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Developing ridge estimators for the extended Poisson-Tweedie regression model: Method, simulation, and application

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ABSTRACT

The extended Poisson-Tweedie (EPT) regression model is one of the count data regression models. It's a more flexible model in count data; since EPT model can be used in two situations: over-dispersion or under-dispersion but under assumption that this model uses second moment to be more flexible model. However, when the predictor (explanatory) variables of the EPT model are highly correlated, this means that there is the multicollinearity problem in the model. It causes inflation of the standard error of the maximum likelihood estimates and makes some of the significant variables insignificant. To handle and reduce the impact of the multicollinearity problem in the EPT model, we developed ridge estimators for this model. A theoretical comparison between the maximum likelihood and the proposed ridge estimators is done. The efficiency of the estimators is evaluated using the mean squared error (MSE). The simulation study and real-life application are used to evaluate the proposed estimators. We examined the performance of seven ridge estimators of the biasing parameter (k) to determine the most appropriate ridge estimator for this model. The simulation and application results showed that the proposed ridge estimator is superior to the maximum likelihood estimator, as it has the smallest MSE.

Introduction

Count data regression models are applied with data that has overdispersion and under-dispersion. The Poisson regression, negative binomial regression, and Conway-Maxwell Poisson (COMP) regression models are all included in these models. There is another model in the count data model family that is very flexible, called the Poisson-Tweedie model. The Poisson model assumes that the mean and variance of the data are equal, and this condition is referred to as equi-dispersion. However, count data can demonstrate other characteristics, such as under-dispersion and over-dispersion. So, the Poisson model makes incorrect standard error (SE) of the regression coefficients in both cases, resulting in potentially misleading inferences. But the Poisson-Tweedie model is more flexible than Poisson model. So, the Poisson-Tweedie model is used to deal with over-dispersion [9,13], but it has drawback, it cannot handle under-dispersed count data. For estimate and inference, this model uses second-moment assumptions, such as expectation and

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variance, and it allows to extend the Poisson-Tweedie model to deal with under-dispersion count data by accepting negative values for the dispersion parameter. So, the extended Poisson-Tweedie (EPT) model is more flexible to use with data suffering from over-dispersion and under-dispersion [9,13].

There is a problem called multicollinearity, it happens when explanatory variables are correlated with each other. It makes the standard error of the estimates is large [14]. To handle this problem, Hoerl and Kennard [14] introduced the ridge estimator for the linear regression model. Månsson and Shukur [22] and Muniz and Kibria [23] proposed the ridge regression estimator (RRE) for the Poisson model. Månsson [21] suggested the RRE for the negative binomial regression model. Türkan and Özel [29] proposed the modified jackknifed Poisson ridge regression estimator. Kaçiranlar and Dawoud [17] proposed some ridge parameters for the Poisson model. Rashad and Algamal [25] developed a new RRE for the Poisson regression model. Yehia [30] proposed the restricted Poisson ridge regression estimator. A new class of ridge-type estimator for the Poisson model is proposed by [33].

Recently, some papers developed the RRE for handling multicollinearity in different recent count regression models. For example, Sami et al. [27] proposed the RRE for the Conway-Maxwell-Poisson (COM-Poisson) model, Amin et al. [6] and Algamal, et al. [32] proposed the RRE and the Modified Jackknifed Ridge Estimator for the Bell regression model respectively, Algamal, et al. [5] proposed the RRE for the zero-inflated Bell regression model, Algamal et al. [4] developed the modified jackknifed ridge estimator for the COM-Poisson model, while the RRE for the Poisson-inverse Gaussian model is proposed by [8].

As far as we know, no previous research paper has examined the issue of multilinearity in the EPT regression model. Therefore, the main objective of our paper is to develop ridge estimators for the EPT regression model. The proposed estimator will efficiently gain an advantage over the existing estimator (maximum likelihood estimator). The superiority of the proposed ridge estimator will be established through simulated datasets and a real-life application.

The rest of this paper is structured as follows. In section 2, the methodology includes the EPT regression and the proposed estimators as well as the theoretical comparison between the proposed estimator with the traditional one. In section 3, it presented estimating the biasing parameter (optimal values). In section 4, we introduced the simulation study to demonstrate the performance of the proposed estimators. We apply the proposed estimator to real-life application, in section 5. In section 6, some conclusions are presented.

Methodology

Poisson-Tweedie distribution

The Poisson-Tweedie distribution includes Tweedie distributed random effects on the observation level of Poisson random variables. The Poisson-Tweedie family is defined by [9,13,16]:

$$Y|Z \sim Poisson(Z) \tag{1}$$

$$Z \sim Tw_{\nu}(\mu,\varphi), \tag{2}$$

where $Tw_{\nu}(\mu,\Phi)$ denotes a Tweedie distribution with probability function given by

$$f_Z(z;\mu,\varphi,\nu) = a\ (z,\varphi,\nu)exp\{(z\psi - k_{\nu}(\psi))\ /\ \varphi\} \tag{3}$$

where $\nu \in (-\infty,0] \cup [1,\infty)$ is an index determining the distribution, $\mu = k_{\nu}(\dot{\psi})$ is the expectation, $\varphi > 0$ is the dispersion parameter, ψ is the canonical parameter and $k_{\nu}(\psi)$ is the cumulant function. Moreover, $Var(Z) = \varphi Var(\mu)$ where $Var(\mu) = k_{\nu}(\ddot{\psi})$ is the variance function. Power variance functions of the form $V(\mu) = \mu^{\nu}$ characterize Tweedie densities. The value of the power parameter corresponds to the positive, non-negative and real values. In these situations, $\mu \in \Omega$, where Ω is the convex support. Finally, the support for $\nu < 0$ corresponds to real values, but the expectation μ is positive. To make $Tw_{\nu}(\mu,\varphi)$ non-negative, we required $\nu \geq 1$.

Apart from the particular situations corresponding to the Gaussian ($\nu = 0$), Poisson ($\varphi = 1$ and $\nu = 1$), non-central gamma ($\nu = 3/2$), gamma ($\nu = 2$), and inverse Gaussian ($\nu = 3$) distributions, the function $a\ (z,\varphi,\nu)$ cannot be stated in a closed form [31]. This distribution is well suited to non-negative data with a probability mass of zero and a right-skewed distribution [31]. The Poisson-Tweedie model is a factorial dispersion model with an over-dispersed probability mass function when $\nu > 1$.

$$f(y;\mu,\varphi,\nu) = \int_0^{\infty} \frac{z^y exp - z}{y!} a\ (z,\varphi,\nu)exp\{(z\psi - k_{\nu}(\psi))\ /\ \varphi\}\ dz \tag{4}$$

When $\nu = 2$, i.e., a Poisson gamma mixture, and when $\nu = 1$, the integral in Eq. (4) is replaced by a sum, and NTA distribution is obtained. The compound Poisson ($1 < \nu < 2$), factorial discrete positive stable ($\nu > 2$), and Poisson-inverse Gaussian ($\nu = 3$) distributions are also special instances. So, the integral in Eq. (4) has no closed-form portion of the specific case corresponding to the negative binomial distribution [31].

Extended Poisson-Tweedie model

The Poisson-Tweedie distribution is a very flexible family of count distributions, but it has two major drawbacks: it cannot handle under-dispersion count data and its probability mass function is an intractable integral, implying that estimation using the maximum

likelihood method is computationally intensive for practical data analysis [9].

Despite these drawbacks, Jørgensen and Kokonendji [16] showed that one can use the factorial cumulant function, $Y \sim PTw_{\nu}(\mu, \varphi)$, $E(Y) = \mu$ and $Var(Y) = \mu + \varphi\mu^{\nu}$. Bonat et al. [9] employed just second-moment assumptions to specify a model such as mean and variance, which expectation and variance are defined as

$$E(y_i) = \mu_i = g^{-1}\left(x_i^T\beta\right) \tag{5}$$

$$Var(y_i) = C_i = \mu_i + \varphi\mu_i^{\nu} \tag{6}$$

where x_i and β are $(p \times 1)$ vectors of known covariates and unknown regression parameters and g is the logarithm link function. $\theta(\beta^T, \gamma^T)^T$, where $\gamma = (\varphi, \nu)$, parametrizes the regression model specified in Eqs. (5) and (6).

The only requirement for a valid model, based on second-moment assumptions, is that $Var(y_i) > 0$, thus

$$\varphi > -\mu_i^{(1-\nu)}. \tag{7}$$

This illustrates that negative values for the dispersion parameter are allowed to some extent. As a result, the Poisson-Tweedie model can be used to handle under-dispersed count data. Furthermore, the extended Poisson-Tweedie model has the same formulation as the popular quasi-binomial and quasi-Poisson models. The extended Poisson-Tweedie regression model is also a reparameterization of the popular quasi-Poisson regression model with $\nu = 1$. It's important to note that, unlike in the Gamma-Count and COM-Poisson distributions, the relation between mean and variance in this case is proportional to the dispersion parameter. As a result, we predict the extended Poisson-Tweedie model to produce regression coefficients that are quite similar to those of the Gamma-Count and COM-Poisson regression models for $\varphi > 0$ and $\nu = 1$.

