1+\pi} - (1+R)(1+\pi)(1-\tau)\right]} \qquad (11)$$

where c_1 and c_2 are given as functions of n_1 by (9). The left hand side of (11) is a monotonically increasing positive function of n_1, so that the n_1 value satisfying (9) and (11) is unique.[14] The optimal choice of n_1 must be at an interior point, for $n_1 = 0$ would mean no capital is ever produced (hence not a Type B equilibrium) while $n_1 = w+T$ would mean $k_1 = 0$, $k_2 = n_1$, so that early consumers would wish to sell their n_1 units of capital but no one would wish to buy.

Equilibrium in the consumption good market requires that[15]

$$\lambda c_1 + (1-\lambda)c_2 = w - n_1 + (1-\lambda\tau)(1+R)n_1 \qquad (12)$$

This relation may be used with (9) to solve for equilibrium transfers T as a function of n_1:

$$T = \frac{\pi}{1+\lambda\pi}(2+\pi-\lambda)w - \frac{\pi}{1+\lambda\pi}[(2+\pi-\lambda) + \lambda(1+R)(1+\pi)^2(1-\tau)]n_1$$

This allows T to be eliminated from (9), yielding:

$$c_1 = (\frac{1+\pi}{1+\lambda\pi})[w + [(1+R)(1-\tau)-1]n_1] \qquad (13a)$$

$$c_2 = (\frac{1}{1+\lambda\pi})[w + [(1+R)(1+\lambda\pi\tau) - 1]n_1] \qquad (13b)$$

So the value of n_1 that is chosen in a Type B equilibrium is given by equation (11) with (13) substituted in for c_1, c_2. Once n_1 has been determined, (5a) and (6) imply that total equilibrium money balances and capital stock are

$$M \equiv m_1 + (1-\lambda)m_2 = \frac{1+\pi}{1+\lambda\pi}(2+\pi-\lambda)w - \frac{1+\pi}{1+\lambda\pi}[(2+\pi-\lambda) + \lambda(1+R)(1+\pi)^2(1-\tau)]n_1 \quad (14)$$

$$K \equiv k_1 + (1-\lambda)k_2 = (1-\lambda\tau)n_1 \quad (15)$$

The division of the equilibrium money stock between m_1 and m_2 is indeterminate, subject to $m_1, m_2 \geq 0$, and likewise for the division of the equilibrium capital stock between k_1 and k_2. However, once the division between m_1 and m_2 is given, k_1 and k_2 are then given (or vice versa) by the budget constraints

$$m_1 + (1+R)(1+\pi)k_1 = \frac{(1+\pi)^2}{1+\lambda\pi}w - \frac{(1+\pi)^2}{1+\lambda\pi}[\lambda\pi(1+R)(1-\tau)+1]n_1$$

$$m_2 + (1+R)(1+\pi)(k_2-n_1) = \frac{1+\pi}{1+\lambda\pi}w - \frac{(1+\pi)}{1+\lambda\pi}[\lambda\pi(1+R)(1-\tau)+1]n_1$$

A "Type C" equilibrium is one in which only capital is held. It is shown in Appendix 1 that in such an equilibrium, no new investment n_2 is ever undertaken in period one, i.e., $n_2 = 0$. As all wealth held over from period one is held as capital, there will never be any

reason for late consumers to sell any of their capital resulting from period zero investment. So (2) and (3) imply

$$k_1 = \frac{w - n_1}{q} \tag{16a}$$

$$k_2 = n_1 + \frac{1+R}{q^2}(w - n_1) \tag{16b}$$

since T = 0 in an equilibrium without money. These in turn imply

$$c_1 = n_1 q(1-\tau) + \frac{1+R}{q}(w-n_1) \tag{17a}$$

$$c_2 = n_1(1+R) + (\frac{1+R}{q})^2(w-n_1) \tag{17b}$$

It is shown in Appendix 1 that in any Type C equilibrium

$$(1+R)^{1/2} < q < (\frac{1+R}{1-\tau})^{1/2} \tag{18}$$

Then (1) is maximized by a choice of n_1 such that

$$\left(\frac{c_2}{c_1}\right)^{\gamma} = \left(\frac{1-\lambda}{\lambda}\right)\beta \frac{\left[1+R-\left(\frac{1+R}{q}\right)^2\right]}{\left[\frac{1+R}{q} - q(1-\tau)\right]} \qquad (19)$$

where c_1 and c_2 are given as functions of n_1 by (17). The left hand side is an increasing function of n_1, so that the choice of n_1 is unique.[16] It is shown in Appendix 1 that the optimal choice of n_1 is always in the range $0 < n_1 < w$.

In fact, using (2) and (3a), the capital market equilibrium condition (5a) reduces to

$$Z = \frac{w-n_1}{q} + (1-\lambda)\frac{1+R}{q^2}(w-n_1) - \lambda(1-\tau)n_1 = 0$$

which implies

$$n_1 = \frac{\left[1 + (1-\lambda)\frac{1+R}{q}\right]}{\left[1 + (1-\lambda)\frac{1+R}{q} + \lambda(1-\tau)q\right]} w \qquad (20)$$

Substitution of (20) into (17) yields

$$c_1 = \frac{(1+R+q)(1-\tau)}{\left[1 + (1-\lambda)\frac{1+R}{q} + \lambda(1-\tau)q\right]} w \qquad (21a)$$

$$c_2 = \frac{(1+R)\left[1 + (1-\lambda\tau)\frac{1+R}{q}\right]}{\left[1 + (1-\lambda)\frac{1+R}{q} + \lambda(1-\tau)q\right]} w \qquad (21b)$$

So the equilibrium value of q is given by (19) and (21) substituted in for c_1, c_2. Once q has been determined, n_1 is given by (20) and k_1 and k_2 are given by (16).

This exhausts the logically possible types of equilibrium. How many of each type can exist for a given economy (w,π,T,R,β,γ)? The Type A equilibrium is clearly unique, when it exists. It is shown in Appendix 1 that the same is true of Types B and C, and furthermore that each type can exist if and only if the other two cannot. Thus one may establish:

Proposition 1. Exactly one equilibrium exists for each set of parameter values.

Type B equilibria with differing values for k_1, k_2, m_1, m_2 but the same values for n_1, q, T, K, M, c_1, and c_2 are counted as a single equilibrium.

It is also shown in Appendix 1 that (for fixed w,R,β,γ) one can define two monotonically increasing functions $\underline{\pi}(\tau)$

and $\bar{\pi}(\tau)$, with $\underline{\pi}(\tau) \leq \bar{\pi}(\tau)$ for all τ, such that Type A equilibrium exists if and only if

$$\pi \leq \underline{\pi}(\tau),$$

Type B equilibrium exists if and only if

$$\underline{\pi}(\tau) < \pi < \bar{\pi}(\tau),$$

and Type C equilibrium exists if and only if

$$\bar{\pi}(\tau) \leq \pi$$

Thus for any level of transaction costs, Type A equilibrium occurs if a sufficiently deflationary monetary policy is pursued, and Type C if a sufficiently inflationary policy is pursued, with Type B occurring for intermediate levels of inflation. The various regions are displayed in Figure 1. As $\pi \to \underline{\pi}(\tau)$ from above, the Type B equilibrium becomes a Type A equilbrium; $K \to 0$ and m_1 and m_2 approach their values in the Type A equilibrium for $\pi = \underline{\pi}(\tau)$. As $\pi \to \bar{\pi}(\tau)$ from below, the Type B equilibrium becomes a Type C

