J. Appl. Math. & Informatics Vol. **39***(2021), No. 3 - 4, pp. 277 - 294*
https://doi.org/10.14317/jami.2021.277

WIJSMAN REGULARLY IDEAL INVARIANT CONVERGENCE OF DOUBLE SEQUENCES OF SETS

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Abstract. In this paper, we introduce the notions of Wijsman regularly invariant convergence types, Wijsman regularly $(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergence, Wijsman regularly $(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-convergence, Wijsman regularly $(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-Cauchy double sequence and Wijsman regularly $(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-Cauchy double sequence of sets. Also, we investigate the relationships among this new notions.

AMS Mathematics Subject Classification : 40A05, 40A35.
Key words and phrases : Double sequence, regularly ideal convergence, invariant convergence, regularly ideal Cauchy sequence, Wijsman convergence.

1. Introduction

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of real sequences has been extended to statistical convergence independently by Fast [23] and Schoenberg [39]. This concept was extended to the double sequences by Mursaleen and Edely [26]. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [24] as a generalization of statistical convergence. Das et al. [8] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Tripathy and Tripathy [41] studied on $\mathcal{I}$-convergent and regularly $\mathcal{I}$-convergent double sequences. Dündar and Altay [10] introduced $\mathcal{I}_2$-convergence and regularly $\mathcal{I}$-convergence of double sequences. Also, Dündar [17] introduced regularly $\mathcal{I}$-convergence and regularly $\mathcal{I}$-Cauchy double sequences of functions. Recently, Dündar and Akın [22] studied regularly ideal convergence of double sequence of sets. Akın [5] investigated regularly ideal invariant convergence of double sequences. A lot of development have been made in this area after the works of [11, 12, 13, 16, 18, 25, 40].

Received August 31, 2020. Revised February 1, 2021. Accepted February 13, 2021.
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Several authors have studied invariant convergent sequences (see, [7, 27, 28, 29, 31, 35, 36, 37, 38, 42]). Recently, the concepts of σ-uniform density of the set $A \subseteq \mathbb{N}$, $\mathcal{I}_\sigma$-convergence and $\mathcal{I}_\sigma^*$-convergence of sequences of real numbers were defined by Nuray et al. [31]. The concept of σ-convergence of double sequences was studied by Çakan et al. [7] and the concept of σ-uniform density of $A \subseteq \mathbb{N} \times \mathbb{N}$ was defined by Tortop and Dündar [42]. Dündar et al. [19] studied ideal invariant convergence of double sequences and some properties.

Now, we recall the basic definitions and concepts (See [4, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 20, 21, 18, 24, 25, 34, 41, 42, 40, 43, 44]).

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

(1) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$, for all n,
(2) $\phi(e) = 1$, where $e = (1, 1, 1, ...)$ and
(3) $\phi(x_{\sigma(n)}) = \phi(x_n)$, for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$, for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$, for all $x \in c$.

In the case σ is translation mappings $\sigma(n) = n+1$, the σ-mean is often called a Banach limit and the space V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences $\hat{c}$.

It can be shown that

$$V_\sigma = \left\{ (x_n) \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \text{ uniformly in } n \right\}.$$

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if
(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$, for each $n \in \mathbb{N}$.

Throughout the paper we take $\mathcal{I}$ as an admissible ideal in $\mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter if and only if
(i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

For any ideal there is a filter $\mathcal{F}(\mathcal{I})$ corresponding with $\mathcal{I}$, given by $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : (\exists A \in \mathcal{I})(M = \mathbb{N} \backslash A)\}$.

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the property (AP), if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

A non-trivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible ideal if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

Throughout the paper, we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$.

$\mathcal{I}_2^0 = \big\{A \subset \mathbb{N}\times\mathbb{N} : (\exists m(A) \in \mathbb{N})(i,j \geq m(A) \Rightarrow (i,j) \notin A)\big\}$. Then, $\mathcal{I}_2^0$ is a strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

An admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{E_1, E_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{F_1, F_2, ...\}$ such that $E_j \Delta F_j \in \mathcal{I}_2^0$, i.e., $E_j \Delta F_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $F = \bigcup_{j=1}^{\infty} F_j \in \mathcal{I}_2$ (hence $F_j \in \mathcal{I}_2$, for each $j \in \mathbb{N}$).

Let $A \subseteq \mathbb{N}$ and

$$s_m = \min_n \big|A \cap \{\sigma(n), \sigma^2(n), \cdots, \sigma^m(n)\}\big|,$$
$$S_m = \max_n \big|A \cap \{\sigma(n), \sigma^2(n), \cdots, \sigma^m(n)\}\big|.$$

If the limits $\underline{V}(A) = \lim\limits_{m\to\infty} \frac{s_m}{m}$ and $\overline{V}(A) = \lim\limits_{m\to\infty} \frac{S_m}{m}$ exist, then they are called a lower and upper σ-uniform density of the set A, respectively. If $\underline{V}(A) = \overline{V}(A)$, then $V(A) = \underline{V}(A) = \overline{V}(A)$ is called σ-uniform density of A.

Denote by $\mathcal{I}_\sigma$ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

Let (X, ρ) be a separable metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by $d(x, A) = \inf\limits_{a\in A} \rho(x, a)$.

Throughout the paper, we let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal, (X, ρ) be a separable metric space and A, A_k be any non-empty closed subsets of X.

A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}$-invariant convergent or $\mathcal{I}_W^\sigma$-convergent to A if for every $\varepsilon > 0$, the set $A(\varepsilon, x) = \{k : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \in \mathcal{I}_\sigma$, that is, $V(A(\varepsilon, x)) = 0$. In this case, we write $A_k \to A(\mathcal{I}_W^\sigma)$.

A sequence $\{A_k\}$ is Wijsman $\mathcal{I}^*$-invariant convergent or $\mathcal{I}_W^{*\sigma}$-convergent to A if there exists a set $M = \{m_1 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I}_\sigma)$ such that for each $x \in X$, $\lim\limits_{k\to\infty} d(x, A_{m_k}) = d(x, A)$.

A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}$-invariant Cauchy sequence or $\mathcal{I}_W^\sigma$-Cauchy sequence if for every $\varepsilon > 0$ and for each $x \in X$, there exists a number $N = N(\varepsilon, x) \in \mathbb{N}$ such that $A(\varepsilon, x) = \big\{k : |d(x, A_k) - d(x, A_N)| \geq \varepsilon\big\} \in \mathcal{I}_\sigma$, that is, $V\big(A(\varepsilon, x)\big) = 0$.

A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}^*$-invariant Cauchy sequence or $\mathcal{I}_W^{*\sigma}$-Cauchy sequence if there exists a set $M = \{m_1 < \cdots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I}_\sigma)$ such that $\lim\limits_{k,p\to\infty} |d(x, A_{m_k}) - d(x, A_{m_p})| = 0$, for each $x \in X$.