Estimation and inference

For the estimate of the regression and dispersion parameters, Bonat et al. [9] merged the quasi-score and Pearson estimating functions, based on ideas from [10,15]. The quasi-score function for β is as follows:

$$\psi_{\beta}(\beta,\ \gamma) = \left(\sum_{i=1}^{n}\frac{\partial\mu_i}{\partial\beta_1}C_i^{-1}(y_i - \mu_i), \cdots,\ \sum_{i=1}^{n}\frac{\partial\mu_i}{\partial\beta_p}C_i^{-1}(y_i - \mu_i)\right)^T \tag{8}$$

where $\frac{\partial\mu_i}{\partial\beta_j} = \mu_i x_{ij}$ for $j = 1, \cdots,\ p$. The entry $(j,\ k)$ of the $p \times p$ sensitivity matrix for ψ_{β} is given as

$$S_{\beta_{jk}} = E\left(\frac{\partial}{\partial\beta_k}\psi_{\beta_j}(\beta,\ \gamma)\right) = -\sum_{i=1}^{n}\mu_i x_{ij}C_i^{-1}x_{ik}\mu_i. \tag{9}$$

The entry $(j,\ k)$ of the $p \times p$ variability matrix for is similarly given by:

$$V_{\beta_{jk}} = Cov\left(\psi_{\beta_j}(\beta,\ \gamma),\ \psi_{\beta_k}(\beta,\ \gamma)\right) = \sum_{i=1}^{n}\mu_i x_{ij}C_i^{-1}x_{ik}\mu_i \tag{10}$$

the Pearson estimating function for the dispersion parameters is presented as follows:

$$\psi_{\gamma}(\gamma,\beta) = \left(-\sum_{i=1}^{n}\frac{\partial C_i^{-1}}{\partial\varphi}\left[(y_i - \mu_i)^2 - C_i\right], -\sum_{i=1}^{n}\frac{\partial C_i^{-1}}{\partial\nu}\left[(y_i - \mu_i)^2 - C_i\right]\right)^T \tag{11}$$

Note that Pearson estimating functions based on squared residuals $(Y_i - \mu_i)^2$ with expected value C_i are unbiased estimating functions.

The dispersion parameters' entry $(j,\ k)$ in the $2\ \times\ 2$ sensitivity matrix is defined by:

$$S_{\gamma_{jk}} = E\left(\frac{\partial}{\partial\gamma_k}\psi_{\gamma_j}(\gamma,\ \beta)\right) = -\sum_{i=1}^{n}\frac{\partial C_i^{-1}}{\partial\gamma_j}C_i\frac{\partial C_i^{-1}}{\partial\gamma_k}C_i \tag{12}$$

where γ_1 and γ_2 refer for either φ or ν.

Similarly, the sensitivity matrix's cross entries are defined as

$$S_{\beta_j\gamma_k} = E\left(\frac{\partial}{\partial\gamma_k}\psi_{\beta_j}(\beta,\ \gamma)\right) = 0 \tag{13}$$

And

$$S_{\gamma_j\beta_k} = E\left(\frac{\partial}{\partial\beta_k}\psi_{\gamma_j}(\gamma,\ \beta)\right) = -\sum_{i=1}^{n}\frac{\partial C_i^{-1}}{\partial\gamma_j}C_i\frac{\partial C_i^{-1}}{\partial\beta_k}C_i \tag{14}$$

Finally, the joint sensitivity matrix for the parameter vector θ is defined as

$$S_\theta = \begin{pmatrix} S_\beta & 0 \\ S_{\gamma\beta} & S_\gamma \end{pmatrix} \tag{15}$$

Eqs. (9), (12-14) define the entries in Eq. (15).

The asymptotic variance of the estimating function estimators was calculated, denoted by $\widehat{\theta}$, which usual form for a parameter vector θ is $J_\theta^{-1} = S_\theta^{-1} V_\theta S_\theta^{-T}$, where $-T$ denotes inverse transpose. The form of the variability matrix θ is

$$V_\theta = \begin{pmatrix} V_\beta & V_{\beta\gamma} \\ V_{\gamma\beta} & V_\gamma \end{pmatrix} \tag{16}$$

where $V_{\gamma\beta} = V_{\beta\gamma}^T$ and V_γ are determined by y_i's third and fourth moments, respectively. To avoid this dependence on higher-order moments, the empirical versions of $V_{\gamma\beta}$ and V_γ are

$$\widetilde{V}_{\gamma_j k} = \sum_{i=1}^{n} \psi_{\gamma_j}(\gamma,\beta)_i \psi_{\gamma_k}(\gamma,\ \beta)_i \ \textit{and}\ \widetilde{V}_{\gamma_j \beta_k} = \sum_{i=1}^{n} \psi_{\gamma_j}(\gamma,\beta)_i \psi_{\beta_k}(\gamma,\ \beta)_i \tag{17}$$

Finally, based on [15], the well-known asymptotic distribution of $\widehat{\theta}$ is

$$\widehat{\theta} \sim N\left(\theta,\ J_\theta^{-1}\right),\ \text{where}\ J_\theta^{-1} = S_\theta^{-1} V_\theta S_\theta^{-T} \tag{18}$$

$\psi_\beta = 0$ and $\psi_\gamma = 0$ to solve the system of equations. The modified chaser algorithm was defined by [15]:

$$\beta^{(i+1)} = \beta^{(i)} - S_\beta^{-1} \psi_\beta\left(\beta^{(i)}, \gamma^{(i)}\right) \tag{19}$$

$$\gamma^{(i+1)} = \gamma^{(i)} - a\ S_\gamma^{-1} \psi_\gamma\left(\beta^{(i+1)}, \gamma^{(i)}\right) \tag{20}$$

where a is the tuning constant to control the step-length. Using iteratively weighted least squares to estimate β as follows:

$$\widehat{\beta}_{EPTML} = (Q)^{-1} X^T \widehat{W} \widetilde{y}^{(i)} \tag{21}$$

Where $Q = X^T \widehat{W} X$, the i^{th}element of $\widetilde{y}^{(i)}$ is $\eta_j^{(i)} + \frac{(y_i - \mu_j^{(i)})}{\partial g^{-1}(\eta_j^{(i)})/\partial \eta_j^{(i)}}$, $\eta_j^{(i)} = x_j^T \widehat{\beta}^{(i)}$, $\mu_j^{(i)} = g^{-1}(x_j^T \widehat{\beta}^{(i)})$ and $W = diag\left(\frac{\mu_1^2}{\partial \mu_1}, \ldots, \frac{\mu_n^2}{\partial \mu_n}\right) = diag\left(\frac{\mu_1}{\theta}, \ldots, \frac{\mu_n}{\theta}\right)$.

The MSE of $\widehat{\beta}_{EPTML}$ is

$$MSE\ (\widehat{\beta}_{EPTML}) = E(\widehat{\beta}_{EPTML} - \beta)^T (\widehat{\beta}_{EPTML} - \beta) = \widehat{\varphi}\, tr\left(UB^{-1}U\right) = \ \widehat{\varphi} \sum_{j=1}^{p} \frac{1}{b_j} \tag{22}$$

Where $tr(.)$ is the trace of the matrix, $B = diag(b_1, \ldots, b_p) = UQU^T$, U represents the orthogonal matrix, whose columns are eigenvectors of Q, $\alpha = (\alpha_1, \ldots, \alpha_p)^T = U^T\beta$, b_j is the jth eigenvalue of the Q matrix, and $\widehat{\varphi}$ is the ML estimator of φ.

Proposed estimator: extended Poisson-Tweedie ridge (EPTR) estimator

Ridge regression was introduced by Hoerl and Kennard [14] to overcome the problem of multicollinearity. it is a shrinkage method that reduces large variance by shrinking all regression coefficients toward zero by adding a positive amount to the diagonal elements of X^TX. The ridge estimator is biased and has smaller mean squared error than MLE.