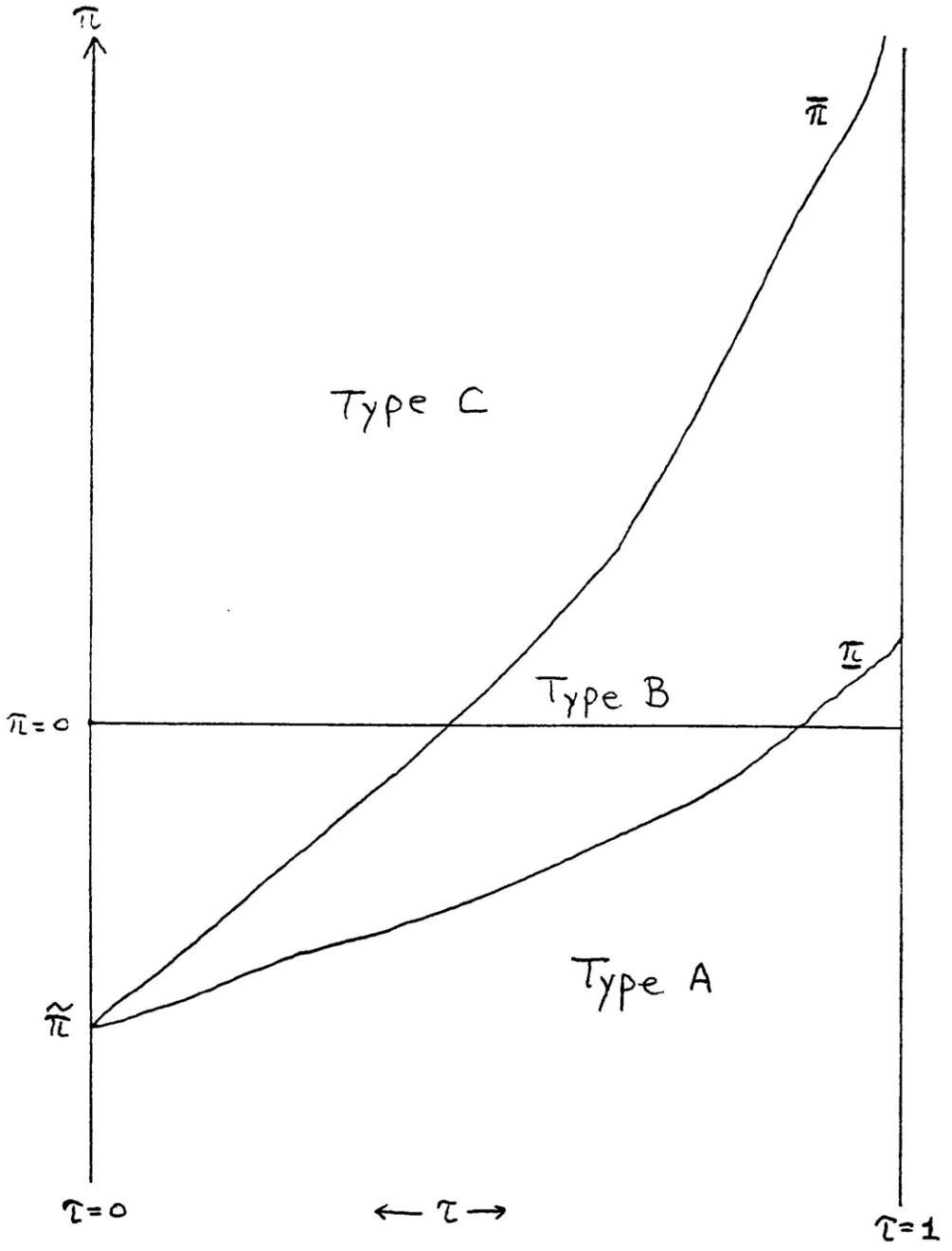


Figure 1.

equilibrium; $M \to 0$ and n_1, k_1, k_2, and q approach their values in the Type C equilibrium for $\pi = \bar{\pi}(\tau)$. Thus the equilibrium values of n, q, T, K, M, c_1, and c_2 are all continuous functions of $(w, \pi, \tau, R, \beta, \gamma)$.

It is also shown in Appendix 1 that

$$\underline{\pi}(0) = \bar{\pi}(0) = \tilde{\pi}$$

where $\tilde{\pi}$ is defined by

$$\frac{1}{(1+\tilde{\pi})^2} = 1+R \qquad (22)$$

That is, if there are no transaction costs, the possible equilibria are Types A and C, with the former occurring if the rate of return on money balances exceeds that available on capital, and the latter occurring if the reverse is true. The equilibrium one-period return on capital when $\tau = 0$ is $q = (1+R)^{1/2}$, as one should expect.[17] The range of inflation rates at which both money and capital are held in equilibrium expands as the size of transaction costs increases, to the extent that <u>no</u> inflation rate is too high when transaction costs approach one hundred percent. The

inflation rate $\underline{\pi}(\tau)$ at which capital begins to be held is also higher the greater the size of transaction costs; the $\underline{\pi}(\tau)$ locus rises above $\pi = 0$ for sufficiently high transaction costs, as shown in Figure 1, if and only if $(\frac{1-\lambda}{\lambda})\beta R < 1$.

3. Optimal Inflation

Suppose R, w, τ, β, and γ are given, but the monetary authority can choose π. How is the resulting steady state affected by the choice of π? For low values of π, Type A equilibrium obtains. As π increases, consumption by early consumers is increased and by late consumers is reduced. When $\underline{\pi}$ is reached, capital begins to be held and the equilibrium is of Type B. As π increases, the equilibrium level of n_1 (and hence the equilibrium capital stock K), increases monotonically,[18] while equilibrium real balances M increase less rapidly than in the Type A region and eventually decrease.[19] When $\bar{\pi}$ is reached, real balances fall to zero and the equilibrium is of Type C. Beyond this point, the inflation rate ceases to be an available policy instrument.

The effect on ex ante welfare W (the same for every agent, in steady state) of an increased rate of money growth is easily calculated. When the rate of money growth is less than $\underline{\pi}$, one has simply

$$\frac{dW}{d\pi} = \lambda\ c_1^{-\gamma}\ \frac{dc_1}{d\pi} + (1-\lambda)\beta\ c_2^{-\gamma}\ \frac{dc_2}{d\pi}$$

$$= \frac{\lambda(1-\lambda)}{(1+\lambda\pi)^2}\ [c_1^{-\gamma} - \beta c_2^{-\gamma}]$$

from (1) and (7). Increasing π increases steady state welfare if and only if $(\frac{c_2}{c_1})^{\gamma}$ is greater than β, where c_1 and c_2 are given by (7). Therefore

$$\frac{dW}{d\pi} \gtrless 0 \text{ as } \pi \lessgtr \pi_0$$

Where π_0 satisfies

$$(\frac{1}{1+\pi_0})^{\gamma} = \beta \qquad (23)$$

Note that a positive rate of time preference ($\beta < 1$) implies that the optimal rate of inflation (in the Type A

region) is positive ($\pi_0 > 0$), and the greater the extent to which the future is discounted, the higher the optimal rate of inflation will be. Thus time preference affects optimal inflation here in exactly the opposite way as in the familiar Friedman (1969) result.

A similar calculation is possible for the Type B region, $\underline{\pi} < \pi < \bar{\pi}$. From (11) one may obtain

$$\frac{dW}{d\pi} = \lambda c_1^{-\gamma} \frac{dc_1}{d\pi} + (1-\lambda)\beta c_2^{-\gamma} \frac{dc_2}{d\pi} \qquad (24)$$

$$= \lambda c_1^{-\gamma} \left\{ \frac{dc_1}{d\pi} + \frac{\left[\frac{1}{1+\pi} - (1+R)(1+\pi)(1-\tau)\right]}{\left[(1+R) - \frac{1}{(1+\pi)^2}\right]} \frac{dc_2}{d\pi} \right\}$$

Equations (11) and (13) may then be differentiated in order to obtain expressions for $\frac{dc_1}{d\pi}$ and $\frac{dc_2}{d\pi}$, which when substituted into (24) yield an expression for $\frac{dW}{d\pi}$. Since the resulting expression is quite complicated, it is not reproduced here. Certain of its properties, however, may easily be displayed. For example, for $\pi = 0$, (24) becomes

$$\frac{dW}{d\pi} = \lambda c_1^{-\gamma} \left\{ \frac{dc_1}{d\pi} + \frac{\left[1 - (1+R)(1-\tau)\right]}{R} \frac{dc_2}{d\pi} \right\}$$

But differentiation of (13) yields

$$\frac{dc_1}{d\pi} = (1-\lambda)c_1 + [(1+R)(1-\tau)-1]\frac{dn_1}{d\pi}$$

$$\frac{dc_2}{d\pi} = -\lambda c_1 + R\frac{dn_1}{d\pi}$$

which imply that

$$\frac{dc_1}{d\pi} + \frac{[1-(1+R)(1-\tau)]}{R}\frac{dc_2}{d\pi} = [1 - \frac{\lambda\tau(1+R)}{R}]c_1$$

so that

$$\frac{dW}{d\pi} \gtrless 0 \text{ as } \tau \lessgtr (\frac{R}{1+R})\frac{1}{\lambda}$$

It is shown in Appendix 1 that a Type B equilibrium is only possible, for zero money growth, when $\tau < (\frac{R}{1+R})\frac{1}{\lambda}$, so $\frac{dW}{d\pi} > 0$ for any Type B equilibrium with $\pi = 0$. That is, an increase in the rate of money growth to some positive value would increase steady state <u>ex ante</u> welfare.

It can be shown that this conclusion holds for any Type B equilibrium with $\pi < 0$. For the right hand side of (11)

is an increasing function of π. This means that either $\frac{dc_2}{d\pi} > 0$, $\frac{dc_1}{d\pi} < 0$, or both. If $\frac{dc_2}{d\pi} > 0$, (24) implies

$$\frac{dW}{d\pi} > \lambda c_1^{-\gamma} \left\{ \frac{dc_1}{d\pi} + \frac{[1 - (1+R)(1-\tau)]}{R} \frac{dc_2}{d\pi} \right\}$$

$$= \lambda c_1^{-\gamma} \left[1 - \frac{\lambda\tau(1+R)}{R} \right] > 0$$

while if $\frac{dc_1}{d\pi} < 0$, (24) implies

$$\frac{dW}{d\pi} > \lambda c_1^{-\gamma} \left\{ \frac{R \left[\frac{1}{1+\pi} - (1+R)(1+\pi)(1-\tau) \right]}{[1 - (1+R)(1-\tau)] \left[(1+R) - \frac{1}{(1+\pi)^2} \right]} \frac{dc_1}{d\pi} \right.$$

$$\left. + \frac{\left[\frac{1}{1+\pi} - (1+R)(1+\pi)(1-\tau) \right]}{\left[(1+R) - \frac{1}{(1+\pi)^2} \right]} \frac{dc_2}{d\pi} \right\}$$

$$= \frac{R \left[\frac{1}{1+\pi} - (1+R)(1+\pi)(1-\tau) \right]}{[1-(1+R)(1-\tau)] \left[(1+R) - \frac{1}{(1+\pi)^2} \right]} \lambda c_1^{-\gamma} \left[1 - \frac{\lambda\tau(1+R)}{R} \right] > 0$$

so that in either case, $\frac{dW}{d\pi} > 0$.

Thus $\pi \leq 0$ imples $\frac{dW}{d\pi} > 0$, both in the Type A region of Figure 1, and in the Type B region. Therefore it follows that:

> Proposition 2. Assuming $\beta < 1$, the optimal rate of inflation, if one exists, will always be positive.