Let $A \subseteq \mathbb{N}\times\mathbb{N}$ and

$$s_{mn} = \min_{k,j} \big|A \cap \big\{\big(\sigma(k), \sigma(j)\big), \big(\sigma^2(k), \sigma^2(j)\big), ..., \big(\sigma^m(k), \sigma^n(j)\big)\big\}\big|,$$
$$S_{mn} = \max_{k,j} \big|A \cap \big\{\big(\sigma(k), \sigma(j)\big), \big(\sigma^2(k), \sigma^2(j)\big), ..., \big(\sigma^m(k), \sigma^n(j)\big)\big\}\big|.$$

If the limits $\underline{V_2}(A) = \lim\limits_{m,n\to\infty} \frac{s_{mn}}{mn}$ and $\overline{V_2}(A) = \lim\limits_{m,n\to\infty} \frac{S_{mn}}{mn}$ exist, then they are called a lower and an upper σ-uniform density of the set A, respectively. If $\underline{V_2}(A) = \overline{V_2}(A)$, then $V_2(A) = \underline{V_2}(A) = \overline{V_2}(A)$ is called the σ-uniform density of A.

Denote by $\mathcal{I}_2^\sigma$ the class of all $A \subseteq \mathbb{N}\times\mathbb{N}$ with $V_2(A) = 0$.

Throughout the paper, we let $\mathcal{I}_2^\sigma \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal and A, B_k, C_k, A_{kj} be any non-empty closed subsets of X.

A double sequence $\{A_{kj}\}$ is said to be bounded if $\sup\limits_{k,j} d(x, A_{kj}) < \infty$, for each $x \in X$. The set of all bounded double sequences of sets will be denoted by L_∞^2.

A double sequence $\{A_{kj}\}$ is said to be Wijsman invariant convergent to A if for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x, A_{\sigma^k(s),\sigma^j(t)}) = d(x, A), \text{ uniformly in } s, t.$$

A double sequence $\{A_{kj}\}$ is said to be Wijsman $\mathcal{I}_2$-invariant convergent or $\mathcal{I}_{W_2}^\sigma$-convergent to A, if for every $\varepsilon > 0$, $A(\varepsilon, x) = \{(k,j) : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\} \in \mathcal{I}_2^\sigma$ that is, $V_2(A(\varepsilon, x)) = 0$. In this case, we write $A_{kj} \to A(\mathcal{I}_{W_2}^\sigma)$.

A double sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2^*$-invariant convergent or $\mathcal{I}_{W_2}^{*\sigma}$-convergent to A if and only if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^\sigma)$ $(\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2^\sigma)$ such that for each $x \in X$ and $(k,j) \in M_2$, $\lim\limits_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$. In this case, we write $\mathcal{I}_{W_2}^{*\sigma} - \lim\limits_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$.

A double sequence $\{A_{kj}\}$ is said to be Wijsman $\mathcal{I}_2$-invariant Cauchy sequence or $\mathcal{I}_{W_2}^\sigma$-Cauchy sequence, if for every $\varepsilon > 0$ and each $x \in X$, there exist $r = r(\varepsilon, x)$, $s = s(\varepsilon, x) \in \mathbb{N}$ such that $A(\varepsilon, x) = \big\{(k,j) : |d(x, A_{kj}) - d(x, A_{rs})| \geq \varepsilon\big\} \in \mathcal{I}_2^\sigma$, that is, $V_2\big(A(\varepsilon, x)\big) = 0$.

A double sequence $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{*\sigma}$-Cauchy if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^\sigma)$ (i.e., $\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2^\sigma$) such that for every $x \in X$ and $(k,j), (p,q) \in M_2$ $\lim\limits_{k,j,p,q\to\infty} |d(x, A_{kj}) - d(x, A_{pq})| = 0$.

A double sequence $\{A_{kj}\}$ is said to be Wijsman regularly convergent ($R(W_2, W)$-convergent) if it is convergent in Pringsheim's sense and for each $x \in X$ the limits $\lim\limits_{k\to\infty} d(x, A_{kj}), (j \in \mathbb{N})$ and $\lim\limits_{j\to\infty} d(x, A_{kj}), (k \in \mathbb{N})$ exist. Note that if $\{A_{kj}\}$ is Wijsman regularly convergent to A, then the limits

$$\lim_{k\to\infty}\lim_{j\to\infty} d(x, A_{kj}) = d(x, A) \text{ and } \lim_{j\to\infty}\lim_{k\to\infty} d(x, A_{kj}) = d(x, A)$$

exist and we write $R(W_2, W) - \lim\limits_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$ or $A_{kj} \overset{R(W_2,W)}{\longrightarrow} A$.

A double sequence $\{A_{kj}\}$ is said to be regularly $(\mathcal{I}_{W_2}, \mathcal{I}_W)$-convergent ($R(\mathcal{I}_{W_2}, \mathcal{I}_W)$-convergent) if it is $\mathcal{I}_{W_2}$-convergent in Pringsheim's sense and for every $\varepsilon > 0$ and each $x \in X$, $\{k \in \mathbb{N} : |d(x, A_{kj}) - d(x, K_j)| \geq \varepsilon\} \in \mathcal{I}$, for some $K_j \in X$ and each

$j \in \mathbb{N}$, and $\{j \in \mathbb{N} : |d(x, A_{kj}) - d(x, L_k)| \geq \varepsilon\} \in \mathcal{I}$, for some $L_k \in X$ and each $k \in \mathbb{N}$.

Note that if $\{A_{kj}\}$ is $R(\mathcal{I}_{W_2}, \mathcal{I}_W)$-convergent to A, then we write $R(\mathcal{I}_{W_2}, \mathcal{I}_W)-\lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$ or $A_{kj} \overset{R(\mathcal{I}_{W_2}, \mathcal{I}_W)}{\longrightarrow} A$.

A double sequence $\{A_{kj}\}$ is said to be regularly $(\mathcal{I}^*_{W_2}, \mathcal{I}^*_W)$-convergent ($R(\mathcal{I}^*_{W_2}, \mathcal{I}^*_W)$-convergent) if there exist the sets $M \in \mathcal{F}(\mathcal{I}_2)$, $M_1 \in \mathcal{F}(\mathcal{I})$ and $M_2 \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{I}_2$, $\mathbb{N} \setminus M_1 \in \mathcal{I}$ and $\mathbb{N} \setminus M_2 \in \mathcal{I}$) such that the limits

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M}} d(x, A_{kj}), \quad \lim_{\substack{k\to\infty \\ k\in M_1}} d(x, A_{kj}) \quad \text{and} \quad \lim_{\substack{j\to\infty \\ j\in M_2}} d(x, A_{kj})$$

exist for each fixed $k \in \mathbb{N}$ and each fixed $j \in \mathbb{N}$, respectively.