To handle the problem that is caused by multicollinearity in the EPT model, we propose the EPTR estimator. The derivation of the proposed EPTR estimator starts by observing that the EPTML method minimizes the weighted sum of squared error (WSSE) approximately. Hence, $\widehat{\beta}_{EPTML}$ can be shown as an optimal estimator according to the WSSE. If we select another estimator $(\widehat{\beta})$ of the parameter β, we can write the WSSE for this estimator as

$$L = (\widetilde{y} - \widehat{\beta})^{\mathrm{T}} (\widetilde{y} - \widehat{\beta}) = (\widetilde{y} - \widehat{\beta}_{EPTML})^T (\widetilde{y} - \widehat{\beta}_{EPTML}) + (\widehat{\beta} - \widehat{\beta}_{EPTML})^T Q\ (\widehat{\beta} - \widehat{\beta}_{EPTML}) = L_{min} + L(\widehat{\beta}), \tag{23}$$

where $L(\widehat{\beta})$ is the WSSE's increase when $\widehat{\beta}_{EPTML}$ is replaced with estimator $\widehat{\beta}$. The $\widehat{\beta}'\widehat{\beta}$'s length is minimized to $L(\widehat{\beta}) = L_0$ in order to find the EPTR estimator. As a Lagrange problem, this may be given as

$$MinimizeF = \widehat{\beta}^T \widehat{\beta} + (1/k)\left((\widehat{\beta} - \widehat{\beta}_{EPTML})^{\mathrm{T}} Q\ (\widehat{\beta} - \widehat{\beta}_{EPTML}) - L_0\right), \tag{24}$$

where (1/k) is called Lagrange multiplier. Differentiating Eq. (24) to $\widehat{\beta}$ and the result equal zero yields

$$\frac{\partial F}{\partial \widehat{\beta}} = 2\widehat{\beta} + (1/k)(2Q\widehat{\beta} - 2Q\widehat{\beta}_{EPTML}). \tag{25}$$

With solving Eq. (25) to $\widehat{\beta}$, we have the proposed EPTR estimator as

$$\widehat{\beta}_{EPTR} = \left(Q + kI_p\right)^{-1} Q\widehat{\beta}_{EPTML};\ k > 0 \tag{26}$$

where I_P $p \times p$ is the identity matrix. The bias and the covariance of EPTR:

$$Bias(\widehat{\beta}_{EPTR}) = -kUB_k^{-1}\alpha \tag{27}$$

$$COV(\widehat{\beta}_{EPTR}) = \widehat{\varphi}UB_k^{-1}BB_k^{-1}U^T \tag{28}$$

$$MSEM(\widehat{\beta}_{EPTR}) = COV(\widehat{\beta}_{EPTR}) + Bias(\widehat{\beta}_{EPTR})Bias(\widehat{\beta}_{EPTR})^T \tag{29}$$

$$= \widehat{\varphi}UB_k^{-1}BB_k^{-1}U^T + \left(-kUB_k^{-1}\alpha\right)\left(-kUB_k^{-1}\alpha\right)^T \tag{30}$$

where $B_k = diag(b_1 + k, \ldots, b_p + k)$. So the MSE of the EPTR estimator, by using the $tr(\cdot)$ operator on Eq. (30) is defined as:

$$MSE(\widehat{\beta}_{EPTR}) = tr(MSEM(\widehat{\beta}_{EPTR})) = \widehat{\varphi}\sum_{j=1}^{p}\frac{b_j}{\left(b_j + k\right)^2} + k^2\sum_{j=1}^{p}\frac{\alpha_j^2}{\left(b_j + k\right)^2}. \tag{31}$$

The following Theorem indicates the superiority of the proposed EPTR estimator.

Theorem 1. *The EPTR estimator is better than EPTML if* $\widehat{\varphi}\left(2 + \frac{k}{b_j}\right) - k\alpha_j^2 > 0\ \forall\, j = 1, .., p$, *where* $k > 0$.

Proof.

The difference among the MSE functions of the EPTML and EPTR estimators:

$$MSE\ (\widehat{\beta}_{EPTML}) - MSE(\widehat{\beta}_{EPTR}) = \widehat{\varphi}\mathrm{B}^{-1} - \widehat{\varphi}U\mathrm{B}_k^{-1}\mathrm{BB}_k^{-1}U^T - \left(kU\mathrm{B}_k^{-1}\alpha\right)\left(kU\mathrm{B}_k^{-1}\alpha\right)^T \tag{32}$$

So, we can rewrite the previous equation as

$$\mathrm{MSE}\ (\widehat{\beta}_{EPTML}) - MSE(\widehat{\beta}_{EPTR}) = \sum_{j=1}^{p}\left(\frac{\widehat{\varphi}}{b_j} - \frac{\widehat{\varphi}b_j + k^2\alpha_j^2}{\left(b_j + k\right)^2}\right), \tag{33}$$

then $\mathrm{MSE}(\widehat{\beta}_{EPTML}) - MSE(\widehat{\beta}_{EPTR}) > 0$ if $\widehat{\varphi}(b_j + k)^2 - \widehat{\varphi}b_j^2 - k^2b_j\alpha_j^2 > 0 \rightarrow 2\widehat{\varphi}b_jk + \widehat{\varphi}k^2 > b_jk^2\alpha_j^2 \rightarrow \widehat{\varphi}(2b_jk + k^2) > b_jk^2\alpha_j^2 \rightarrow \widehat{\varphi}\left(2 + \frac{k}{b_j}\right) > k\alpha_j^2$. Then $\mathrm{MSE}(\widehat{\beta}_{EPTML}) > MSE(\widehat{\beta}_{EPTR})$ if $\widehat{\varphi}\left(2 + \frac{k}{b_j}\right) - k\alpha_j^2 > 0\ \forall\, j = 1, .., p$.

Simulation study

Simulation design

The Monte Carlo Simulation Study is designed to evaluate the performance of the proposed estimators. Also, designed to explore the flexibility of the extended Poisson-Tweedie models to tackle over and under-dispersion count data. So, this simulation is designed to evaluate the performance of the proposed estimators with the extended Poisson-Tweedie model to deal with multicollinearity in the case of over- and under-dispersion. The performance of the proposed estimators is compared with each other using MSE under different conditions (see Table 1). R software was used to conduct our simulation study based on the "*flexcm*" package.

We conducted the simulation experiments using different values of n, p, ρ, and φ as follows:

Step 1. The predictor (explanatory) variables were generated as follows [19,20,24,28,34,35]:

$$x_{ij} = \xi_{ij}\left(1 - \rho^2\right)^{1/2} + \rho\xi_{ip},\ i = 1, \ldots, n; j = 1, \ldots, p \tag{34}$$

Table 1
Simulation factors.

Factor	Notation	Values
Sample size	n	30, 50, 100, 150, 200, 300, 400
Number of regression parameters	p	2, 4, 6
Dispersion parameter	φ	4,8
Degree of correlation	ρ	0.85, 0.90, 0.95, 0.99

where ξ_{ij} are generated from standard uniform and ρ is the correlation between the predictor variables. Since we are interested in the effect of multicollinearity, the correlation value can be considered the most important value.

Step 2. The response variable (y_i) is generated as follows:

$$y_i \sim Tw_v(\mu_i, \varphi);\ \mu_i = \exp(x_i^T\beta),\ i = 1, \cdots, n. \tag{35}$$

We choose the values of β vector (with intercept) under the assumption that $\sum_{j=1}^{p}\beta_j^2 = 1; \beta_1 = \ldots = \beta_p$ according to [7,12,18,20,26].

Step 3. We used the simulated MSE criterion to evaluate the performance of the proposed EPTR estimator by using the different combination n, p, ρ, φ and the output data (y_i) is recurrent $A = 1000$ times to calculate the MSE criterion as follows:

$$MSE(\widehat{\beta}) = \frac{1}{A}\sum_{t=1}^{A}(\widehat{\beta}_t - \beta)^T(\widehat{\beta}_t - \beta) \tag{36}$$

where $\widehat{\beta}_t$ is the estimate of β in the t^{th} replication.

Based on previous studies on ridge estimation in different regression models, we can use the following estimators for k parameter in the EPTR estimator

$$\widehat{k}_1 = \frac{p\widehat{\varphi}}{\sum_{j=1}^{p}\widehat{\alpha}_j^2}, \tag{37}$$

where $\widehat{\alpha}_j$ is an element of the estimated vector $\widehat{\alpha} = (\widehat{\alpha}_1, \ldots, \widehat{\alpha}_p)^T = U^T\widehat{\beta}_{EPTML}$.

$$\widehat{k}_2 = \frac{\widehat{\varphi}}{\max_j(\widehat{\alpha}_j^2)}, \tag{38}$$

$$\widehat{k}_3 = \left(\prod_{j=1}^{p}\frac{\widehat{\varphi}}{(1+\widehat{k}_2)\widehat{\alpha}_j^2}\right)^{1/p}, \tag{39}$$

Table 2
MSEs of EPTML and EPTR estimators when $p = 2$, $\varphi = 4$.