If later consumption is not discounted, it ceases to be true that $\pi_0 > 0$. It follows from (22) that

$$\pi_0 \gtrless 0 \text{ as } \beta \lessgtr 1$$

It therefore follows that if $\beta > 1$, the optimal rate of inflation will be negative for sufficiently high transaction costs. But there will still be an intermediate region of transaction costs (for which the optimal steady state is a Type B equilbrium) in which the optimal rate of inflation is positive. This is because the above demonstration that $\frac{dW}{d\pi} > 0$ for any Type B equilibrium with $\pi \leq 0$ nowhere assumes anything about the size of β. In

fact, as will be explained in the following section, this result does not depend on any properties of agents' preferences at all.

4. Beyond the "Money Triangle"

Since the result that moderate inflation can be desirable is an unfamiliar one, it is worth examining in some detail. The result does not depend simply upon the irreversibility of the investment decision (i.e., transaction costs on the sale of capital), but depends crucially upon the presence of uncertainty about the time at which one will wish to consume. Suppose that the technological possibilities are as described above, but that all agents consume in both periods one and two. A quasi-concave utility function $W(c_1, c_2)$ represents their preferences for early and late consumption; there is no uncertainty about these preferences. It is shown in Appendix 1 that there are again three possible types of steady state equilbirium. In Type A equilibrium, both period one consumption and period two consumption are financed out of cash balances. In Type B equilibrium, period one consumption is financed out of cash balances,

while period two consumption is financed out of the proceeds of investment projects undertaken in period zero, so that while capital exists, none is ever sold. In Type C equilibrium, period two consumption is financed out of the proceeds of investment projects undertaken in period zero, while period one consumption is financed either by the sale of capital in period one or from the proceeds of capital purchased in period zero (these two strategies provide equal single-period returns). It is easily shown that a unique equilibrium exists for any economy specified by R, w, τ, π, and a utility function $W(c_1, c_2)$, and that it is Type A if and only if $\pi \leq \tilde{\pi}$, Type B if and only if $\tilde{\pi} < \pi < \pi^*(\tau)$, and Type C if and only if $\pi^*(\tau) \leq \pi$, where $\tilde{\pi}$ is defined by (22) and $\pi^*(\tau)$ by

$$\frac{1}{(1+\pi^*)^2} = (1+R)(1-\tau) \tag{25}$$

These regions are shown in Figure 2, corresponding to Figure 1 in the case of the model with uncertainty.

What is optimal monetary policy in this certainty model? In the case of a Type A equilibrium, one has

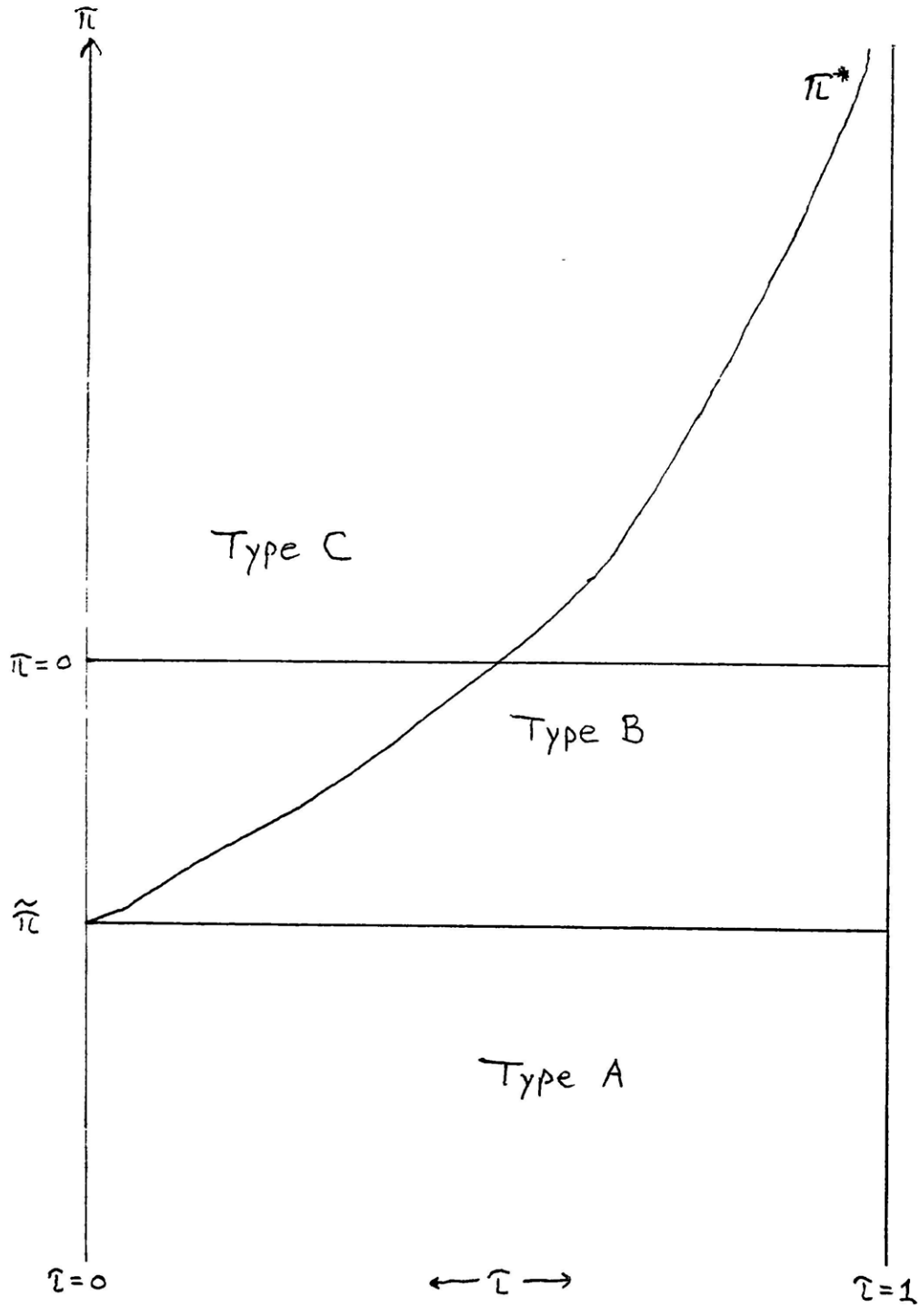


Figure 2.

$$c_1 = \frac{m_1}{1+\pi} - m_2 = \frac{w+T}{1+\pi} - m_2 \qquad (26a)$$

$$c_2 = \frac{m_2}{1+\pi} \qquad (26b)$$

Agents therefore choose m_2 so as to maximize $W(c_1,c_2)$, substituting (26) for c_1 and c_2, which implies

$$\frac{W_1}{W_2} = \frac{1}{1+\pi} \qquad (27)$$

Since in equilibrium

$$T = (\frac{\pi}{1+\pi})M = (\frac{\pi}{1+\pi})\ (m_1 + m_2) \qquad (28)$$

the equilibrium consumption pattern is found to be

$$c_1 = w - \frac{m_2}{1+\pi} \qquad (29a)$$

$$c_2 = \frac{m_2}{1+\pi} \qquad (29b)$$

Therefore, for any choice of π by the monetary authority, the allocation of consumption in Type A equilibrium (if possible) is given by a point (c_1, c_2) on the line satisfying

$$c_1 + c_2 = w \tag{30}$$

and (27). The optimal allocation among those satisfying (30) is the one for which $W_1 = W_2$, so that $\pi = 0$ is the optimal rate of inflation.[20] It is clear that in this model the sole effect of inflation is that of a distorting tax, which prevents agents from being able to attain through private transactions the true marginal rate of transformation between first and second period consumption that is possible for the economy. The welfare loss due to a small amount of inflation will be a second order effect of the familiar sort.[21]

This contrasts with the result in the case of uncertainty, where the optimal rate of inflation, π_0, is positive if $\beta < 1$. The latter result is easily understood in terms of the of the ability of inflation to correct for the absence of an insurance market. If agents could purchase insurance contracts which promised a payment c_1

contingent upon one's being an early consumer and c_2 contingent upon one's being a late consumer, it is clear that the contract that would dominate all others would satisfy

$$\left(\frac{c_2}{c_1}\right)^{\gamma} = \beta \qquad (31)$$

and the social budget constraint[22]

$$\lambda c_1 + (1-\lambda)c_2 = w \qquad (32)$$

A Type A equilibrium, in the absence of an intermediary that can provide such contracts, satisfies the budget constraint, but agents have no choice among the allocations satisfying this constraint. Rather, the allocation between c_1 and c_2 is entirely determined by the monetary authority's choice of π, in the manner indicated by equations (7). Since (7) implies

$$\frac{c_2}{c_1} = \frac{1}{1+\pi}$$

it is clear that $\frac{c_2}{c_1}$ is too high for all $\pi \leq \pi_o$. In particular, it is too high if $\pi = 0$ (if $\beta < 1$). Inflation redistributes from late consumers to early consumers (because inflation reduces the value of money holdings more the longer one holds them) and so achieves something closer to the desired degree of insurance. Equation (7) indicates that for $\pi = \pi_o$, the optimal insurance contract is achieved.