Note that if $\{A_{kj}\}$ is $R(\mathcal{I}^*_{W_2}, \mathcal{I}^*_W)$-convergent to A, then for each $x \in X$ the limits $\lim_{k\to\infty} \lim_{j\to\infty} d(x, A_{kj})$ and $\lim_{j\to\infty} \lim_{k\to\infty} d(x, A_{kj})$ exist and are equal to A and we write $R(\mathcal{I}^*_{W_2}, \mathcal{I}^*_W) - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$ or $A_{kj} \overset{R(\mathcal{I}^*_{W_2}, \mathcal{I}^*_W)}{\longrightarrow} A$.

A double sequence $\{A_{kj}\}$ is said to be regularly $(\mathcal{I}_{W_2}, \mathcal{I}_W)$-Cauchy ($R(\mathcal{I}_{W_2}, \mathcal{I}_W)$-Cauchy) if it is $\mathcal{I}_2$-Cauchy in Pringsheim's sense and for every $\varepsilon > 0$ and each $x \in X$, there exist $m_j = m_j(\varepsilon, x), n_k = n_k(\varepsilon, x) \in \mathbb{N}$ such that

$\{k \in \mathbb{N} : |d(x, A_{kj}) - d(x, A_{m_j j})| \geq \varepsilon\} \in \mathcal{I}, (j \in \mathbb{N})$,

$\{j \in \mathbb{N} : |d(x, A_{kj}) - d(x, A_{kn_k})| \geq \varepsilon\} \in \mathcal{I}, (k \in \mathbb{N})$.

A double sequence $\{A_{kj}\}$ is said to be regularly $(\mathcal{I}^*_{W_2}, \mathcal{I}^*_W)$-Cauchy ($R(\mathcal{I}^*_{W_2}, \mathcal{I}^*_W)$-Cauchy) if there exist the sets $M \in \mathcal{F}(\mathcal{I}_2)$, $M_1 \in \mathcal{F}(\mathcal{I})$ and $M_2 \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N}\times\mathbb{N}\setminus M \in \mathcal{I}_2$, $\mathbb{N}\setminus M_1 \in \mathcal{I}$ and $\mathbb{N}\setminus M_2 \in \mathcal{I}$), and for every $\varepsilon > 0$ and each $x \in X$ there exist $N = N(\varepsilon, x), s = s(\varepsilon, x), t = t(\varepsilon, x), m_j = m_j(\varepsilon, x), n_k = n_k(\varepsilon, x) \in \mathbb{N}$ such that whenever $k, j, s, t, m_j, n_k \geq N$, we have

$$|d(x, A_{kj}) - d(x, A_{st})| < \varepsilon, \text{ for } (m, n), (s, t) \in M,$$
$$|d(x, A_{kj}) - d(x, A_{m_j j})| < \varepsilon, \text{ for each } k \in M_1 \text{ and each } j \in \mathbb{N},$$
$$|d(x, A_{kj}) - d(x, A_{kn_k}| < \varepsilon, \text{ for each } j \in M_2 \text{ and each } k \in \mathbb{N}.$$

Lemma 1.1. [42] *Let $\{A_{kj}\}$ be bounded sequence. If $\{A_{kj}\}$ is $\mathcal{I}^{\sigma}_{W_2}$-convergent to A, then $\{A_{kj}\}$ is Wijsman invariant convergent to A.*

Lemma 1.2. [42] *Let $0 < p < \infty$.*

(i) *If $A_{kj} \to A([W_2V_\sigma]_p)$, then $A_{kj} \to A(\mathcal{I}^{\sigma}_{W_2})$.*

(ii) *If $\{A_{kj}\} \in L^2_\infty$ and $A_{kj} \to A(\mathcal{I}^{\sigma}_{W_2})$, then $A_{kj} \to A([W_2V_\sigma]_p)$.*

(iii) *If $\{A_{kj}\} \in L^2_\infty$, then $\{A_{kj}\}$ is $\mathcal{I}^{\sigma}_{W_2}$-convergent to A if and only if $A_{kj} \to A([W_2V_\sigma]_p)$.*

Lemma 1.3. [42] *If a sequence $\{A_{kj}\}$ is $\mathcal{I}^{*\sigma}_{W_2}$-convergent to A, then this sequence is $\mathcal{I}^{\sigma}_{W_2}$-convergent to A.*

Lemma 1.4. [42] *Let $\mathcal{I}_2^\sigma$ has property $(AP2)$. If $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^\sigma$-convergent to A, then $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{*\sigma}$-convergent to A.*

Lemma 1.5. [42] *If a double sequence $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^\sigma$-convergent, then $\{A_{kj}\}$ is an $\mathcal{I}_{W_2}^\sigma$-Cauchy double sequence of sets.*

Lemma 1.6. [42] *If a double sequence $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{*\sigma}$-Cauchy double sequence, then $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^\sigma$-Cauchy double sequence of sets.*

2. Main Results

Now, we introduce the notions of Wijsman regularly invariant convergence, Wijsman regularly strongly invariant convergence, Wijsman regularly p-strongly invariant convergence, Wijsman regularly $(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergence, Wijsman regularly $(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-convergence, Wijsman regularly $(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-Cauchy double sequence, Wijsman regularly $(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-Cauchy double sequence of sets and investigate the relationship among them.

Definition 2.1. A double sequence $\{A_{kj}\}$ is said to be Wijsman regularly invariant convergent ($R(W^\sigma, W_2^\sigma)$-convergent) if it is Wijsman invariant convergent in Pringsheim's sense and the following limits hold:

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m} d(x, A_{\sigma^k(s),\sigma^j(t)}) = d(x, B_j), \text{ uniformly in } s,$$

for some $B_j \in X$, each $j \in \mathbb{N}$ and each $x \in X$, and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} d(x, A_{\sigma^k(s),\sigma^j(t)}) = d(x, C_k), \text{ uniformly in } t,$$

for some $C_k \in X$, each $k \in \mathbb{N}$ and each $x \in X$.

Note that if $\{A_{kj}\}$ is $R(W^\sigma, W_2^\sigma)$-convergent to A, the following limits hold:

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{mn} \sum_{k=0}^{m} \sum_{j=0}^{n} d(x, A_{\sigma^k(s),\sigma^j(t)}) = d(x, A), \text{ uniformly in } s, t$$

and

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{mn} \sum_{j=0}^{n} \sum_{k=0}^{m} d(x, A_{\sigma^k(s),\sigma^j(t)}) = d(x, A), \text{ uniformly in } s, t,$$

for each $x \in X$. In this case, for each $x \in X$, we write

$$R(W^\sigma, W_2^\sigma) - \lim_{m,n\to\infty} \sum_{k,j=0,0}^{m,n} d(x, A_{\sigma^k(s),\sigma^j(t)}) = d(x, A) \text{ or } A_{kj} \overset{R(W^\sigma, W_2^\sigma)}{\longrightarrow} A,$$

uniformly in s, t.