n	ρ	EPTML	EPTR						
			$\widehat{k}_1$	$\widehat{k}_2$	$\widehat{k}_3$	$\widehat{k}_4$	$\widehat{k}_5$	$\widehat{k}_6$	$\widehat{k}_7$
30	0.85	3.9567	1.6962	2.1594	1.6118	0.7989	0.5218	0.9665	0.1307
	0.90	9.1717	3.7251	4.9643	2.2991	0.4407	0.3012	0.8961	0.1562
	0.95	12.6302	5.1214	6.8705	2.7574	0.4129	0.2341	0.8755	0.1293
	0.99	45.7273	18.9317	25.4772	4.9902	0.2457	0.0957	0.4396	0.0950
50	0.85	2.3558	1.0514	1.3124	1.1933	0.7078	0.5585	0.8234	0.1767
	0.90	4.5971	1.8522	2.4400	1.5172	0.6503	0.3709	0.7918	0.0889
	0.95	9.3535	3.8554	5.1364	2.1923	0.4729	0.2090	0.7564	0.0686
	0.99	33.9151	13.6696	18.3864	3.9248	0.2579	0.0712	0.4468	0.0581
100	0.85	1.5624	0.6792	0.8347	0.9133	0.6656	0.5723	0.6682	0.1829
	0.90	1.5034	0.6049	0.7543	0.8404	0.5868	0.5062	0.6034	0.1607
	0.95	5.8604	2.1992	2.9917	1.5378	0.4790	0.2488	0.6914	0.0558
	0.99	21.7349	7.5074	10.8772	2.2281	0.2874	0.0736	0.4071	0.0259
150	0.85	0.8418	0.4015	0.4667	0.6005	0.4924	0.4619	0.4706	0.2137
	0.90	1.5701	0.6778	0.8388	0.8955	0.6387	0.5403	0.6415	0.1637
	0.95	2.4741	0.9625	1.2547	1.1069	0.7034	0.5164	0.7210	0.0994
	0.99	14.0902	5.0286	7.1414	1.9685	0.3814	0.1120	0.5041	0.0212
200	0.85	0.6094	0.3050	0.3486	0.4736	0.4092	0.3968	0.3857	0.2281
	0.90	0.9334	0.4252	0.5011	0.6459	0.5202	0.4862	0.4990	0.2124
	0.95	1.8441	0.7193	0.9197	0.9421	0.6463	0.5227	0.6455	0.1221
	0.99	9.6749	3.4248	4.8466	1.6994	0.4404	0.1604	0.5701	0.0221
300	0.85	0.4322	0.2411	0.2687	0.3606	0.3339	0.3287	0.3055	0.2161
	0.90	0.8006	0.3808	0.4421	0.5933	0.4951	0.4749	0.4728	0.2423
	0.95	1.7088	0.7660	0.9418	0.9738	0.6987	0.5714	0.6903	0.1484
	0.99	7.4310	2.8838	3.9491	1.8250	0.6851	0.2971	0.7918	0.0299
400	0.85	0.2885	0.1591	0.1824	0.2387	0.2298	0.2265	0.2019	0.1590
	0.90	0.6247	0.3178	0.3593	0.4875	0.4263	0.4141	0.3959	0.2427
	0.95	1.2968	0.5709	0.6953	0.8152	0.6229	0.5553	0.6099	0.1864
	0.99	5.9663	2.2080	3.0520	1.5233	0.6221	0.2933	0.7022	0.0302

$$\widehat{k}_4 = \min_j\left(\frac{1}{\widehat{m}_j}\right), \tag{40}$$

where $\widehat{m}_j = \sqrt{\frac{\max_j(\widehat{\varphi}.b_j)}{\widehat{\varphi}(n-p)+\widehat{\alpha}_j^2 \max_j(b_j)}}$; $j = 1,\ldots,p$.

$$\widehat{k}_5 = \underset{j}{\text{median}}\left(\frac{1}{\widehat{m}_j}\right), \tag{41}$$

$$\widehat{k}_6 = \left(\frac{1}{p}\sum_{j=1}^{p}\frac{\widehat{\varphi}}{\widehat{\alpha}_j^2}\right)^{1/p}, \tag{42}$$

$$\widehat{k}_7 = \max_j\left(\frac{\widehat{\varphi}}{\sqrt{1/\widehat{\alpha}_j^2}}\right). \tag{43}$$

We can present the main results of the simulation stated in Tables 2,3,4,5,6,7 as follows:

1. The performance of the EPTML estimator is the worst among all the suggested estimators, as expected. So, it is not recommended for regression coefficient estimation in the EPT model when multicollinearity exists.
2. The estimators' MSEs increase if the correlation degree (ρ), the number of explanatory variables (p), and the dispersion parameter (φ) are increased.
3. The estimators' MSEs decrease when the sample size (n) is increased when the other factors are kept fixed.
4. In case where the model has multicollinearity, the proposed EPTR estimator with all seven suggested estimators for the biasing parameter (k) is better than the PML estimator for all ρ, p, φ, and n values.
5. The performance of the proposed EPTR estimator with $\widehat{k}_7$ followed by $\widehat{k}_5$, $\widehat{k}_4$ and $\widehat{k}_6$ is better than the EPTML and the EPTR with other suggested estimators for k.

Table 3
MSEs of EPTML and EPTR estimators when $p = 2$, $\varphi = 8$.

n	ρ	EPTML	EPTR						
			$\widehat{k}_1$	$\widehat{k}_2$	$\widehat{k}_3$	$\widehat{k}_4$	$\widehat{k}_5$	$\widehat{k}_6$	$\widehat{k}_7$
30	0.85	6.9598	3.2110	4.0587	2.8561	1.5253	0.9539	1.7887	0.1532
	0.90	26.101	7.9144	10.6898	4.1006	0.8660	0.5768	1.7015	0.2531
	0.95	21.4341	9.5855	12.4654	5.0357	0.7919	0.4278	1.7592	0.2236
	0.99	78.1284	36.2192	46.7136	10.2476	0.5558	0.1742	1.1065	0.1856
50	0.85	4.4116	2.0875	2.6167	2.1717	1.3430	0.9991	1.5250	0.1730
	0.90	8.1406	3.5479	4.6109	2.6038	1.2667	0.6577	1.4166	0.1018
	0.95	17.8325	8.4483	10.7977	4.5154	0.9989	0.3862	1.6081	0.1094
	0.99	56.4599	25.7124	32.9548	8.2023	0.6262	0.1404	1.1926	0.1214
100	0.85	2.7348	1.2026	1.5127	1.5547	1.2002	0.9953	1.1486	0.1462
	0.90	2.7874	1.1166	1.4479	1.4634	1.0741	0.8876	1.0503	0.1292
	0.95	10.6129	4.2121	5.7010	2.4736	0.9195	0.4285	1.1564	0.0697
	0.99	39.3135	15.0550	21.0238	3.6855	0.5795	0.1257	0.6887	0.0532
150	0.85	1.5439	0.7017	0.8472	1.0826	0.8923	0.8207	0.8358	0.1878
	0.90	2.9537	1.2810	1.6252	1.5944	1.1824	0.9455	1.1270	0.1195
	0.95	4.4076	1.7591	2.3258	1.8291	1.2875	0.8779	1.1915	0.0697
	0.99	26.4459	10.2778	14.2256	3.1987	0.7625	0.1851	0.8287	0.0349
200	0.85	1.1092	0.5179	0.6111	0.8578	0.7349	0.7034	0.6839	0.2216
	0.90	1.7160	0.7513	0.9189	1.1696	0.9535	0.8678	0.8901	0.1750
	0.95	3.4201	1.3442	1.7655	1.6315	1.2092	0.9185	1.1116	0.0808
	0.99	17.7438	6.7229	9.3335	2.7264	0.8591	0.2640	0.9124	0.0276
300	0.85	0.8043	0.4030	0.4633	0.6745	0.6125	0.5994	0.5576	0.2348
	0.90	1.4477	0.6283	0.7707	1.0438	0.8783	0.8279	0.8091	0.2041
	0.95	3.2621	1.4669	1.8616	1.7157	1.3122	1.0043	1.1994	0.0937
	0.99	13.4337	5.0263	7.0507	2.5710	1.2217	0.4822	1.1490	0.0230
400	0.85	0.4995	0.2359	0.2735	0.4265	0.3898	0.3837	0.3502	0.1773
	0.90	1.1588	0.5368	0.6426	0.8878	0.7743	0.7400	0.7055	0.2207
	0.95	2.3420	0.9477	1.2169	1.3962	1.1085	0.9591	1.0189	0.1261
	0.99	10.6946	4.0449	5.6040	2.2666	1.1201	0.4635	1.0351	0.0196

Table 4
MSEs of EPTML and EPTR estimators when $p = 4$, $\varphi = 4$.