Similar results obtain in the case of Type B equilibrium, although in that case the welfare gains from moderate inflation exist in the case of consumption uncertainty, regardless of the size of β. In the certainty model, one has

$$k_1 = m_2 = n_2 = 0$$

$$k_2 = n_1$$

$$c_1 = \frac{m_1}{1+\pi}$$

$$c_2 = n_1(1+R)$$

That is, initial wealth is divided between real money balances m_1 and investment n_1 according to how much early consumption as opposed to late is desired. Agents choose m_1 and n_1 to maximize $W(c_1,c_2)$, subject to the budget constraint

$$m_1 + n_1 \leq w + T$$

Therefore consumption is allocated so that

$$\frac{W_1}{W_2} = (1+R)(1+\pi) \qquad (31)$$

The money market equilibrium condition (for a steady state) corresponding to (6) is

$$m_1 = \left(\frac{1+\pi}{\pi}\right) T$$

Using this to substitute for T is the budget constraint yields the feasible consumption set

$$c_1 + \frac{c_2}{1+R} \leq w \qquad (32)$$

The steady state consumption allocation is that which satisfies (31) and (32) with equality. The welfare effect of moving to a different steady state money growth rate is therefore

$$\frac{dW}{d\pi} = W_1 \frac{dc_1}{d\pi} + W \frac{dc_2}{d\pi} \tag{33}$$

$$= [W_1 - (1+R)W_2] \frac{dc_1}{d\pi} = (1+R)\pi W_2 \frac{d}{d\pi} \left(\frac{m_1}{1+\pi}\right)$$

It is clear from (33) that at $\pi = 0$, $\frac{dW}{d\pi} = 0$, and that this is the optimal rate of inflation. For again the only effect of inflation is that of a distorting tax. All equilibria (as long as one remains within the region of Type B equilibrium) occur on the boundary of the feasible set (32), which set remains invariant under changes in monetary policy. The effect of a change in the inflation rate is merely to change the apparent trade-off between early and late consumption perceived by agents. The point on the boundary of the feasible set which reaches the highest indifference curve is that where the boundary is tangent to an indifference curve, i.e., where

$$\frac{W_1}{W_2} = 1+R$$

This is the point agents will choose if they are able to achieve the economy's marginal rate of transformation between early and late consumption, point X in Figure 3. This occurs only when $\pi = 0$. For $\pi > 0$, agents face a $c_1 - c_2$ tradeoff PP in Figure 3, instead of the economy's tradeoff AA, and so choose point Y, which achieves for them a lower welfare. A similar but reversed distortion occurs if $\pi < 0$. Again, the welfare loss due to a small amount of inflation is a second-order effect.[23]

The case of consumption uncertainty, analyzed above, is different. There Type B equilibrium is characterized by (11) and (13). Equation (13) with n_1 varying over its range indicates that the feasible set of (c_1, c_2) allocations is given by

$$\left[\frac{R+(1+R)\lambda\pi\tau}{(1+R)(1+\pi)\tau}\right] c_1 + \left[\frac{1-(1+R)(1-\tau)}{(1+R)\tau}\right] c_2 \leq w \qquad (34a)$$

$$c_1 \leq \frac{1+\pi}{1+\lambda\pi} w \qquad (34b)$$

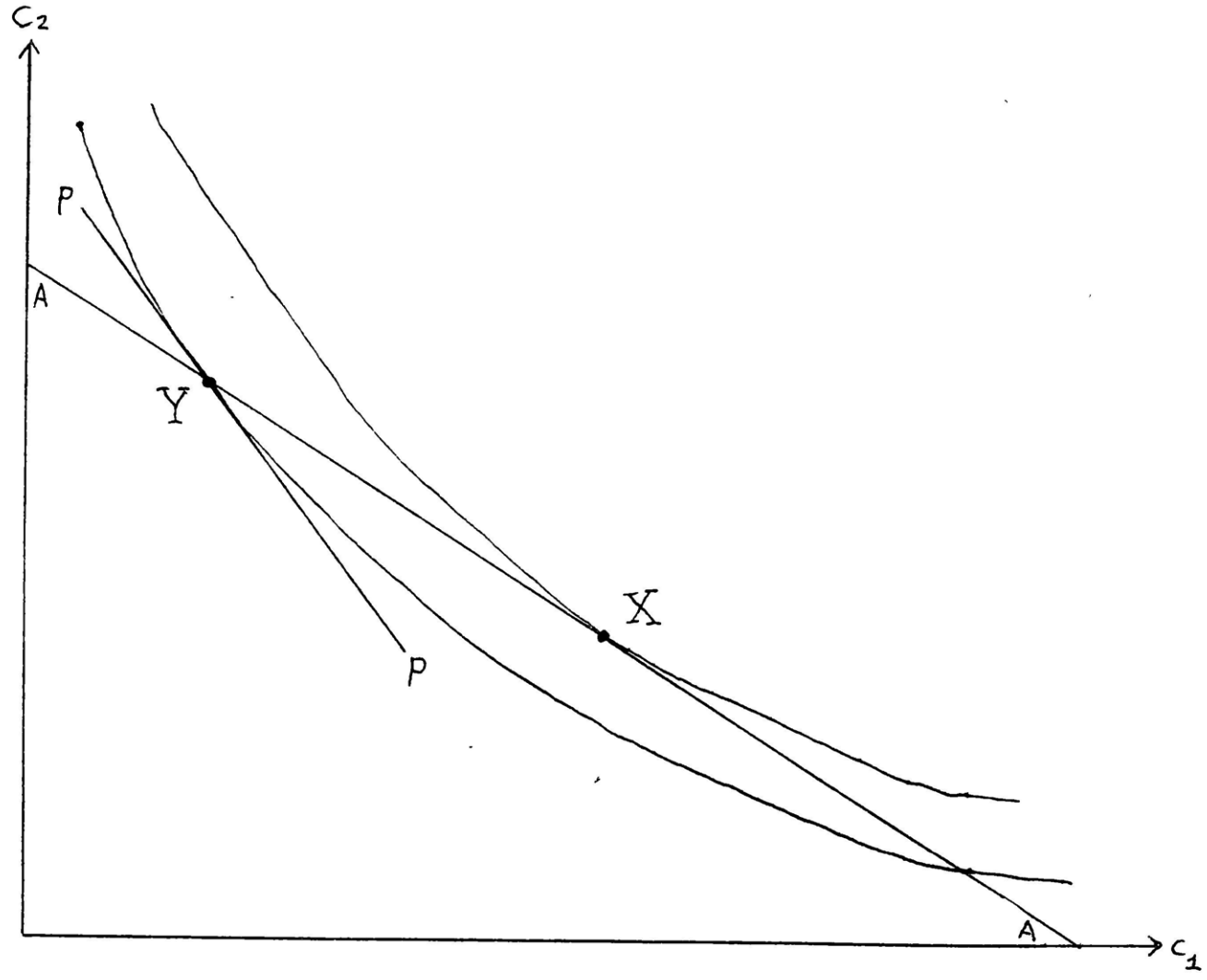


Figure 3.

$$c_2 \leq \frac{1+R}{1+\lambda\pi}\left[\frac{\lambda(1+\pi)^2(1-\tau) + (1+\lambda\pi\tau)(2-\pi-\lambda)}{(2+\pi-\lambda)+\lambda(1+R)(1+\pi)^2(1-\tau)}\right] w \quad (34c)$$

In fact, Type B equilibrium always occurs on the feasible segment where (34a) holds with equality, since this is just the agents' budget constraint with T replaced by its equilibrium value. So the consumption allocation in a Type B equilibrium is given by the intersection of (11) and (34a) holding with equality. Now (34a) implies that the social cost of early consumption in terms of foregone late consumption is

$$\frac{R + (1+R)\lambda\pi\tau}{(1+\pi)[1-(1+R)(1-\tau)]} \quad (35a)$$

while the cost faced by agents is

$$\frac{[(1+R) - \frac{1}{(1+\pi)^2}]}{[\frac{1}{1+\pi} - (1+R)(1+\pi)(1-\tau)]} \quad (35b)$$

The numerator of (35a) exceeds the numerator of (35b) if and only if $\pi < 0$; the denominator of (35a) exceeds the denominator of (35b) if and only if $\pi > 0$. Hence

$$(35a) \gtrless (35b) \quad \text{as} \quad \pi \lessgtr 0$$

It therefore obtains, in this case as in the certainty case, that the social marginal rate of transformation between early and late consumption equals the rate available to agents only when $\pi = 0$. When $\pi < 0$, increased inflation (or decreased deflation) shifts the cost to agents of early consumption closer to the social cost, while when $\pi > 0$, increased inflation shifts it further from the social cost. If this were the only effect of monetary policy, $\pi = 0$ would again be optimal. But in this case, unlike the certainty case, the feasibility constraint (34a) is affected by a change in the inflation rate. The coefficient on c_2 in (34a) is unaffected, whereas the coefficient on c_1 is increased if and only if

$$\tau > \left(\frac{R}{1+R}\right) \frac{1}{\lambda}$$

Hence increased inflation (apart from its effect on the perceived cost of early consumption) loosens the feasibility constraint if τ is less than that value, tightens it if τ is greater. As noted above, Type B

equilibrium occurs for $\pi \leq 0$ only when τ is less than this value, so in these cases inflation loosens the constraint.

Thus the total effect of a change in monetary policy is the sum of two effects, shown in Figure 4. When $\pi = 0$, the economy's feasibility constraint is AA, and the budget constraint of agents is also AA, so that they choose consumption allocation X. The effect of $\pi > 0$ is to shift the economy's feasibility constraint out to BB, and also to cause agents to face budget constraint PP instead of BB. The differing slope of PP from that of BB means a welfare loss, the sort indicated by the conventional "money triangle", but these losses only increase quadratically in π. The other effect, the loosening of the economy's feasibility constraint, produces a welfare gain that is of first order in π, so that for small enough positive π this effect dominates the "triangle" losses. Hence the consumption allocation chosen in the case of moderate inflation, point Z in Figure 4, will be preferred to point X. Note that this conclusion does not require the functional form assumed in (1); the same result would follow in the case of any quasi-concave utility function $W(c_1, c_2)$.