Definition 2.2. A double sequence $\{A_{kj}\}$ is said to be Wijsman regularly strongly invariant convergent ($R[W^\sigma, W_2^\sigma]$-convergent) if it is Wijsman strongly invariant convergent in Pringsheim's sense and the following limits hold:

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m} |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, B_j)| = 0, \quad \text{uniformly in } s,$$

for some $B_j \in X$, each $j \in \mathbb{N}$ and each $x \in X$, and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, C_k)| = 0, \quad \text{uniformly in } t,$$

for some $C_k \in X$, each $k \in \mathbb{N}$ and each $x \in X$.

Note that if $\{A_{kj}\}$ is $R[W^\sigma, W_2^\sigma]$-convergent to A, the following limits hold:

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{mn} \sum_{k=0}^{m} \sum_{j=0}^{n} |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, A)| = 0, \text{ uniformly in } s, t$$

and

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{mn} \sum_{j=0}^{n} \sum_{k=0}^{m} |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, A)| = 0, \text{ uniformly in } s, t,$$

for each $x \in X$. In this case, for each $x \in X$, we write

$$R[W^\sigma, W_2^\sigma] - \lim_{m,n\to\infty} \sum_{k,j=0,0}^{m,n} |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, A)| = 0 \text{ or } A_{kj} \overset{R[W^\sigma, W_2^\sigma]}{\longrightarrow} A,$$

uniformly in s, t.

Definition 2.3. Let $0 < p < \infty$. A double sequence $\{A_{kj}\}$ is said to be Wijsman regularly p-strongly invariant convergent ($R[W^\sigma, W_2^\sigma]_p$-convergent), if it is Wijsman p-strongly invariant convergent in Pringsheim's sense and the following limits hold:

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m} |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, B_j)|^p = 0, \text{ uniformly in } s,$$

for some $B_j \in X$, each $j \in \mathbb{N}$ and each $x \in X$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, C_k)|^p = 0, \text{ uniformly in } t,$$

for some $C_k \in X$, each $k \in \mathbb{N}$ and each $x \in X$.

Note that if $\{A_{kj}\}$ is $R[W^\sigma, W_2^\sigma]_p$-convergent to A, the following limits hold:

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{mn} \sum_{k=0}^{m} \sum_{j=0}^{n} |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, A)|^p = 0, \text{ uniformly in } s, t$$

and

$$\lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{mn}\sum_{j=0}^{n}\sum_{k=0}^{m}|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)|^p=0,\ \text{uniformly in } s,t,$$

for each $x\in X$. In this case, for each $x\in X$, we write

$$R[W^\sigma,W_2^\sigma]_p-\lim_{m,n\to\infty}\sum_{k,j=0,0}^{m,n}|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)|=0 \text{ or } A_{kj}\overset{R[W^\sigma,W_2^\sigma]_p}{\longrightarrow}A,$$

uniformly in s,t.

Definition 2.4. A double sequence $\{A_{kj}\}$ is said to be Wijsman regularly ideal invariant convergent ($R(\mathcal{I}_W^\sigma,\mathcal{I}_{W_2}^\sigma)$-convergent), if it is Wijsman ideal invariant convergent in Pringsheim's sense and for every $\varepsilon>0$ and each $x\in X$, the following limits hold:

$$\{k\in\mathbb{N}:|d(x,A_{kj})-d(x,B_j)|\geq\varepsilon\}\in\mathcal{I}_\sigma,$$

for some $B_j\in X$, each $j\in\mathbb{N}$ and each $x\in X$, and

$$\{j\in\mathbb{N}:|d(x,A_{kj})-d(x,C_k)|\geq\varepsilon\}\in\mathcal{I}_\sigma,$$

for some $C_k\in X$, each $k\in\mathbb{N}$ and each $x\in X$.

Note that if $\{A_{kj}\}$ is $R(\mathcal{I}_W^\sigma,\mathcal{I}_{W_2}^\sigma)$-convergent to A, then for each $x\in X$, we write

$$R(\mathcal{I}_W^\sigma,\mathcal{I}_{W_2}^\sigma)-\lim d(x,A_{kj})=d(x,A) \text{ or } A_{kj}\overset{R(\mathcal{I}_W^\sigma,\mathcal{I}_{W_2}^\sigma)}{\longrightarrow}A.$$

Theorem 2.5. *Assume that $\{A_{kj}\}$ is a bounded double sequence. If $\{A_{kj}\}$ is $R(\mathcal{I}_W^\sigma,\mathcal{I}_{W_2}^\sigma)$-convergent, then $\{A_{kj}\}$ is $R(W^\sigma,W_2^\sigma)$-convergent.*

Proof. Let $\{A_{kj}\}$ is a bounded double sequence and $\{A_{kj}\}$ is $R(\mathcal{I}_W^\sigma,\mathcal{I}_{W_2}^\sigma)$-convergent to A. Then, $\{A_{kj}\}$ is Wijsman ideal invariant convergent in Pringsheim's sense and for every $\varepsilon>0$ and each $x\in X$, the followings hold:

$$\{k\in\mathbb{N}:|d(x,A_{kj})-d(x,B_j)|\geq\varepsilon\}\in\mathcal{I}_\sigma$$

for some $B_j\in X$, each $j\in\mathbb{N}$ and each $x\in X$, and

$$\{j\in\mathbb{N}:|d(x,A_{kj})-d(x,C_k)|\geq\varepsilon\}\in\mathcal{I}_\sigma$$

for some $C_k\in X$, each $k\in\mathbb{N}$ and each $x\in X$. Since $\{A_{kj}\}$ is Wijsman ideal invariant convergent in Pringsheim's sense, then by Lemma 1.1 $\{A_{kj}\}$ is Wijsman invariant convergent to A.

Now, let $\varepsilon>0$. We estimate

$$u(m,s)=\left|\frac{1}{m}\sum_{k=0}^{m}d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,B_j)\right|,\ \text{uniformly in }\ s,$$

for some $B_j\in X$, each $j\in\mathbb{N}$ and each $x\in X$. Then, we have

$$u(m,s)\leq u^1(m,s)+u^2(m,s)$$

where

$$u^1(m,s)=\frac{1}{m}\sum_{\substack{k=0\\ |d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,B_j)|\geq\varepsilon}}^{m}|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,B_j)|$$

and

$$u^2(m,s)=\frac{1}{m}\sum_{\substack{k=0\\ |d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,B_j)|<\varepsilon}}^{m}|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,B_j)|,$$

uniformly in s, for some $B_j\in X$, each $j\in\mathbb{N}$ and each $x\in X$. Therefore, we have $u^2(m,s)<\varepsilon$, for every $s=1,2,\ldots$. The boundedness of $\{A_{kj}\}$ implies that there exists $K>0$ such that for each $x\in X$,

$$|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,B_j)|\leq K,\ (k,s=1,2,\ldots),$$

then this implies that

$$\begin{aligned}u^1(m,s) &\leq \frac{K}{m}\Big|\big\{1\leq k\leq m:|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,B_j)|\geq\varepsilon\big\}\Big|\\ &\leq K\frac{\max\limits_s\Big|\big\{1\leq k\leq m:|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,B_j)|\geq\varepsilon\big\}\Big|}{m}\\ &= K\frac{S_m}{m}\end{aligned}$$

and so, $\{A_{kj}\}$ is Wijsman σ-convergent to B_j.