n	ρ	EPTML	EPTR						
			$\hat{k}_1$	$\hat{k}_2$	$\hat{k}_3$	$\hat{k}_4$	$\hat{k}_5$	$\hat{k}_6$	$\hat{k}_7$
30	0.85	13.2994	4.9792	8.0497	3.1113	4.2031	2.4398	2.5604	0.2026
	0.90	25.2727	10.2606	16.0497	5.3546	3.3673	1.5639	2.7379	0.1327
	0.95	40.7291	16.4223	25.6670	8.1429	3.2316	1.0219	2.9735	0.0642
	0.99	219.7111	90.9574	140.7966	29.1613	0.9890	0.1174	1.4724	0.0476
50	0.85	6.9523	2.7706	4.2436	2.3222	3.4193	2.5424	2.2809	0.3918
	0.90	9.6071	3.6706	5.7999	2.6279	3.8642	2.5605	2.4394	0.2659
	0.95	16.4300	6.2397	9.9779	3.6944	4.0687	2.1235	2.7084	0.1203
	0.99	89.0436	32.7629	53.8991	11.4476	1.9300	0.3942	2.0137	0.0226
100	0.85	3.2078	1.3133	1.9276	1.4653	2.2976	2.0056	1.5788	0.5824
	0.90	5.2588	1.9825	3.0899	1.8825	2.9527	2.3496	1.9692	0.4031
	0.95	9.2983	3.4190	5.5142	2.6415	3.9018	2.6504	2.5774	0.2033
	0.99	37.5395	13.3603	22.1620	6.0620	3.4942	1.0760	2.5341	0.0235
150	0.85	2.1622	0.8646	1.2927	1.0630	1.7125	1.5771	1.1915	0.6208
	0.90	2.8988	1.1544	1.7252	1.3640	2.1579	1.9116	1.4861	0.5745
	0.95	7.3918	2.7340	4.3408	2.1914	3.5138	2.5042	2.2059	0.2551
	0.99	29.5948	10.2164	17.2696	5.0192	3.8866	1.3780	2.7117	0.0311
200	0.85	1.4753	0.6231	0.8987	0.8494	1.2647	1.1969	0.9287	0.6082
	0.90	2.3486	0.9648	1.4068	1.2409	1.8756	1.7218	1.3433	0.6422
	0.95	4.6244	1.6967	2.6682	1.7932	2.8835	2.3635	1.9347	0.3860
	0.99	23.0114	7.9590	13.3988	4.2304	3.9933	1.7394	2.7657	0.0440
300	0.85	0.9882	0.5560	0.6590	0.6461	0.8997	0.8731	0.7104	0.4680
	0.90	1.7528	0.7746	1.1025	1.0275	1.4939	1.4046	1.1169	0.6714
	0.95	2.8541	1.0799	1.6445	1.3052	2.1076	1.8748	1.4295	0.5226
	0.99	16.7415	6.0340	9.9205	3.8165	4.0463	2.1348	2.8257	0.0700
400	0.85	0.7778	0.5098	0.5472	0.5224	0.7202	0.7047	0.5786	0.3947
	0.90	1.1661	0.6046	0.7424	0.7318	1.0483	1.0098	0.8021	0.5127
	0.95	2.2110	0.8750	1.3189	1.1236	1.7832	1.6338	1.2471	0.5828
	0.99	10.3128	3.2550	5.6967	2.3295	3.5896	2.2741	2.2309	0.1143

Table 5
MSEs of EPTML and EPTR estimators when $p = 4$, $\varphi = 8$.

n	ρ	EPTML	EPTR						
			$\hat{k}_1$	$\hat{k}_2$	$\hat{k}_3$	$\hat{k}_4$	$\hat{k}_5$	$\hat{k}_6$	$\hat{k}_7$
30	0.85	25.5101	10.4329	16.2004	6.3481	7.8971	4.6036	5.7815	0.1526
	0.90	56.3170	23.2356	34.4135	11.1939	7.1875	3.2680	6.5871	0.1356
	0.95	82.5546	36.8237	55.1561	18.0367	6.9131	2.0562	7.7852	0.0774
	0.99	380.2680	179.2142	258.4073	62.6245	2.6469	0.2563	4.5638	0.0989
50	0.85	13.9195	6.2675	9.1417	5.2241	6.7077	4.8723	5.3719	0.2986
	0.90	19.1666	8.3036	12.3627	5.8454	7.6214	4.8784	5.7157	0.1936
	0.95	32.4578	14.5001	21.5150	8.5184	8.4896	4.2120	6.8452	0.0874
	0.99	173.8716	73.3788	112.7837	26.0098	4.6532	0.8660	5.5365	0.0437
100	0.85	6.0507	2.6068	3.7843	2.9529	4.2877	3.7770	3.3130	0.4520
	0.90	10.1863	4.1937	6.3428	3.8638	5.6704	4.4191	4.3205	0.2615
	0.95	17.9137	7.3902	11.2777	5.6377	7.7278	5.0778	5.9747	0.1165
	0.99	72.9076	30.1155	46.4756	12.8197	7.7667	2.2771	6.4696	0.0189
150	0.85	4.1034	1.6594	2.5489	2.0145	3.2254	3.0119	2.4219	0.5059
	0.90	5.5755	2.3088	3.4443	2.7599	4.1061	3.6773	3.1496	0.4305
	0.95	14.4083	5.8320	8.9920	4.4237	6.7901	4.7700	4.9897	0.1357
	0.99	55.0517	20.6020	33.3350	9.2825	7.7097	2.3846	5.9932	0.0184
200	0.85	2.8304	1.1627	1.7030	1.7346	2.3997	2.2892	1.9105	0.6004
	0.90	4.4581	1.8326	2.7266	2.4672	3.5203	3.2585	2.7859	0.5018
	0.95	8.7401	3.3593	5.2941	3.4287	5.4349	4.5107	4.0631	0.2164
	0.99	44.1215	16.5992	26.7763	8.5760	7.8719	3.0775	6.3555	0.0219
300	0.85	1.9135	0.8401	1.2020	1.3427	1.7248	1.6791	1.4466	0.6459
	0.90	3.2739	1.3867	2.0350	1.9957	2.7504	2.5902	2.2112	0.5902
	0.95	5.3822	2.0451	3.1610	2.3510	3.9268	3.4859	2.7823	0.3403
	0.99	32.1456	12.5390	20.0310	6.6280	7.9882	3.9967	5.8976	0.0293
400	0.85	1.4937	0.6855	0.9592	1.0562	1.3718	1.3452	1.1331	0.6720
	0.90	2.2779	0.9337	1.3900	1.4989	2.0258	1.9607	1.6274	0.6552
	0.95	4.2584	1.6541	2.5464	2.2311	3.3565	3.0975	2.5790	0.4327
	0.99	19.0533	6.2046	10.9726	3.7669	6.5574	4.0041	4.3712	0.0438

Table 6
MSEs of EPTML and EPTR estimators when $p = 6$, $\varphi = 4$.

n	ρ	EPTML	EPTR						
			$\widehat{k}_1$	$\widehat{k}_2$	$\widehat{k}_3$	$\widehat{k}_4$	$\widehat{k}_5$	$\widehat{k}_6$	$\widehat{k}_7$
30	0.85	15.3335	5.4552	9.6895	3.3896	8.3228	4.9633	4.2277	0.4648
	0.90	29.4709	11.2928	19.7158	5.4498	8.4811	3.7712	4.3055	0.2737
	0.95	59.7354	21.1156	38.4771	8.7611	11.0283	3.1852	5.2300	0.1011
	0.99	220.8704	83.4513	146.2832	27.5851	5.9100	0.4993	3.3293	0.0152
50	0.85	8.2223	3.2439	5.3845	2.5875	5.7848	4.4254	3.4830	0.8084
	0.90	13.7056	5.2504	8.8228	3.8155	8.5826	5.4256	4.7523	0.4486
	0.95	30.5590	11.1062	19.6045	5.7803	9.9409	4.3933	5.2977	0.1883
	0.99	129.0132	48.5248	83.9896	18.4340	7.7791	0.9950	4.6374	0.0203
100	0.85	4.4565	1.7454	2.8835	1.7682	3.7427	3.3112	2.5770	1.0088
	0.90	6.2097	2.4227	4.0189	2.1882	4.7985	3.9939	3.1438	0.8828
	0.95	11.6549	4.3444	7.5086	3.1589	7.2763	5.0585	4.2667	0.4791
	0.99	74.5017	25.7734	46.4491	10.4600	9.7875	1.8827	4.9862	0.0231
150	0.85	2.8179	1.1529	1.8376	1.3729	2.5443	2.3819	1.9240	1.0652
	0.90	4.1084	1.5854	2.6062	1.7488	3.4785	3.1150	2.4661	1.0163
	0.95	7.3497	2.6888	4.6474	2.3552	5.6154	4.5322	3.5195	0.6686
	0.99	45.5186	17.0454	29.8983	7.9438	10.6625	3.4787	5.6977	0.0603
200	0.85	2.1920	0.9088	1.4488	1.1223	2.0056	1.9162	1.5626	1.0178
	0.90	2.8916	1.1330	1.8394	1.4399	2.5990	2.4306	1.9758	1.0464
	0.95	5.9761	2.2617	3.8083	2.1478	4.8977	4.1351	3.1952	0.8121
	0.99	31.8884	11.0181	20.0200	5.8528	10.5660	4.3084	5.6312	0.0857
300	0.85	1.2343	0.8110	0.8742	0.7641	1.1833	1.1641	1.0092	0.5468
	0.90	1.8129	0.9865	1.1985	1.0246	1.6985	1.6452	1.3835	0.7482
	0.95	4.1777	1.5800	2.6269	1.8561	3.6624	3.2934	2.6387	0.9545
	0.99	19.6309	6.5564	12.0872	4.1788	9.5331	5.0563	5.0550	0.1637
400	0.85	1.0431	0.7568	0.7965	0.6556	1.0072	0.9942	0.8757	0.5284
	0.90	1.5010	0.9104	1.0346	0.8934	1.4277	1.3931	1.1927	0.6576
	0.95	3.0389	1.1634	1.9084	1.4627	2.7454	2.5476	2.0429	1.0178
	0.99	14.4528	4.8934	8.9316	3.6152	8.3551	5.3082	4.9020	0.2610