How can monetary policy loosen the feasibility

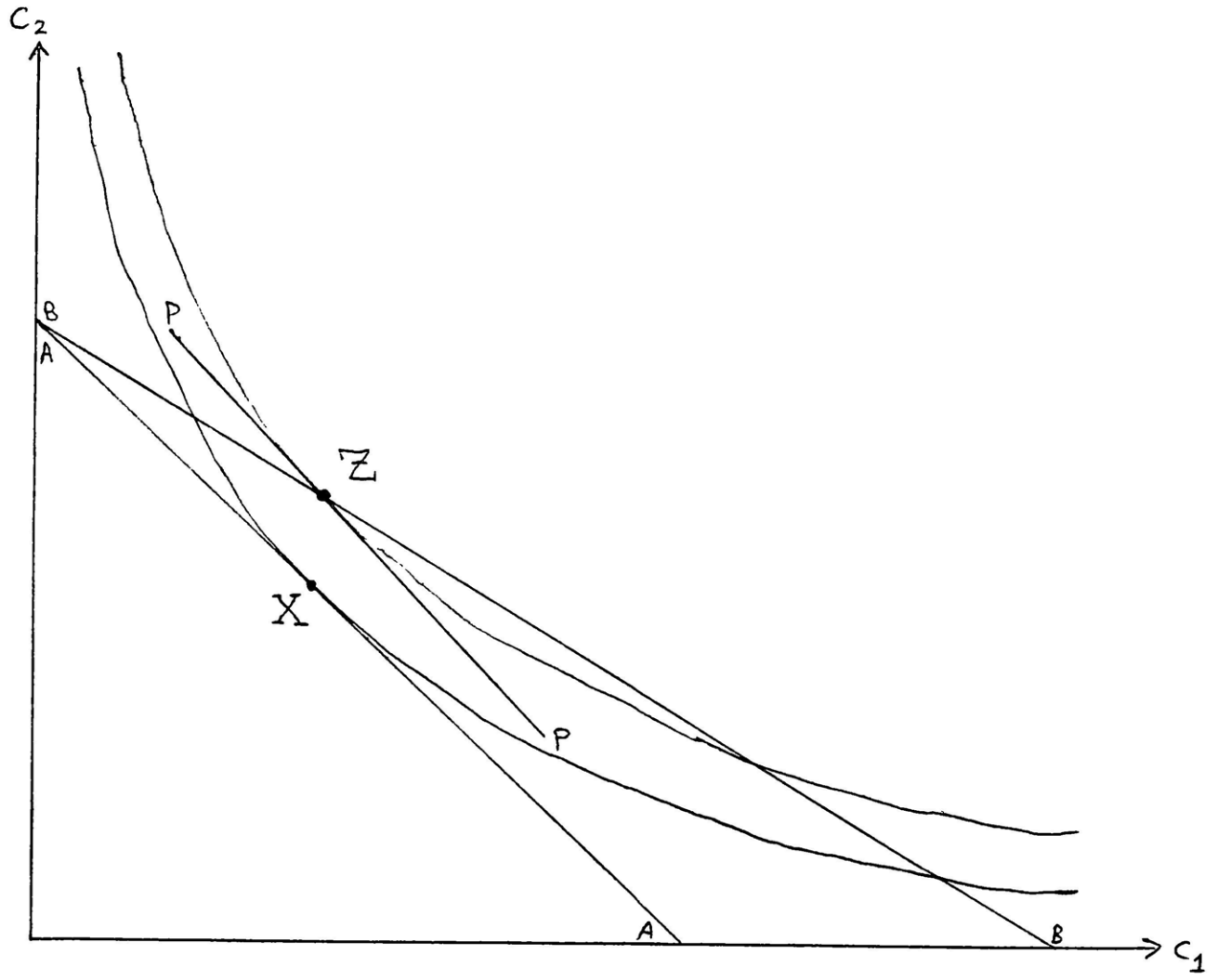


Figure 4.

constraint in the case of consumption? This happens because, as partial differentiation of (13) shows,

$$-\frac{\frac{\partial c_2}{\partial \pi}}{\frac{\partial c_1}{\partial \pi}} < -\frac{\frac{\partial c_2}{\partial n_1}}{\frac{\partial c_1}{\partial n_1}} \qquad (36)$$

when $\tau < (\frac{R}{1+R})\frac{1}{\lambda}$. It follows from (12) that

$$-\frac{\frac{\partial c_2}{\partial \pi}}{\frac{\partial c_1}{\partial \pi}} = \frac{\lambda}{1-\lambda} \qquad (37)$$

This is the rate of transformation between period one consumption and and period two consumption possible for the economy as a whole, assuming that the constraints that all investment must be undertaken by private parties who do not know when they will wish to consume, and that early consumers must sell their capital, incurring transaction costs, cannot be gotten around. If intermediaries existed who could provide insurance contracts against consumption

uncertainty (but who could not get around the constraints just mentioned), the equilibrium allocation would satisfy

$$\frac{W_1}{W_2} = \frac{\lambda}{\beta(1-\lambda)} \left(\frac{c_2}{c_1}\right)^{\gamma} = \frac{\lambda}{1-\lambda} \tag{38}$$

On the other hand it follows from (34a) that

$$- \frac{\frac{\partial c_2}{\partial n_1}}{\frac{\partial c_1}{\partial n_1}} = \frac{[(1+R)(1+\lambda\pi\tau) - 1]}{(1+\pi)[1-(1+R)(1-\tau)]} \tag{39}$$

This is the rate of transformation between period one consumption and period two consumption that could be obtained if all agents were to agree to change the amount of illiquid investment in their portfolios, what is called the "social" rate of transformation above. So even if agents cooperated to defeat the distorting effect of the inflation tax and all chose the jointly optimal level of n_1, instead of the individually optimal level given by (11), they would attain an allocation satisfying

$$\frac{W_1}{W_2} = = \frac{\lambda}{\beta(1-\lambda)} \left(\frac{c_2}{c_1}\right)^{\gamma} = \frac{[(1+R)(1+\lambda\pi\tau) - 1]}{(1+\pi)[1 - (1+R)(1-\tau)]} \quad (40)$$

rather than the first-best (38). Because the right hand side of (39) and (40) is smaller than the right hand side of (37) and (38), the redistribution from late consumers to early consumers brought about by inflation moves the allocation (c_1, c_2) to a point outside the $\pi = 0$ feasibility constraint.

Thus the first-order welfare gains from moderate inflation result because inflation, by redistributing purchasing power from late consumers (who hold liquid assets for a longer time) to early consumers, provides insurance against uncertainty about the time at which one will wish to consume, which agents cannot provide for themselves. The standard argument about inflation as a distorting tax correctly concludes that zero money growth is necessary to make the slope of PP equal to that of BB in Figure 4, but neglects the fact that the slope of BB itself (the feasibility constraint (34a)) is not equal to the marginal rate of transformation available to the economy through redistribution of purchasing power. Hence it does

not follow that equalizing the slopes of BB and PP by reducing inflation to zero increases welfare.

A similar result is demonstrated in Appendix 2 in the case of the model of money demand presented by Miller and Orr (1966). In that model, like the one presented here, agents make portfolio decisions in the face of uncertainty about the timing of future expenditures, and transaction costs are incurred upon liquidation of the higher-yield asset. Again the optimal rate of inflation is shown to be positive, because the redistributive aspect of inflationary policy provides desirable insurance against an unanticipated need for early expenditures.

It is therefore not enough merely to recognize that as money balances are held, they must supply some kind of liquidity services, and then to argue on traditional price-theoretic grounds that perfectly anticipated inflation must reduce welfare, as in the case of any distorting tax. It depends on the kind of liquidity services supplied by money balances. In the case of the model of precautionary demand presented here, the market imperfections that create the demand for liquidity also imply a welfare gain from moderate rates of inflation that outweighs the effect of the distorting tax, for low enough rates of inflation.[24]

Appendix 1

In this appendix certain details are supplied of the derivations of Type B and C equilibria in section 2, as well as a proof of Proposition 1. Consider again a Type B equilibrium. A late consumer considering how to invest his liquid assets $(\frac{m_1}{1+\pi})+k_1(1+R))$ in period one can obtain a return of $\frac{1}{1+\pi} = \frac{1+R}{q}$ on either money balances or purchased capital, whereas if he invests in new capital which he plans to sell the following period, his return will be $q(1-\tau)$. This must be less than $\frac{1}{1+\pi}$. If it were greater, money balances would never be held either in period zero or period one, as they would be dominated in return by an investment in new capital which one planned to sell the next period. And if it were equal to $\frac{1}{1+\pi}$, all wealth in period zero would be put in new investment (since it would yield as good a return as the other possibilities if one turned out to be an early consumer, and a better return if one turned out to be a late consumer), and late consumers would simply hold this capital to maturity. But then early

consumers would wish to sell their capital in period one while no one would wish to buy, so that no equilibrium is possible. Hence a Type B equilibrium can occur only if the inflation rate is such that

$$\frac{1}{(1+\pi)^2} > (1+R)(1-\tau)$$

In this case $n_2 = 0$, as asserted in the text, and a late consumer in period one is indifferent between holding his liquid wealth as money balances or buying capital with it. Furthermore, the capital n_1 that a late consumer carries over into period one he continues to hold, since he earns a greater return on it this way (1+R per unit of capital) than he could by selling it and investing the proceeds ($(1+R)(1-\tau)$ at most). Hence, as asserted in the text, $k_2 \geq n_1$.