Similarly, we can show that $\{A_{kj}\}$ is Wijsman σ-convergent to C_k. Hence, $\{A_{kj}\}$ is $R(W^\sigma,W_2^\sigma)$-convergent. □

Theorem 2.6. *Let $0<p<\infty$. Then,*

(i) *If $\{A_{kj}\}$ is $R[W^\sigma,W_2^\sigma]_p$-convergent, then $\{A_{kj}\}$ is $R(\mathcal{I}_W^\sigma,\mathcal{I}_{W_2}^\sigma)$-convergent.*

(ii) *If $\{A_{kj}\}\in L_\infty^2$ and $\{A_{kj}\}$ is $R(\mathcal{I}_W^\sigma,\mathcal{I}_{W_2}^\sigma)$-convergent, then $\{A_{kj}\}$ is $R[W^\sigma,W_2^\sigma]_p$-convergent.*

(iii) *If $\{A_{kj}\}\in L_\infty^2$, then $\{A_{kj}\}$ is $R[W^\sigma,W_2^\sigma]_p$-convergent if and only if $\{A_{kj}\}$ is $R(\mathcal{I}_W^\sigma,\mathcal{I}_{W_2}^\sigma)$ is convergent.*

Proof. (*i*) Let $x=\{A_{kj}\}$ is $R[W^\sigma,W_2^\sigma]_p$-convergent. Then, it is Wijsman p-strongly invariant convergent in Pringsheim's sense and the following limits holds:

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m}|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,B_j)|^p=0,\quad\text{uniformly in } s,$$

for some $B_j\in X$, each $j\in\mathbb{N}$ and each $x\in X$, and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,C_k)|^p=0,\quad\text{uniformly in } t,$$

for some $C_k \in X$, each $k \in \mathbb{N}$ and each $x \in X$. Since $\{A_{kj}\}$ is Wijsman p-strongly invariant convergent in Pringsheim's sense, then by Lemma 1.2 $\{A_{kj}\}$ is $\mathcal{I}^{\sigma}_{W_2}$-convergent.

Also, for every $\varepsilon > 0$, some $B_j \in X$, each $j \in \mathbb{N}$ and each $x \in X$ we can write

$$\begin{aligned}
&\sum_{k=1}^{m} \left|d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, B_j)\right|^p \\
&\qquad \geq \sum_{\substack{k=1 \\ |d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,B_j)|\geq\varepsilon}}^{m} |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, B_j)|^p \\
&\qquad \geq \varepsilon^p \left|\left\{k \leq m : |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, B_j)| \geq \varepsilon\right\}\right| \\
&\qquad \geq \varepsilon^p \max_s \left|\left\{k \leq m : |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, B_j)| \geq \varepsilon\right\}\right|
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{m}\sum_{k=1}^{m} \left|d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, B_j)\right|^p \\
&\qquad \geq \varepsilon^p \frac{\max_s \left|\left\{k \leq m : |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, B_j)| \geq \varepsilon\right\}\right|}{m} \\
&\qquad = \varepsilon^p \frac{S_m}{m},
\end{aligned}$$

for every $s = 1, 2, \ldots$. This implies

$$\lim_{m\to\infty} \frac{S_m}{m} = 0$$

and so $\{A_{kj}\}$ is $\mathcal{I}^{\sigma}_{W}$-convergent to B_j.

Similarly, we can show that $\{A_{kj}\}$ is $\mathcal{I}^{\sigma}_{W}$-convergent to C_k. Hence, $\{A_{kj}\}$ is $R(\mathcal{I}^{\sigma}_{W}, \mathcal{I}^{\sigma}_{W_2})$-convergent.

(ii) Let $\{A_{kj}\} \in L^2_\infty$ and $\{A_{kj}\}$ is $R(\mathcal{I}^{\sigma}_{W}, \mathcal{I}^{\sigma}_{W_2})$-convergent. Then, $\{A_{kj}\}$ is Wijsman ideal invariant convergent in Pringsheim's sense and for every $\varepsilon > 0$ and $x \in X$ the followings hold:

$$\{k \in \mathbb{N} : |d(x, A_{kj}) - d(x, B_j)| \geq \varepsilon\} \in \mathcal{I}_\sigma,$$

for some $B_j \in X$, each $j \in \mathbb{N}$ and each $x \in X$, and

$$\{j \in \mathbb{N} : |d(x, A_{kj}) - d(x, C_k)| \geq \varepsilon\} \in \mathcal{I}_\sigma,$$

for some $C_k \in X$, each $k \in \mathbb{N}$ and each $x \in X$. Since $\{A_{kj}\}$ is Wijsman ideal invariant convergent in Pringsheim's sense, then by Lemma 1.2, $\{A_{kj}\}$ is Wijsman p-strongly invariant convergent. Let $0 < p < \infty$ and $\varepsilon > 0$. Since $\{A_{kj}\}$ is bounded, $\{A_{kj}\}$ implies that there exists $K > 0$ such that

$$|d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, B_j)| \leq K, \ (j \in \mathbb{N}),$$

for all $k, s \in \mathbb{N}$, some $B_j \in X$ and each $x \in X$. Then, for every $s = 1, 2, \ldots$we have

$$\begin{aligned}
&\frac{1}{m}\sum_{k=1}^{m}\left|d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, B_j)\right|^p \\
&\qquad = \frac{1}{m}\sum_{\substack{k=1 \\ |d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,B_j)|\geq\varepsilon}}^{m} |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, B_j)|^p \\
&\qquad\quad + \frac{1}{m}\sum_{\substack{k=1 \\ |d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,B_j)|<\varepsilon}}^{m} |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, B_j)|^p \\
&\qquad \leq K\frac{\max_s \left|\left\{k \leq m : |d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, B_j)| \geq \varepsilon\right\}\right|}{m} + \varepsilon^p \\
&\qquad \leq K\frac{S_m}{m} + \varepsilon^p.
\end{aligned}$$

Hence, we obtain

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{m}\left|d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, B_j)\right|^p = 0, \text{ uniformly in } s$$

and so, $\{A_{kj}\}$ is Wijsman p-strongly invariant convergent to B_j.

Similarly, we show that $\{A_{kj}\}$ is Wijsman p-strongly invariant convergent to C_k. Hence, $\{A_{kj}\}$ is $R[W^\sigma, W_2^\sigma]_p$-convergent.