Table 7
MSEs of EPTML and EPTR estimators when $p = 6$, $\varphi = 8$.

n	ρ	EPTML	EPTR						
			$\widehat{k}_1$	$\widehat{k}_2$	$\widehat{k}_3$	$\widehat{k}_4$	$\widehat{k}_5$	$\widehat{k}_6$	$\widehat{k}_7$
30	0.85	30.7256	12.0887	20.3306	7.7718	15.5137	9.2113	9.8808	0.3601
	0.90	64.7003	28.3071	46.3721	13.2523	17.3942	7.7319	11.7824	0.2034
	0.95	125.5904	50.8608	85.3092	21.7029	22.7366	6.1598	14.4063	0.0875
	0.99	422.8635	184.4101	296.6343	64.7647	13.9270	1.0112	10.4627	0.0242
50	0.85	16.8203	7.4869	11.8287	5.8016	11.7132	8.9587	8.2400	0.6172
	0.90	26.6769	11.5816	18.2619	8.5154	16.8512	10.7022	11.1971	0.3042
	0.95	58.9942	25.0810	40.4555	13.9027	19.7339	8.8054	13.7684	0.1311
	0.99	254.3081	114.3964	177.8730	47.2278	20.1488	2.4765	14.8152	0.0211
100	0.85	8.9042	3.8014	6.0464	3.9856	7.4033	6.5990	5.7095	0.8647
	0.90	12.1106	5.2719	8.3293	4.8566	9.3009	7.8551	7.0052	0.6618
	0.95	23.1577	9.7671	15.8888	7.0005	14.4201	10.0296	9.9650	0.2797
	0.99	145.6237	58.2328	96.5926	23.4920	22.3776	4.0255	13.9896	0.0149
150	0.85	5.5587	2.3683	3.7197	2.9889	4.9890	4.6953	4.0735	1.0151
	0.90	8.1246	3.3747	5.4204	3.7738	6.8194	6.1588	5.3405	0.8127
	0.95	14.7581	5.9943	9.9091	5.1153	11.1790	9.0550	7.9881	0.3864
	0.99	88.6535	38.2991	61.9402	18.0224	22.7404	7.0372	15.3423	0.0302
200	0.85	4.3665	1.8340	2.9135	2.4601	3.9706	3.7975	3.3136	1.0706
	0.90	5.6977	2.2969	3.6907	3.0799	5.0827	4.7682	4.1398	0.9359
	0.95	11.6646	4.7300	7.7463	4.7460	9.4788	8.1159	7.1294	0.5036
	0.99	62.7656	24.2513	41.5749	12.4298	21.0085	8.1186	13.6712	0.0358
300	0.85	2.4433	1.0782	1.6002	1.6739	2.3319	2.2900	2.0603	0.9986
	0.90	3.5803	1.4389	2.3146	2.1618	3.3353	3.2276	2.8233	1.1165
	0.95	8.2240	3.1721	5.3278	3.7622	7.1165	6.4752	5.5382	0.6401
	0.99	37.7756	13.4666	24.1098	8.1386	17.9201	9.6829	11.5210	0.0641
400	0.85	1.9864	1.0877	1.3701	1.4459	1.9120	1.8865	1.7180	0.9094
	0.90	2.9720	1.2279	1.9671	1.9022	2.8157	2.7452	2.4234	1.1125
	0.95	5.9903	2.3196	3.8487	3.1146	5.3595	4.9982	4.3396	0.8167
	0.99	28.8476	10.5318	18.5946	7.3452	16.4248	10.2505	11.1487	0.1020

6. The proposed EPTR estimator with the other estimators ($\widehat{k}_7$, $\widehat{k}_6$, $\widehat{k}_5$ and $\widehat{k}_4$) of k gives better results, especially for low values of ρ. While the proposed EPTR estimator with the other estimators ($\widehat{k}_7$, $\widehat{k}_5$, $\widehat{k}_4$ and $\widehat{k}_6$) of k gives better results, especially for high values of ρ.
7. Finally, the proposed EPTR estimator with any suggested estimator of (k) is the best when the model contains the multicollinearity problem.

Another criterion of comparable performance of the proposed EPTR estimator is the relative efficiency (RE). Based on the estimated values of MSE from Eq. (36), the RE of EPTR estimator is calculated by [1,3,26,36,37]:

$$RE\left(\widehat{\beta}_{EPTR(k_a)}\right) = \frac{MSE\left(\widehat{\beta}_{EPTML}\right)}{MSE\left(\widehat{\beta}_{EPTR(k_a)}\right)},\ a = 1,\ 2,\ \ldots,\ 7 \tag{44}$$

where $\widehat{\beta}_{EPTR(k_a)}$ is the estimated value of the EPTR estimator with different k (from k_1 to k_7). Moreover, the root mean squared error (RMSE) has long been used as a standard statistical tool for assessing the performance of each estimator. Based on the estimated values of MSE from Eq. (36), the RMSE for the EPTML and EPTR estimators are calculated by

$$\text{RMSE}\left(\widehat{\beta}_{EPTML}\right) = \sqrt{MSE\left(\widehat{\beta}_{EPTML}\right)}, \tag{45}$$

$$\text{RMSE}\left(\widehat{\beta}_{EPTR(k_a)}\right) = \sqrt{MSE\left(\widehat{\beta}_{EPTR(k_a)}\right)},\ a = 1,\ 2,\ \ldots,\ 7. \tag{46}$$

Fig. 1 shows the criteria of RMSE and RE versus the factor ρ for EPTML and all k's of the proposed EPTR estimator. It is obvious that the performance of all k's for the proposed EPTR is better than the traditional EPTML over the values of the factor ρ, where all k's for the proposed EPTR give lower RMSE and RE values than that of the EPTML. The estimator $\widehat{k}_7$ for the EPTR estimator has the best performance, with the lowest RMSE and RE among all other k's in the EPTR and the traditional EPTML estimator.

Fig. 2 shows the criteria of RMSE and RE versus the factor n for EPTML and all k's of the proposed EPTR estimator. It is obvious that the performance of all k's for the proposed EPTR is better than the traditional EPTML over the values of the factor n, where all k's for the proposed EPTR give lower RMSE and RE values than that of the EPTML. The estimator $\widehat{k}_7$ for the EPTR estimator has the best performance, with the lowest RMSE and RE among all other k's in the EPTR and the traditional EPTML estimator.

Fig. 3 shows the criteria of RMSE and RE versus the factor φ for EPTML and all k's of the proposed EPTR estimator. It is obvious that the performance of all k's for the proposed EPTR is better than the traditional EPTML over the values of the factor φ, where all k's for the proposed EPTR give lower RMSE and RE values than that of the EPTML. The estimator $\widehat{k}_7$ for the EPTR estimator has the best performance, with the lowest RMSE and RE among all other k's in the EPTR and the traditional EPTML estimator.

Fig. 4 shows the criteria of RMSE and RE versus the factor p for EPTML and all k's of the proposed EPTR estimator. It is obvious that the performance of all k's for the proposed EPTR is better than the traditional EPTML over the values of the factor p, where all k's for the proposed EPTR give lower RMSE and RE values than that of the EPTML. The estimator $\widehat{k}_7$ for the EPTR estimator has the best performance, with the lowest RMSE and RE among all other k's in the EPTR and the traditional EPTML estimator.

Real-Life application

In this section, Cameron-Trivedi [11] dataset is used to estimate a recreation demand function. This data is from a survey on the number of recreational boating trips to Somerville Lake, East Texas, in 1980. Recently, Abonazel and Dawoud [2] also used this data as an application on the Poisson regression model. The response variable (y) is the number of recreational boating trips to Somerville Lake in 1980. While the explanatory variables are: $X1$: visiting Lake Conroe fees, $X2$: visiting Somerville Lake fees, and $X3$: visiting Houston Lake fees. In our application, the sample size (n) is 179 observations after removing the outliers. To examine the existence of multicollinearity in this data, the coefficients of correlation between X's, the variance inflation factor (*VIF*), and the value of the condition number $\left(CN = \sqrt{\frac{b_{max}}{b_{min}}}\right)$ are calculated, where b_{max} and b_{min} are the maximum and minimum values of the eigenvalues of $X^T\widehat{W}X$ matrix. The correlation coefficients are $Cor(X1,\ X2) = 0.971, Cor(X1,\ X3) = 0.981,$ and $Cor(X2,\ X3) = 0.942$. The values of the VIF are VIF (X1) = 43.84, VIF(X2) = 14.62, and VIF(X3) = 24.64. While the value of the *CN* is 29.101. The results of correlation coefficients, *VIF*, and *CN* confirm the presence of severe multicollinearity. Table 8 displays the estimated coefficients of the EPT regression model and estimated MSE values of EPTML and EPTR estimators.