The inflation rate must also be such that

$$1+R > \frac{1}{(1+\pi)^2}$$

If not, holding money balances for two periods yields as high a return as is available from producing capital with which to produce goods, and no one will ever hold capital. Every agent will choose $n_1 = n_2 = 0$, and then equilibrium is possible only if $k_1 = k_2 = 0$ as well, so that only a

Type A equilibrium is possible. Thus conditions (10) have been established.

Similarly, consider again a Type C equilibrium. In the absence of money, equilibrium can occur only with

$$q < \left(\frac{1+R}{1-\tau}\right)^{1/2}$$

For if $q \geq \left(\frac{1+R}{1-\tau}\right)^{1/2}$, the return on investment in order to sell capital the following period, $q(1-\tau)$, would be as large as the return on buying capital in order to sell the final product the following period, $\frac{1+R}{q}$, and so agents would choose $k_1 = 0$ (since investment promises at least as good a return for an early consumer, and a better return for a late consumer). But $k_1 = 0$ would imply $k_2 \leq n_1$, because of (3), and so capital would be in excess of supply:

$$Z = (1-\lambda)(k_2-n_1) - \lambda(1-\tau)n_1 - (1-\lambda)(1-\tau)n_2 < 0$$

When this condition holds, agents choose $n_2 = 0$, as asserted in the text. This establishes (17).

And if $q \leq (1+R)^{1/2}$, it is clear that both s_1 and c_2 would be decreasing functions of n_1, so that $n_1 = 0$ would be chosen; but then capital would be in excess demand (no

one wants to produce capital, all agents want to hold all their wealth as capital). So equilibrium also requires

$$q > (1+R)^{1/2}$$

Thus conditions (18) have been established.

As noted in the text, Type A equilibrium is clearly unique, when it exists. The condition for its existence is that agents behaving as they are required to in the Type A equilibrium not wish to invest in the production of capital, assuming that any capital they had to sell is valued at the shadow price of capital. In the Type A equilibrium that shadow price is $(1+R)(1+\pi)$, since capital and money balances are perfect substitutes. A marginal unit of investment by late consumers in period one would increase c_2 at the rate $(1+R)(1+\pi)(1-\tau)$, but the reduction in money balances used to pay for it would reduce c_2 at the rate $1/1+\pi$, so for this not to be undertaken one must have

$$\left(\frac{1}{1+\pi}\right)^2 \geq (1+R)(1-\tau) \qquad (1.1)$$

A marginal unit of investment in period zero would affect c_1 in this manner, and would increase c_2 at the rate $(1+R)$, while the reduction in money balances used to pay for it

would reduce c_2 at the rate $\left(\frac{1}{1+\pi}\right)^2$, so for this not to be undertaken one must have

$$\lambda c_1^{-\gamma}\left[(1+R)(1+\pi)(1-\tau)-\frac{1}{1+\pi}\right] + (1-\lambda)\beta c_2^{-\gamma}\left[1+R-\frac{1}{(1+\pi)^2}\right] \leq 0$$

where c_1 and c_2 are given by (7). Clearly if

$$\left(\frac{1}{1+\pi}\right)^2 \geq 1+R$$

a Type A equilibrium exists. If not, the condition for existence, in addition to (1.1), is

$$\left(\frac{1}{1+\pi}\right)^{\gamma} > \left(\frac{1-\lambda}{\lambda}\right)\beta \frac{\left[1+R-\frac{1}{(1+\pi)^2}\right]}{\left[\frac{1}{1+\pi}-(1+R)(1+\pi)(1-\tau)\right]} \qquad (1.2)$$

using (7). Now the right hand side is an increasing function of π, while the left hand side is a decreasing function of π, so condition (1.2) has the form $\pi < \underline{\pi}$, where $\underline{\pi}$ depends on R, w, τ, λ, β, and γ. Note that this inequality will imply condition (1.1), so that it completely describes the economies which can have a Type A

equilibrium. It is easily verified that as τ varies, for fixed values of the other parameters describing the economy, $\underline{\pi}(\tau)$ is a strictly increasing function, for the left hand side of (1.2) has no τ-dependence while the right hand side is a decreasing function of τ. It is also clear from (1.2) that $\underline{\pi}(1) \geq 0$ if and only if

$$\frac{(1-\lambda)}{\lambda} \beta R \leq 1$$

and that $\underline{\pi}(0)=\tilde{\pi}$, as also stated in the text.

Similar propositions are true of Type B equilibrium. It was established above that (10) must hold. Then equations (13) imply

$$\frac{\partial}{\partial n_1} \left(\frac{c_2}{c_1}\right) = \frac{(1+R)(1+\pi) w\tau}{(1+\lambda\pi) c_1^2} > 0$$

Therefore the n_1 that jointly satisfies (11) and (13) is unique, if one exists within the acceptable range. This implies that the Type B equilibrium will be unique (up to the indeterminacy in the allocation of total K and M between k_1, k_2 and m_1, m_2) if it exists.

The condition for existence of Type B equilibrium is that (11) (with (13) substituted in for c_1 and c_2) have a solution satisfying

$$0 < n_1 < \frac{2 + \pi - \lambda}{(2+\pi-\lambda) + \lambda(1+R)(1+\pi)^2(1-\tau)} w$$

The lower bound is obvious, and is also necessary in order for the equilibrium value of K to be positive, by (15). The upper bound on n_1 is necessary in order for the equilibrium demand for money balances M to be positive, by (14). The lower bound amounts to the requirement that $(\frac{c_2}{c_1})^{\gamma}$ be less than the right hand side of (11) when c_1 and c_2 take the values given in (13) with $n_1 = 0$. But (13) becomes (7) for $n_1 = 0$, and so the lower bound is exactly the condition that a Type A equilibrium not exist, i.e., $\underline{\pi} < \pi$. The upper bound amounts to the requirement that $(\frac{c_2}{c_1})^{\gamma}$ be greater than the right hand side of (11) when c_1 and c_2 take the values given in (13) with n_1 at its upper bound, i.e., when

$$c_1 = \frac{(1+\pi)(2+\pi)(1+R)(1-\tau)}{(2+\pi-\lambda) + \lambda(1+R)(1+\pi)^2(1-\tau)} w$$

$$c_2 = \frac{(1+R)(2+\pi-\lambda\tau)}{(2+\pi-\lambda) + \lambda(1+R)(1+\pi)^2(1-\tau)} w$$

The upper bound is therefore

$$\left[\frac{(2+\pi-\lambda\tau)}{(1+\pi)(2+\pi)(1-\tau)}\right]^{\gamma} > \left(\frac{1-\lambda}{\lambda}\right) \beta \frac{\left[(1+R) - \frac{1}{(1+\pi)^2}\right]}{\left[\frac{1}{1+\pi} - (1+R)(1+\pi)(1-\tau)\right]} \quad (1.3)$$

The left hand side of (1.3) is a decreasing function of π, while the right hand side is an increasing function. Hence it follows that this condition may be written $\pi < \bar{\pi}$, where $\bar{\pi}$ is a function of R, w, τ, λ, β, and γ, and the complete conditions for existence of a Type B equilibrium become $\underline{\pi} < \pi < \bar{\pi}$. It is easily verified that as τ varies, for fixed values of the other parameters, $\bar{\pi}(\tau)$ is a strictly increasing function, for the left hand side of (1.3) is an increasing function of τ while the right hand side is a decreasing function of τ. It is also clear from (1.3) that $\bar{\pi}(0) = \tilde{\pi}$, and that as $\tau \to 1$, $\bar{\pi}(\tau) \to \infty$, as asserted in the text.

The Type C equilibrium value of q is given, as shown in the text, by equation (19) with (21) substituted in for c_1 and c_2. Equations (21) imply

$$\frac{\partial}{\partial q}\left(\frac{c_2}{c_1}\right) < 0$$

Therefore the left hand side of (19) is a decreasing function of q, while the right hand side is an increasing function of q, and so the solution, if one exists, is unique. Since the right hand side of (19) varies between 0 and $+\infty$ as q varies between $(1+R)^{1/2}$ and $\left(\frac{1+R}{1-\tau}\right)^{1/2}$, there is certain to exist a solution within this interval. Hence for every economy, specified by R, w, τ, λ, β, and γ, there exists a unique Type C equilibrium.

The condition for a Type C equilibrium to exist despite the existence of fiat money is that in the Type C equilibrium described above, agents do not wish to hold any money. Since money balances and capital are perfect substitutes, and capital is purchased for its one-period return in a Type C equilibrium, the condition holds if and only if $\left(\frac{1}{1+\pi}\right) < \frac{1+R}{q}$. Therefore a Type C equilibrium exists if and only if the rate of money growth satisfies

$$\pi > \frac{q^*}{1+R} - 1$$

where q^* is the equilibrium value of q determined by R, w, τ, λ, β, and γ in the manner described above. But a comparison of the definition of q^* with that of $\bar{\pi}$ shows that this condition is equivalent to $\pi > \bar{\pi}$. Hence each type of equilibrium exists if and only if the other two do not, and Proposition 1 is established.