(iii) This is immedate consequence of (i) and (ii). □

Definition 2.7. A double sequence $\{A_{kj}\}$ is said to be regularly $(\mathcal{I}_W^{\sigma*}, \mathcal{I}_{W_2}^{\sigma*})$-convergent ($R(\mathcal{I}_W^{\sigma*}, \mathcal{I}_{W_2}^{\sigma*})$-convergent) if and only if there exist the sets $M \in \mathcal{F}(\mathcal{I}_2^\sigma)$, $M_1 \in \mathcal{F}(\mathcal{I}_\sigma)$ and $M_2 \in \mathcal{F}(\mathcal{I}_\sigma)$ (i.e., $\mathbb{N}\times\mathbb{N}\setminus M \in \mathcal{I}_2^\sigma$, $\mathbb{N}\setminus M_1 \in \mathcal{I}_\sigma$ and $\mathbb{N}\setminus M_2 \in \mathcal{I}_\sigma$) such that for each $x \in X$, the following limits hold:

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M}} d(x, A_{kj}),\ \lim_{\substack{k\to\infty \\ k\in M_1}} d(x, A_{kj})\ (j \in \mathbb{N}) \text{ and } \lim_{\substack{j\to\infty \\ j\in M_2}} d(x, A_{kj})\ (k \in \mathbb{N}).$$

Note that if $\{A_{kj}\}$ is $R(\mathcal{I}_W^{\sigma*}, \mathcal{I}_{W_2}^{\sigma*})$-convergent to A, then for each $x \in X$, we write

$$R(\mathcal{I}_W^{\sigma*}, \mathcal{I}_{W_2}^{\sigma*}) - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A) \text{ or } d(x, A_{kj}) \overset{R(\mathcal{I}_W^{\sigma*}, \mathcal{I}_{W_2}^{\sigma*})}{\longrightarrow} A.$$

Theorem 2.8. *If a double sequence $\{A_{kj}\}$ is $R(\mathcal{I}_W^{\sigma*}, \mathcal{I}_{W_2}^{\sigma*})$-convergent, then $\{A_{kj}\}$ is $R(\mathcal{I}_W^{\sigma}, \mathcal{I}_{W_2}^{\sigma})$-convergent.*

Proof. Let $\{A_{kj}\}$ be $R(\mathcal{I}_W^{\sigma*}, \mathcal{I}_{W_2}^{\sigma*})$-convergent. Then, $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{\sigma*}$-convergent and so, by Lemma 1.3 $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{\sigma}$-convergent. Also, there exist the sets

$M_1, M_2 \in \mathcal{F}(\mathcal{I}_\sigma)$ such that for each $x \in X$,

$$(\forall \varepsilon > 0)\ (\exists k_0 \in \mathbb{N})\ (\forall k \geq k_0)\ (k \in M_1)\ |d(x, A_{kj}) - d(x, B_j)| < \varepsilon,$$

for some B_j and each $j \in \mathbb{N}$, and

$$(\forall \varepsilon > 0)\ (\exists j_0 \in \mathbb{N})\ (\forall j \geq j_0)\ (j \in M_2)\ |d(x, A_{kj}) - d(x, C_k)| < \varepsilon,$$

for some C_k and each $k \in \mathbb{N}$. Hence, we have

$$\begin{aligned} A(\varepsilon) &= \{k \in \mathbb{N} : |d(x, A_{kj}) - d(x, B_j)| \geq \varepsilon\} \subset H_1 \cup \{1, \ldots, (k_0 - 1)\},\ (j \in \mathbb{N}), \\ B(\varepsilon) &= \{j \in \mathbb{N} : |d(x, A_{kj}) - d(x, C_k)| \geq \varepsilon\} \subset H_2 \cup \{1, \ldots, (j_0 - 1)\},\ (k \in \mathbb{N}), \end{aligned}$$

for $H_1, H_2 \in \mathcal{I}_\sigma$. Since $\mathcal{I}_\sigma$ is an admissible ideal we get

$$H_1 \cup \{1, 2, \ldots, (k_0 - 1)\} \in \mathcal{I}_\sigma \text{ and } \ H_2 \cup \{1, 2, \ldots, (j_0 - 1)\} \in \mathcal{I}_\sigma$$

and therefore $A(\varepsilon) \in \mathcal{I}_\sigma$ and $B(\varepsilon) \in \mathcal{I}_\sigma$. This shows that the double sequence $\{A_{kj}\}$ is $R(\mathcal{I}^\sigma_W, \mathcal{I}^\sigma_{W_2})$-convergent. □

Theorem 2.9. *Let $\mathcal{I}_\sigma$ has property (AP) and $\mathcal{I}^\sigma_2$ has property $(AP2)$. If a double sequence $\{A_{kj}\}$ is $R(\mathcal{I}^\sigma_W, \mathcal{I}^\sigma_{W_2})$-convergent, then $\{A_{kj}\}$ is $R(\mathcal{I}^{\sigma*}_W, \mathcal{I}^{\sigma*}_{W_2})$-convergent.*

Proof. Let a double sequence $\{A_{kj}\}$ be $R(\mathcal{I}^\sigma_W, \mathcal{I}^\sigma_{W_2})$-convergent. Then, $\{A_{kj}\}$ is $\mathcal{I}^\sigma_{W_2}$-convergent and so, by Lemma 1.4 $\{A_{kj}\}$ is $\mathcal{I}^{\sigma*}_{W_2}$-convergent. Also, for each $\varepsilon > 0$ and each $x \in X$ we have

$$A(\varepsilon) = \{k \in \mathbb{N} : |d(x, A_{kj}) - d(x, B_j)| \geq \varepsilon\} \in \mathcal{I}_\sigma,$$

for some B_j and each $j \in \mathbb{N}$, and

$$B(\varepsilon) = \{j \in \mathbb{N} : |d(x, A_{kj}) - d(x, C_k)| \geq \varepsilon\} \in \mathcal{I}_\sigma,$$

for some C_k and each $k \in \mathbb{N}$.

Now, for each $x \in X$ we put

$$\begin{aligned} A_1 &= \{k \in \mathbb{N} : |d(x, A_{kj} - B_j)| \geq 1\}, \\ A_t &= \left\{k \in \mathbb{N} : \frac{1}{t} \leq |d(x, A_{kj}) - d(x, B_j)| < \frac{1}{t-1}\right\}, \end{aligned}$$

for $t \geq 2$, some B_j and each $j \in \mathbb{N}$. It is clear that $A_m \cap A_n = \emptyset$, for $m \neq n$ and $A_m \in \mathcal{I}_\sigma$, for each $m \in \mathbb{N}$. By the property (AP) there is a countable family of sets $\{B_1, B_2, \ldots\}$ in $\mathcal{I}_\sigma$ such that $A_n \triangle B_n$ is a finite set for each $n \in \mathbb{N}$ and $B = \bigcup_{n=1}^{\infty} B_n \in \mathcal{I}_\sigma$.