From Table 8, it can be noted that:

1. The proposed EPTR estimator with all different suggested estimator of biasing parameters ($\widehat{k}_1$, $\widehat{k}_2$, $\widehat{k}_3$, $\widehat{k}_4$, $\widehat{k}_5$, $\widehat{k}_6$, $\widehat{k}_7$) has the lowest MSE values than the traditional EPTML estimator, where these estimators are the best-performing.
2. The EPTML estimator has the lowest performance of all estimators.
3. The best performing goes to the proposed EPTR estimator with $\widehat{k}_6$ followed by $\widehat{k}_3$ then $\widehat{k}_1$ and $\widehat{k}_7$.

From this data set, we can confirm whether the required condition of Theorem 1 is satisfied or not.

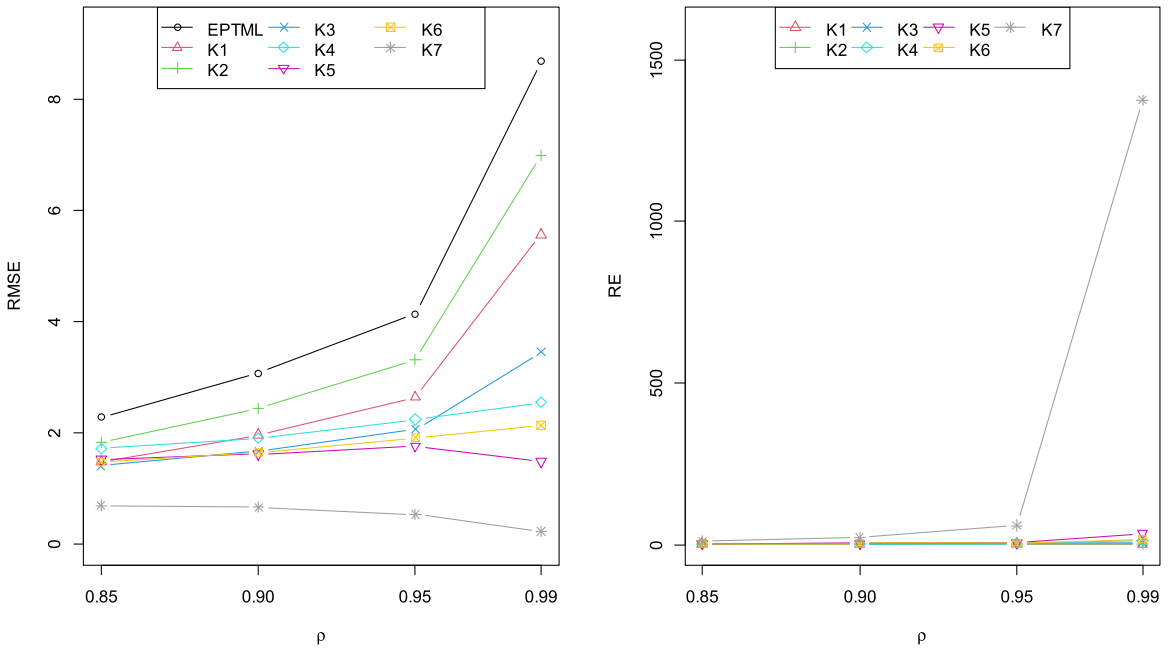


Fig. 1. RMSE and RE of EPTML and EPTR estimators versus ρ.

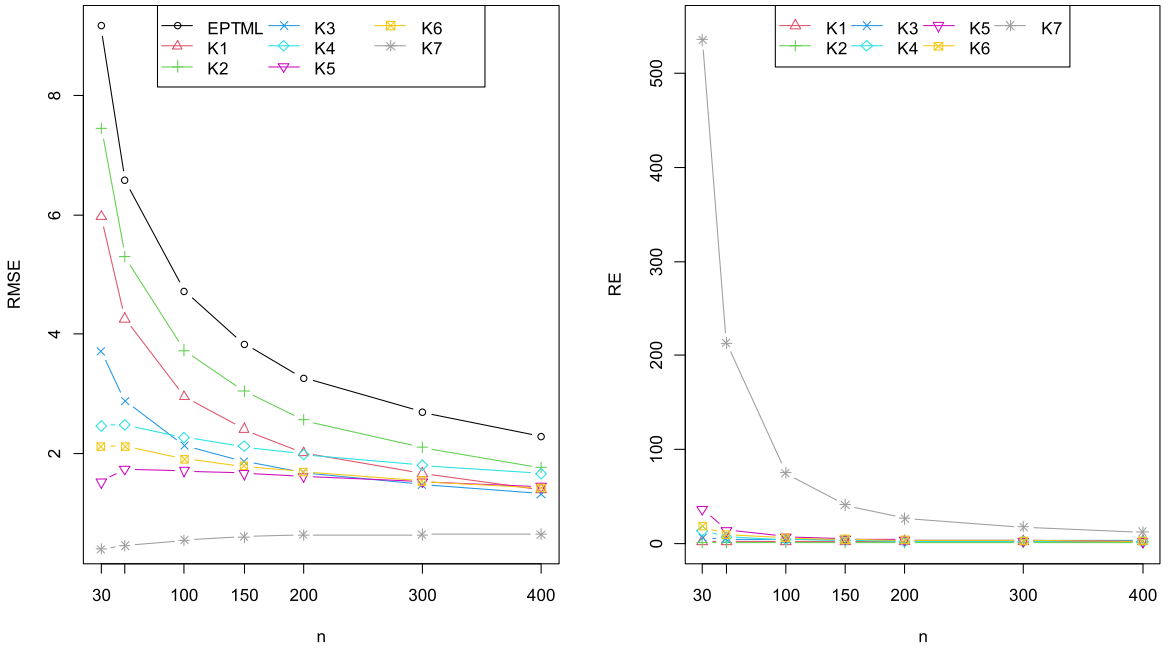


Fig. 2. RMSE and RE of EPTML and EPTR estimators versus n.

Table 9 displays the results of verifying this condition for each k. We found that the required condition is satisfied for all different suggested estimator of biasing parameters ($\widehat{k}_1$, $\widehat{k}_2$, $\widehat{k}_3$, $\widehat{k}_4$, $\widehat{k}_5$, $\widehat{k}_6$, $\widehat{k}_7$).

Fig. 5 presents the MSE values of the EPTR estimator and the terms of the squared bias and covariance values against k from 0 to 20. The MSE values of the EPTR estimator are smaller than the MSE values of the EPTML estimator for all values of k from 0 to 20. Also, the squared bias values of the EPTR estimator are increasing slowly, while the covariance term is decreasing over the values of k. The sum of the squared bias and the covariance for the proposed EPTR estimator makes its MSE values so much lower than those of the traditional EPTML estimator over all values of k.

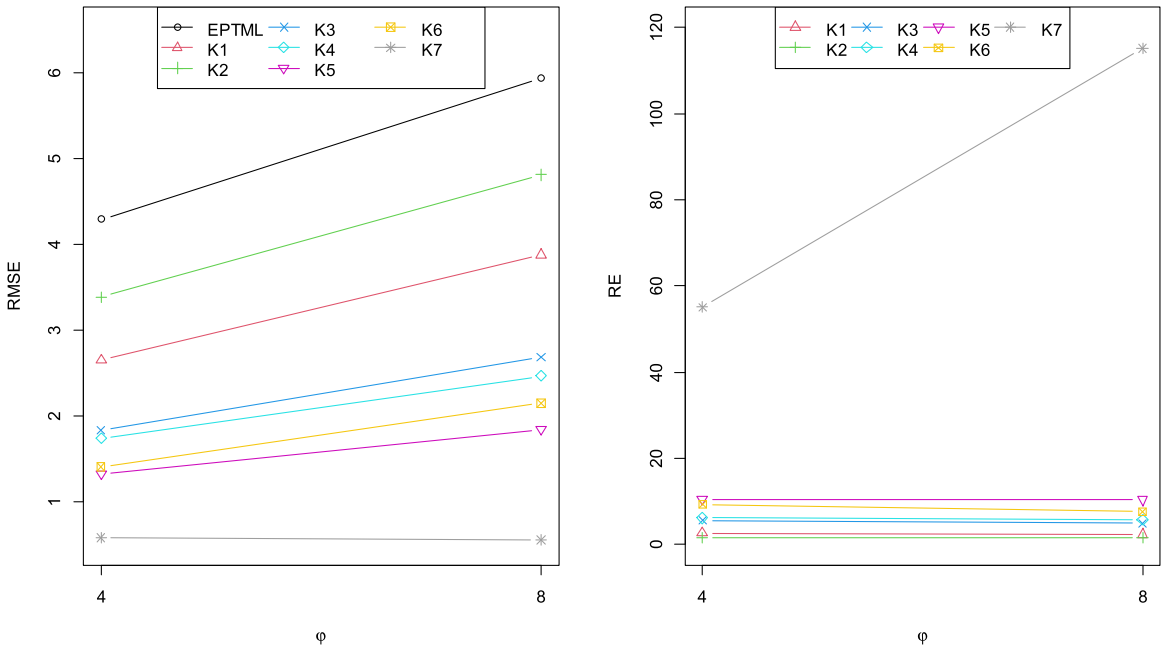


Fig. 3. RMSE and RE of EPTML and EPTR estimators versus φ.