Similarly, three types of equilibrium may be distinguished in the case of the certainty model discussed in section 4. Since in this case agents can plan their future consumption with complete certainty, period one consumption is financed entirely out of whatever form of wealth earns the highest one-period return, and period two consumption is financed entirely out of whatever form of wealth earns the highest two-period return. It therefore follows that if capital is sold, there must be agents seeking a one-period return by investing in order to sell the capital, and others seeking a one-period return by purchasing capital. Hence these two one-period returns must be equal, $q(1-\tau) = \frac{1+R}{q}$, and the equilibrium price of capital must be $q = \left(\frac{1+R}{1-\tau}\right)^{1/2}$. It then follows that first-

period consumption will be financed entirely out of cash balances if $\frac{1}{1+\pi} > q(1-\pi)$, i.e., if $\pi < \pi^*$, and entirely out of the sale of capital or the proceeds from capital purchased in period zero if $\frac{1}{1+\pi} < q(1-\tau)$, i.e., if $\pi > \pi^*$. It also follows that second period consumption will be financed entirely out of cash balances if $\pi < \tilde{\pi}$, since it follows that $(\frac{1}{1+\pi})^2 > (1+R)$, and entirely out of the proceeds of investment projects entered into in period zero if $\pi > \tilde{\pi}$, since it follows in that case that $1+R > (\frac{1}{1+\pi})^2$ and $1+R > [q(1-\tau)]^2$.

Hence it follows that if $\tilde{\pi} < \pi$, only cash balances are held (Type A); if $\tilde{\pi} < \pi < \pi^*$, first period consumption is financed out of investment projects entered into in period zero (Type B); and if $\pi^* < \pi$, cash balances are not held (Type C). If $\pi = \tilde{\pi}$, first period consumption is financed entirely out of cash balances, but second period consumption may be financed out of any combination of cash balances and period zero investment, so that there is a continuum of equilibria with Type A at one extreme ($n_1 = 0$) and Type B at the other ($m_2 = 0$). If $\pi = \pi^*$, second period consumption is financed entirely out of the proceeds of

period zero investment, but first period consumption may be financed out of any combination of cash balances, sale of capital, or proceeds of capital purchased in period zero, limited only by the capital market equilibrium condition, so that there is a continuum of equilibria with Type B at one extreme ($k_1 = 0$) and Type C at the other $m_1 = 0$). The regions in which various types of equilibria occur are indicated in Figure 2.

Appendix 2

It is argued in the text that the desirability of moderate inflation results from the conjunction of uncertainty about the timing of one's expenditures and transaction costs associated with the sale of higher-yield but illiquid forms of wealth. Yet a reader might fear that the result in the case of the model presented in the text depends on other special features of that model, such as the finite-lived agents or the transaction costs proportional to the amount of capital sold. Hence this appendix demonstrates a similar result in the case of a substantially different model, which nonetheless shares the two features of timing uncertainty and transaction costs.

Consider the model of the demand for money by firms presented by Miller and Orr (1966), a stochastic version of the Baumol-Tobin model discussed in the introduction. A firm must use cash for its transactions; in some it receives cash, while in others it must pay out cash. The timing of these transactions is stochastic; specifically, in the absence of any financial transactions, the firm's cash balances m follow a random walk

$$dm = \sigma\, dz \qquad (2.1)$$

where σ^2 is the instantaneous variance of the firm's cash

balances. Cash may be exchanged for an asset earning a constant riskless return of r, but a fixed brokerage fee γ is incurred each time a firm exchanges some of its cash balances for the asset or some of its asset holdings for cash balances. Firms choose when to convert cash into the asset and vice versa, so as to minimize their costs of managing the cash balances they need for transactions; these costs consist of a cost γ each time a financial transaction occurs, and the loss of interest payments on the wealth held in the form of cash balances, at a constant rate r. Miller and Orr show that the optimal policy for a firm is an "S-s" inventory policy : whenever cash balances fall to zero, they are returned to some level z; whenever they rise to some upper limit h, they are also returned to level z; and otherwise they are allowed to fluctuate. Under such a policy, a firm's average cash balances will be

$$m^* = \frac{h+z}{3} \qquad (2.2)$$

and the mean time between financial transactions will be

$$D = \frac{z(h-z)}{\sigma^2} \qquad (2.3)$$

Its expected costs per unit time are therefore

$$C = \frac{\gamma}{D} + rm^* = \frac{\gamma\sigma^2}{z(h-z)} + \frac{r}{3}(h+z) \qquad (2.4)$$

The firm chooses h and z to minimize (2.4), so that

$$h = 3z$$
$$z = \left(\frac{3\gamma\sigma^2}{4r}\right)^{1/3}$$

and the average money balances held by a firm are thus

$$m^* = \frac{4}{9}h = \left(\frac{16\gamma\sigma^2}{9r}\right)^{1/3}$$

Suppose a monetary authority exists, which prints new money (or takes money out of circulation) so as to increase (or decrease) the money supply at a constant rate π. The new money is handed out to all firms at a constant rate T, lump sum (or they are all taxed at a constant rate, lump sum, in the case of negative π). Then cash balances of a given firm will fluctuate according to an Ito process

$$dm = T - \pi m + \sigma dz \qquad (2.6)$$

Money market equilibrium demands that $T = \pi m^*$, where m^* denotes the average cash balances, so that (2.6) becomes

$$dm = -\pi(m-m^*) + \sigma dz \qquad (2.7)$$

This means that the distribution function for cash balances across firms f(m,t) will evolve according to the differential equation

$$\frac{\partial f}{\partial t} = \pi(m-m^*)\frac{\partial f}{\partial m} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial m^2}$$

Hence the steady state distribution of cash balances f(m) will satisfy

$$\frac{\sigma^2}{2}\frac{d^2 f}{dm^2} + \pi(m-m^*)\frac{df}{dm} = 0 \qquad (2.8)$$

Setting $y \equiv (m-m^*)$, this becomes

$$\frac{d^2 f}{dy^2} = -\frac{2\pi}{\sigma^2} y \frac{df}{dy}$$

and the solution is of the form

$$f(y) = B_1 + A_1 y - A_1 \frac{\pi}{3\sigma^2} y^3 + O(\pi^2), \text{ for } -m^* < y < z-m^*$$
$$f(y) = B_2 + A_2 y - A_2 \frac{\pi}{3\sigma^2} y^3 + O(\pi^2), \text{ for } z-m^* < y < h-m^*$$

where terms of order $O(\pi^2)$ are neglected because we are

only interested in the effect on the steady state with $\pi=0$ of the introduction of a small amount of inflation or deflation. The coefficients A_1, A_2, B_1, B_2 are fixed by the boundary conditions

(i) $f(-m^*) = f(h-m^*) = 0$, and

(ii) $f(z-m^*)$ has the same value approached from either direction,

and the normalization conditon

$$\text{(iii)} \quad \int_{-m^*}^{h-m^*} f(y)dy = 1.$$

Using (2.5) to write

$$m^* = \frac{4}{9}h + m_1\pi h + O(\pi^2) \qquad (2.9)$$

One finds the coefficients to be

$$A_1 = \frac{6}{h^2} + \frac{7}{9}\frac{\pi}{\sigma^2} + O(\pi^2)$$

$$A_2 = -\frac{3}{h^2} - \frac{7}{18}\frac{\pi}{\sigma^2} + O(\pi^2)$$

$$B_1 = \frac{8}{3h} + \frac{6m_1\pi}{h} + \frac{124}{729}\frac{\pi h}{\sigma^2} + O(\pi^2)$$

$$B_2 = \frac{5}{3h} - \frac{3m_1\pi}{h} + \frac{65}{1458}\frac{\pi h}{\sigma^2} + O(\pi^2)$$

Then average money balances are given by

$$m^* = \int_{-m^*}^{h-m^*} (y+m^*)\, f(y)dy$$

which implies that for consistency one must have

$$\int_{-m^*}^{h-m^*} yf(y)dy = 0$$

This implies

$$m_1 = -\frac{4}{729}\frac{h^2}{\sigma^2}$$

so that, recalling (2.9),

$$\frac{\partial m^*}{\partial \pi} = -\frac{4}{729}\frac{h^3}{\sigma^2} \qquad (2.10)$$

Diffusion process (2.17) also implies that the expected time D(m) until the next financial transaction, as a function of current money balances -- i.e., expected time to absorption by either the barrier at m=0 or that at m=h -- obeys the differential equation

$$\frac{\sigma^2}{2} \frac{d^2D}{dm^2} - \pi (m-m^*) \frac{dD}{dm} = -1 \qquad (2.11)$$

Again setting $y \equiv (m-m^*)$, this becomes

$$\frac{d^2D}{dy^2} = \frac{2\pi}{\sigma^2} y \frac{dD}{dy} - \frac{2}{\sigma^2}$$

and the solution is of the form

$$D(y) = B + Ay - \frac{y^2}{\sigma^2} + \frac{\pi}{3\sigma^2} Ay^3 - \frac{\pi}{3\sigma^4} y^4 + O(\pi^2)$$

The coefficients A and B are fixed by the boundary conditions that $D(-m^*) = D(h-m^*) = 0$. Again using (2.9), the coefficients are found to be

$$A = \frac{1}{9} \frac{h}{\sigma^2} - 2m_1 \frac{\pi h}{\sigma^2} - \frac{20}{2187} \frac{\pi h^3}{\sigma^4} + O(\pi^2)$$

$$B = \frac{20}{81} \frac{h^2}{\sigma^2} + \frac{m_1}{9} \frac{\pi h^2}{\sigma^2} + \frac{400}{19,683} \frac{\pi h^4}{\sigma^4} + O(\pi^2)$$

The average time between financial transactions will be the expected time immediately after a transaction, or

$$D(z-m^*) = \frac{2}{9}\frac{h^2}{\sigma^2} + \frac{14}{729}\frac{\pi h^4}{\sigma^4} + O(\pi^2)$$

Note that this agrees with (2.3) for the case $\pi = 0$. One thus obtains

$$\frac{\partial D}{\partial \pi} = \frac{14}{729}\frac{h^4}{\sigma^4} \qquad (2.12)$$