We prove that

$$\lim_{\substack{k \to \infty \\ k \in M}} d(x, A_{kj}) = d(x, B_j), \text{ some } B_j \text{ and each } j \in \mathbb{N},$$

for $M = \mathbb{N}\backslash B \in \mathcal{F}(\mathcal{I}_\sigma)$. Let $\delta > 0$ be given. Choose $t \in \mathbb{N}$ such that $1/t < \delta$. Then, we have for each $x \in X$,

$$\{k \in \mathbb{N} : |d(x, A_{kj}) - d(x, B_j)| \geq \delta\} \subset \bigcup_{n=1}^{t} x_n, \text{ for some } B_j \text{ and each } j \in \mathbb{N}.$$

Since $A_n \triangle B_n$ is a finite set for $n \in \{1, 2, \ldots, t\}$, there exists $k_0 \in \mathbb{N}$ such that

$$\Big(\bigcup_{n=1}^{t} B_n\Big) \cap \{k : k \geq k_0\} = \Big(\bigcup_{n=1}^{t} A_n\Big) \cap \{k : k \geq k_0\}.$$

If $k \geq k_0$ and $k \notin B$, then

$$k \notin \bigcup_{n=1}^{t} B_n \text{ and so } k \notin \bigcup_{n=1}^{t} A_n.$$

Thus, we have

$$|d(x, A_{kj}) - d(x, B_j)| < \frac{1}{t} < \delta,$$

for some B_j, each $j \in \mathbb{N}$ and each $x \in X$. This implies that

$$\lim_{k\to\infty} d(x, A_{kj}) = d(x, B_j).$$

Hence, we have

$$\mathcal{I}_\sigma^* - \lim_{k\to\infty} d(x, A_{kj}) = d(x, B_j),$$

for some B_j, each $j \in \mathbb{N}$ and each $x \in X$.

Similarly, for the set

$$B(\varepsilon) = \{j \in \mathbb{N} : |d(x, A_{kj}) - d(x, C_k)| \geq \varepsilon\} \in \mathcal{I}_\sigma,$$

we have

$$\mathcal{I}_\sigma^* - \lim_{j\to\infty} d(x, A_{kj}) = d(x, C_k),$$

for some C_k, each $m \in \mathbb{N}$ and each $x \in X$. Hence, a double sequence $\{A_{kj}\}$ is $R(\mathcal{I}_W^{\sigma*}, \mathcal{I}_{W_2}^{\sigma*})$-convergent. □

Definition 2.10. A double sequence $\{A_{kj}\}$ is said to be regularly $(\mathcal{I}_W^\sigma, \mathcal{I}_{W_2}^\sigma)$-Cauchy sequence ($R(\mathcal{I}_W^\sigma, \mathcal{I}_{W_2}^\sigma)$-Cauchy sequence), if it is $\mathcal{I}_{W_2}^\sigma$-Cauchy in Pringsheim's sense and also for every $\varepsilon > 0$ and each $x \in X$ there exist numbers $m_j = m_j(\varepsilon), n_k = n_k(\varepsilon) \in \mathbb{N}$ such that

$$A_1(\varepsilon) = \big\{k \in \mathbb{N} : |d(x, A_{kj}) - d(x, A_{m_j j})| \geq \varepsilon\big\} \in \mathcal{I}_\sigma, (j \in \mathbb{N})$$

and

$$A_2(\varepsilon) = \big\{j \in \mathbb{N} : |d(x, A_{kj}) - d(x, A_{kn_k})| \geq \varepsilon\big\} \in \mathcal{I}_\sigma, (k \in \mathbb{N})$$

holds.

Theorem 2.11. *If a double sequence $\{A_{kj}\}$ is $R(\mathcal{I}_W^\sigma, \mathcal{I}_{W_2}^\sigma)$-convergent, then $\{A_{kj}\}$ is $R(\mathcal{I}_W^\sigma, \mathcal{I}_{W_2}^\sigma)$-Cauchy sequence.*

Proof. Let $\{A_{kj}\}$ be $R(\mathcal{I}_W^\sigma, \mathcal{I}_{W_2}^\sigma)$-convergent. Then, $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^\sigma$-convergent and by Lemma 1.5, it is $\mathcal{I}_{W_2}^\sigma$-Cauchy sequence. Also, for every $\varepsilon > 0$ and each $x \in X$ we have

$$A_1\Big(\frac{\varepsilon}{2}\Big) = \Big\{k \in \mathbb{N} : |d(x, A_{kj}) - d(x, B_j)| \geq \frac{\varepsilon}{2}\Big\} \in \mathcal{I}_\sigma,$$

for some B_j and each $j \in \mathbb{N}$, and

$$A_2\Big(\frac{\varepsilon}{2}\Big) = \Big\{j \in \mathbb{N} : |d(x, A_{kj}) - d(x, C_k)| \geq \frac{\varepsilon}{2}\Big\} \in \mathcal{I}_\sigma,$$

for some C_k and each $k \in \mathbb{N}$. Since $\mathcal{I}_\sigma$ is an admissible ideal, for each $x \in X$ the sets

$$A_1^c\Big(\frac{\varepsilon}{2}\Big) = \Big\{k \in \mathbb{N} : |d(x, A_{kj}) - d(x, B_j)| < \frac{\varepsilon}{2}\Big\},$$

for some B_j and each $j \in \mathbb{N}$, and

$$A_2^c\Big(\frac{\varepsilon}{2}\Big) = \Big\{j \in \mathbb{N} : |d(x, A_{kj}) - d(x, C_k)| < \frac{\varepsilon}{2}\Big\},$$

for some C_k and each $k \in \mathbb{N}$, are nonempty and belong to $\mathcal{F}(\mathcal{I}_\sigma)$. For $m_j \in A_1^c(\frac{\varepsilon}{2})$, $(j \in \mathbb{N}$ and $m_j > 0)$ we have

$$|d(x, A_{m_j j}) - d(x, B_j)| < \frac{\varepsilon}{2},$$

for some B_j, each $j \in \mathbb{N}$ and each $x \in X$. Now, for each $\varepsilon > 0$ we define the set

$$B_1(\varepsilon) = \{k \in \mathbb{N} : |d(x, A_{kj}) - d(x, A_{m_j j})| \geq \varepsilon\},\ (j \in \mathbb{N}),$$

where $m_j = m_j(\varepsilon) \in \mathbb{N}$. We must prove $B_1(\varepsilon) \subset A_1(\frac{\varepsilon}{2})$. Let $k \in B_1(\varepsilon)$. Then, for $m_j \in A_1^c(\frac{\varepsilon}{2})$, $(j \in \mathbb{N}$ and $m_j > 0)$ we have

$$\begin{aligned}\varepsilon \leq |d(x, A_{kj}) - d(x, A_{m_j j})| &\leq |d(x, A_{kj}) - d(x, B_j)| + |d(x, A_{m_j j}) - d(x, B_j)| \\ &< |d(x, A_{kj}) - d(x, B_j)| + \frac{\varepsilon}{2},\end{aligned}$$

for some B_j and each $j \in \mathbb{N}$. This shows that

$$\frac{\varepsilon}{2} < |d(x, A_{kj}) - d(x, B_j)|$$

and so $k \in A_1(\frac{\varepsilon}{2})$. Hence, we have $B_1(\varepsilon) \subset A_1(\frac{\varepsilon}{2})$.