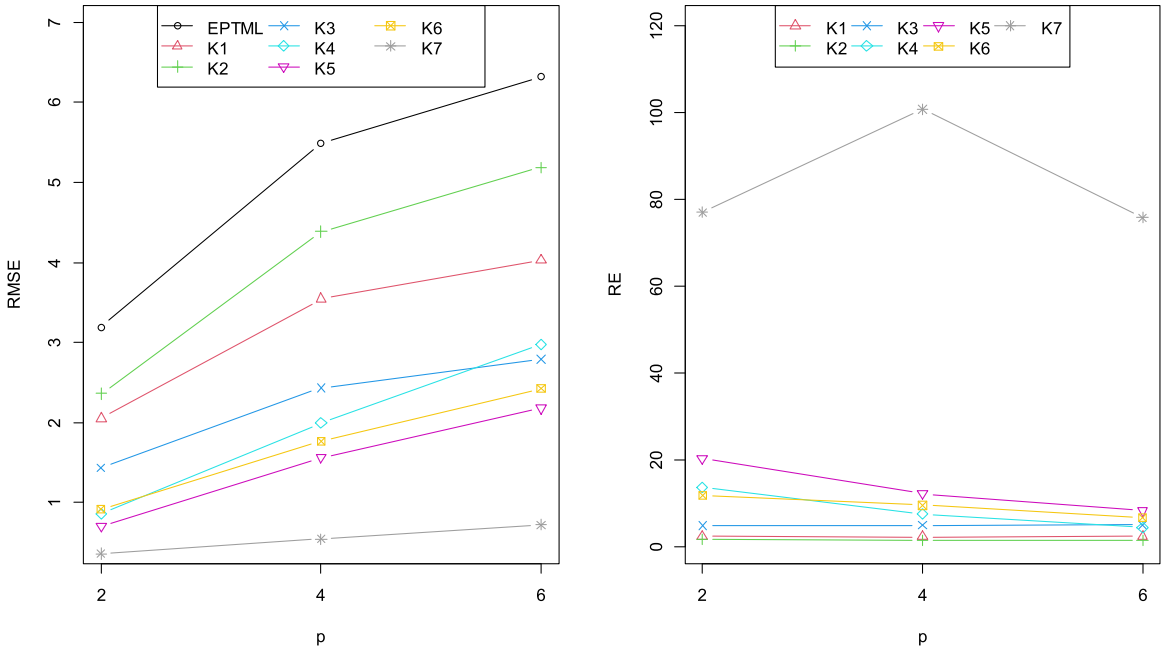


Fig. 4. RMSE and RE of EPTML and EPTR estimators versus p.

Conclusion

The presence of multicollinearity makes the EPTML estimator inefficient and unstable, where it exhibits the worst performance. Therefore, we developed in this paper the ridge (EPTR) estimator for the EPT model to deal with multicollinearity. Theoretically, we indicated the condition that is necessary for the outperformance of the proposed EPTR estimator over the EPTML. A simulation study is also conducted to get a good view of the proposed EPTR estimator's performance with different suggested seven estimators of k. The results give an indication that the proposed EPTR estimator with all suggested estimators of k gives better results than the traditional EPTML estimator when the problem of multicollinearity is present. Then, we compared the proposed EPTR estimator's performance

Table 8
Estimated coefficients and MSE for EPTML and EPTR estimators.

Estimator	Biasing Parameter	$X1$	$X2$	$X3$	$\widehat{k}$	MSE
EPTML	——	−0.10256	1.45816	−0.56723	——	5.040829
EPTR	$\widehat{k}_1$	0.182015	0.78297	−0.17434	3.304208	1.011289
	$\widehat{k}_2$	0.115986	1.045478	−0.37058	1.282233	1.180859
	$\widehat{k}_3$	0.17682	0.808358	−0.19443	3.021378	0.997384
	$\widehat{k}_4$	0.070807	1.170085	−0.45038	0.7481	1.618877
	$\widehat{k}_5$	0.07612	1.157081	−0.44264	0.796179	1.556548
	$\widehat{k}_6$	0.157954	0.892863	−0.25987	2.246726	0.991921
	$\widehat{k}_7$	0.191472	0.734138	−0.13522	3.93495	1.052434

Table 9
Results of verifying the required condition for Theorem 1.

Biasing Parameter ($\widehat{k}$)	Condition	$j=1$	$j=2$	$j=3$	Satisfied or not
$\widehat{k}_1$	$\widehat{\varphi}\left(2+\frac{\widehat{k}_1}{b_j}\right)-\widehat{k}_1\widehat{\alpha}_j^2$	4.7281	1.5096	18.5417	satisfied
$\widehat{k}_2$	$\widehat{\varphi}\left(2+\frac{\widehat{k}_2}{b_j}\right)-\widehat{k}_2\widehat{\alpha}_j^2$	5.1488	3.8998	10.5093	satisfied
$\widehat{k}_3$	$\widehat{\varphi}\left(2+\frac{\widehat{k}_3}{b_j}\right)-\widehat{k}_3\widehat{\alpha}_j^2$	4.7869	1.8439	17.4181	satisfied
$\widehat{k}_4$	$\widehat{\varphi}\left(2+\frac{\widehat{k}_4}{b_j}\right)-\widehat{k}_4\widehat{\alpha}_j^2$	5.2599	4.5312	8.3874	satisfied
$\widehat{k}_5$	$\widehat{\varphi}\left(2+\frac{\widehat{k}_5}{b_j}\right)-\widehat{k}_5\widehat{\alpha}_j^2$	5.2499	4.4744	8.5784	satisfied
$\widehat{k}_6$	$\widehat{\varphi}\left(2+\frac{\widehat{k}_6}{b_j}\right)-\widehat{k}_6\widehat{\alpha}_j^2$	4.9481	2.7597	14.3408	satisfied
$\widehat{k}_7$	$\widehat{\varphi}\left(2+\frac{\widehat{k}_7}{b_j}\right)-\widehat{k}_7\widehat{\alpha}_j^2$	4.5968	0.7639	21.0473	satisfied

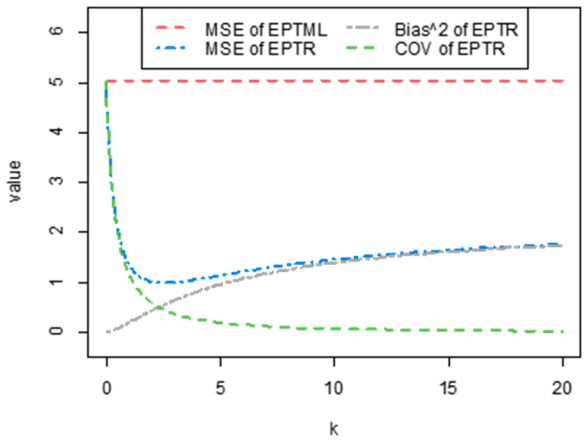


Fig. 5. MSE, squared bias (Bias^2), and covariance (COV) values versus k.

and EPTML estimator by using a numerical example. The application and simulation results are like those of the theoretical part. Thus, if the EPT model contains the multicollinearity problem, we recommend that practitioners use the proposed EPTR estimator with any of the suggested biasing parameter estimators, and especially $\widehat{k}_7$ and $\widehat{k}_6$, to estimate the regression coefficients of the model. In future work, we can provide a robust ridge estimator for the EPT model as an extension of Abonazel and Dawoud [2] or suggest other flexible biased estimators for this model such as Liu, Liu-type, or modified two-parameter Liu estimators.

Credit author statement

All authors contributed to the research.

Funding statement

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Declaration of Competing Interest

We hereby declare that this manuscript is original and there is no competing interest.

Data availability

Dataset of the real-life application is available as a supporting file.

Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.sciaf.2023.e02006.

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