So if firms continue to manage their cash balances according to the (h,z) rule that was optimal for $\pi = 0$, but a small constant rate of money growth is introduced, the average money balances held by firms will be reduced, according to (2.10), and the average time between financial transactions will be increased, according to (2.12). If firms react optimally, there will also be changes in the values of h and z (and changes in m^* and D resulting from those changes), but there is no need to calculate these, for by the envelope theorem the welfare effect of a small increase in the rate of inflation is simply

$$\frac{dC}{d\pi} = -\frac{\gamma}{D^2}\frac{\partial D}{\partial \pi} + r\frac{\partial m^*}{\partial \pi} \qquad (2.13)$$

Because $\frac{\partial D}{\partial \pi} > 0$ and $\frac{\partial m^*}{\partial \pi} < 0$, it is clear that moderate inflation will reduce firms' costs of managing the cash balances they need for transactions. Using (2.3), (2.5), (2.10), and (2.12), the exact welfare gain is seen to be

$$- \frac{dC}{d\pi} = \frac{7}{18}\gamma + r\left[\frac{4}{729}\ \frac{h^3}{\sigma^2}\right]$$
$$= \frac{1}{2}\gamma$$

Hence a result like that obtained in the text holds here as well. The optimal rate of inflation is some rate greater than zero. It is clear that uncertainty about the timing of one's expenditures, so that different agents turn out to have different degrees of need for liquidity, is essential to the result. As argued in the text, it is the redistributive aspect of perfectly anticipated inflation that accounts for the welfare gain -- by redistributing cash balances from those with less need for liquidity to those with more, inflation provides insurance which all would have desired <u>ex ante</u>. Inflation adds to the cash balances of those firms that currently have $m < m^*$, while depleting the cash balances of those firms that currently have $m > m^*$, and in so doing helps all firms maintain a more steady level of cash balances, so that costly financial transactions need occur less frequently.

FOOTNOTES

[1] The terminology is, of course, that of Keynes (1936, chap. 13).

[2] Hahn (1982, pp. 26-27) urges the development of a theory of liquidity with these features as well.

[3] A very similar model is presented by Diamond and Dybvig (1981), although their purpose is to explain the role of banks. In their model, as in that of Goldman, the return on the illiquid asset in the case of an investor who sells early is exogenously given. The present treatment differs in presenting a full equilibrium model, in which the available return to an investor who sells early is determined in the market for such intermediate goods.

[4] Stochastic opportunities to purchase consumption goods are important, for example, in the search model of Diamond (1982).

[5] One may suppose that capital is specific to a particular production process and not easily portable, whereas the final produce, is easy to transport and is a perfect substitute for goods produced elsewhere. This is a property of most real investment projects, though less a property of the kinds of financial assets held by households. The model might therefore more appropriately have been cast as a model of liquidity demand on the part of firms, although, like most formal models in monetary

theory, it is presented in terms of a decision by consumers to hold cash balances.

It is also necessary to assume that no financial intermediaries exist, such as the banks in the Diamond-Dybvig model, that invest depositors' funds in the illiquid investment projects and offer in exchange a "demand deposit" contract, which allows depositors a positive return even if they wish to withdraw their funds early. If it were possible for early consumers to costlessly borrow against the future yield of their illiquid investment projects, the liquidity demand discussed here would not exist. One might suppose that the quality of the capital is hard to assess - lenders will not accept it as collateral because they cannot distinguish an investor who wishes to sell early because he wishes to consume early from an investor who wishes to sell out because he has discovered his plant is faulty - whereas the quality of the final product is easily determined. In this case the transaction costs may be taken to be costs of appraising the quality of the capital, costs which would have to be undertaken by a lender as well as a buyer. Perhaps leading to an early consumer would leave him with no incentive to bring the project properly to fruition, so that lenders, aware of the moral hazard problem, will not lend. Or perhaps there are simply significant transaction costs associated with borrowing.

[6] Throughout this essay, only perfectly anticipated monetary policy, and perfectly anticipated inflation (or deflation), is considered.

[7] See, e.g., Bailey (1956) or Fischer (1981).

[8] For recent examples, see Jovanovic (1982), Rotemberg (1982), and Fischer (1983).

[9] For another example, see Gale (1983). In his model, firms are uncertain about the demand for their product, because they do not all sell in a common market. A policy of government purchases, applied uniformly to all markets, has the effect of reducing demand fluctuations. The reduced uncertainty allows greater employment. Gale shows that in such a model one can derive the traditional Keynesian government expenditure "multiplier", without any assumption of non-market-clearing prices or irrational behavior.

[10] Since preferences are assumed to be state-dependent, the reference to "expected utility maximization" involves more than assuming the von Neumann-Morgenstern axioms to be satisfied. However, Kreps (1979) presents a set of axioms for preferences which imply the possibility of representing them as resulting from maximization of the expected value of a state-dependent utility function.

[11] This functional form for $W(c_1,c_2)$ is quite restrictive, but possesses the advantages of being familiar, having parameters with clear interpretations, and allowing the simple result stated as Proposition 1. This special utility function is not necessary for the most

important result of this paper - the welfare gains from moderate inflation - as is shown in section 4.

[12] What the monetary authority chooses, of course, is not π directly but a rate of nominal money growth, and each period's price level is then determined by the demand for money. The formulation given in the text is chosen so as to suppress all variables which are not the same every period, since I am interested only in steady states.

[13] Total supply of nominal money in period t is Mp_{t-1} held over from period t-1 plus Tp_t of new money distributed. The condition for market clearing is therefore $Mp_{t-1} + Tp_t = Mp_t$, or

$$\frac{T}{M} = \frac{p_t - p_{t-1}}{p_t} = \frac{\pi}{1+\pi}$$

[14] This unique n_1 value is a utility maximum, since utility (1) is a concave function of n_1:

$$\frac{d^2W}{dn_1^2} = \lambda(-\gamma c_1^{-\gamma-1})\left[\frac{1}{1+\pi} - (1+R)(1+\pi)(1-\tau)\right]^2$$
$$+ (1-\gamma)\beta(-\gamma c_2^{-\gamma-1})\left[(1+R) - \frac{1}{(1+\pi)^2}\right]^2 < 0$$

[15] This relation is implied by (5) and (6), in accordance with Walras' Law.

[16] Again, this n_1 value is a utility maximum because utility is a concave function of n_1:

$$\frac{d^2W}{dn_1^2} = \lambda(-\gamma c_1^{-\gamma-1})[\frac{1+R}{q} - q(1-\tau)^2 + (1-\lambda)\beta(-\gamma c_2^{-\gamma-1})[(1+R)$$

$$- (\frac{1+R}{q})^2]^2 < 0$$

[17] The upper and lower bounds on q provided by (18) coincide in this case. Note that the bounds in (18) must hold with inequality only when $\tau > 0$.

[18] The right hand side of (11) is an increasing function of π, while (13) implies

$$\frac{\partial c_1}{\partial \pi} > 0, \quad \frac{\partial c_1}{\partial n_1} < 0, \quad \frac{\partial c_2}{\partial \pi} < 0, \quad \frac{\partial c_2}{\partial n_1} > 0$$

so that differentiation of (11) indicates that $\frac{dn_1}{d\pi} > 0$.

[19] In the Type A region, $M = \frac{1+\pi}{(1+\lambda\pi)}(2+\pi-\lambda)w$, so that equilibrium real balances increase as π increases. In the Type B region, M is given by (14), which is less than the Type A real balances would be (for the same rate of money growth) by a term which is monotonically increasing in π. Hence the rate of increase of M with increasing π is reduced by the holding of capital, even if $\frac{dM}{d\pi}$ does not at once become negative.

Note that $M(\pi)$ does not have the properties of the "money demand curve" invoked in "money triangle" welfare calculations, such as Bailey (1956). It is not uniformly downward-sloping, like a demand curve, because the increased transfer payments associated with an increased rate of money growth increase the budget which consumers

allocate between cash balances, other sorts of wealth, and consumption.

[20] Of course, as shown in Figure 2, if investment is possible (and $R > 0$), Type A equilibrium never exists for $\pi = 0$, no matter how large the transaction costs. Still, among Type A equilibria (if investment projects are ignored) the one with $\pi = 0$ is the best, a result which may usefully be contrasted with that which obtains in the case of the model with uncertainty.

[21] It follows from (29) that

$$\frac{dW}{d\pi} = W_1 \frac{dc_1}{d\pi} + W_2 \frac{dc_2}{d\pi} = (W_2 - W_1) \frac{d}{d\pi}\left(\frac{m_2}{1+\pi}\right)$$

Hence (17) implies that at $\pi = 0$, $\frac{dW}{d\pi} = 0$. However, a second-order effect may be calculated, namely

$$\frac{1}{\pi} \frac{dW}{d\pi}\Bigg|_{\pi = 0} = \frac{W_1^2}{W_{11} - 2W_{12} + W_{22}} < 0$$

which is negative because of the quasi-concavity of $W(c_1, c_2)$.

[22] The possibility of productive investment is ignored here. Compare equation (12) above.

[23] Differentiation of (31) and (32) and substitution into (33) yields

$$\frac{1}{\pi}\frac{dW}{d\pi}\Bigg|_{\pi = 0} = [W_{11}(W_1)^{-2} - 2W_{12}(W_1W_2)^{-1} + W_{22}(W_2)^{-2}]^{-1} < 0$$

which is negative because of the quasi-concavity of $W(c_1, c_2)$.

[24] Of course, the present paper considers only the comparative welfare properties of alternative steady states. A different problem is faced by a monetary authority considering a change in the rate of money growth. The dynamic effects of such a change and their welfare evaluation, are not taken up here.

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