Similarly, for each $\varepsilon > 0$ and $n_k \in A_2^c(\frac{\varepsilon}{2})$ $(k \in \mathbb{N}$ and $n_k > 0)$ we have

$$|d(x, A_{kn_k}) - d(x, C_k)| < \frac{\varepsilon}{2},$$

for some C_k, each $k \in \mathbb{N}$ and each $x \in X$. Therefore, it can be seen that

$$B_2(\varepsilon) \subset A_2(\frac{\varepsilon}{2}),$$

where

$$B_2(\varepsilon) = \{j \in \mathbb{N} : |d(x, A_{kj}) - d(x, A_{kn_k})| \geq \varepsilon\},\ (k \in \mathbb{N}),$$

where $n_k = n_k(\varepsilon) \in \mathbb{N}$.

Therefore, we have $B_1(\varepsilon) \in \mathcal{I}_\sigma$ and $B_2(\varepsilon) \in \mathcal{I}_\sigma$. This shows that $\{A_{kj}\}$ is $R(\mathcal{I}^\sigma_W, \mathcal{I}^\sigma_{W_2})$-Cauchy sequence. □

Definition 2.12. A double sequence $\{A_{kj}\}$ is regularly $(\mathcal{I}^{\sigma*}_W, \mathcal{I}^{\sigma*}_{W_2})$-Cauchy sequence ($R(\mathcal{I}^{\sigma*}_W, \mathcal{I}^{\sigma*}_{W_2})$-Cauchy sequence) if there exist the sets $M \in \mathcal{F}(\mathcal{I}^\sigma_2)$, $M_1 \in \mathcal{F}(\mathcal{I}_\sigma)$ and $M_2 \in \mathcal{F}(\mathcal{I}_\sigma)$ (that is $\mathbb{N} \times \mathbb{N} \backslash M = H \in \mathcal{I}^\sigma_2$, $\mathbb{N} \backslash M_1 \in \mathcal{I}_\sigma$ and $\mathbb{N} \backslash M_2 \in \mathcal{I}_\sigma$) and for every $\varepsilon > 0$ and each $x \in X$, there exist $N = N(\varepsilon)$, $s = s(\varepsilon)$, $t = t(\varepsilon)$, $m_j = m_j(\varepsilon)$, $n_k = n_k(\varepsilon) \in \mathbb{N}$ such that whenever $k, j, s, t, m_j, n_k \geq N$, we have

$$|d(x, A_{kj}) - d(x, A_{st})| < \varepsilon, \ (for\ (k, j), (s, t) \in M),$$
$$|d(x, A_{kj}) - d(x, A_{m_j j})| < \varepsilon, \ (for\ each\ k \in M_1\ and\ each\ j \in \mathbb{N}),$$
$$|d(x, A_{kj}) - d(x, A_{kn_k})| < \varepsilon, \ (for\ each\ j \in M_2\ and\ each\ k \in \mathbb{N}).$$

Theorem 2.13. *If a double sequence $\{A_{kj}\}$ is $R(\mathcal{I}^{\sigma*}_W, \mathcal{I}^{\sigma*}_{W_2})$-Cauchy sequence, then $\{A_{kj}\}$ is $R(\mathcal{I}^\sigma_W, \mathcal{I}^\sigma_{W_2})$-Cauchy sequence.*

Proof. Since a double sequence $\{A_{kj}\}$ is $R(\mathcal{I}^{\sigma*}_W, \mathcal{I}^{\sigma*}_{W_2})$-Cauchy sequence, then $\{A_{kj}\}$ is $\mathcal{I}^{\sigma*}_{W_2}$-Cauchy sequence and so it is $\mathcal{I}^\sigma_{W_2}$-Cauchy sequence by Lemma 1.6. Also, since $\{A_{kj}\}$ is $R(\mathcal{I}^{\sigma*}_W, \mathcal{I}^{\sigma*}_{W_2})$-Cauchy sequence, there exist the sets $M_1 \in \mathcal{F}(\mathcal{I}_\sigma)$ and $M_2 \in \mathcal{F}(\mathcal{I}_\sigma)$ (that is, $\mathbb{N} \backslash M_1 \in \mathcal{I}_\sigma$ and $\mathbb{N} \backslash M_2 \in \mathcal{I}_\sigma$) and for every $\varepsilon > 0$ and each $x \in X$, there exist $N = N(\varepsilon)$, $m_j = m_j(\varepsilon)$, $n_k = n_k(\varepsilon) \in \mathbb{N}$ such that we have

$$|d(x, A_{kj}) - d(x, A_{m_j j})| < \varepsilon, \ (for\ each\ k \in M_1\ and\ each\ j \in \mathbb{N}),$$
$$|d(x, A_{kj}) - d(x, A_{kn_k})| < \varepsilon, \ (for\ each\ j \in M_1\ and\ each\ k \in \mathbb{N}),$$

whenever $k, j, m_j, n_k \geq N$. Therefore, $H_1 = \mathbb{N} \backslash M_1 \in \mathcal{I}_\sigma$ and $H_2 = \mathbb{N} \backslash M_2 \in \mathcal{I}_\sigma$ we have

$$A_1(\varepsilon) = \{k \in \mathbb{N} : |d(x, A_{kj}) - d(x, A_{m_j j})| \geq \varepsilon\} \subset H_1 \cup \{1, \ldots, (N-1)\}, (j \in \mathbb{N})$$

for each $k \in M_1$ and each $x \in X$, and

$$A_2(\varepsilon) = \{j \in \mathbb{N} : |d(x, A_{kj}) - d(x, A_{kn_k})| \geq \varepsilon\} \subset H_2 \cup \{1, \ldots, (N-1)\}, (k \in \mathbb{N})$$

for each $j \in M_2$ and each $x \in X$. Since $\mathcal{I}_\sigma$ is an admissible ideal,

$$H_1 \cup \{1, 2, \ldots, (N-1)\} \in \mathcal{I}_\sigma \text{ and } H_2 \cup \{1, 2, \ldots, (N-1)\} \in \mathcal{I}_\sigma,$$

and so, $A_1(\varepsilon) \in \mathcal{I}_\sigma$ and $A_2(\varepsilon) \in \mathcal{I}_\sigma$. Hence, $\{A_{kj}\}$ is $R(\mathcal{I}^\sigma_W, \mathcal{I}^\sigma_{W_2})$-Cauchy sequence. □

3. Conclusions and Future Work

We investigated the concepts of Wijsman regularly invariant convergence typies and Wijsman regularly ideal invariant convergence and Cauchy sequence typies. These concepts can also be studied for the lacunary sequence in the future.

4. Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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