Erzincan Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
2018, 11(2), 237-242

Erzincan University
Journal of Science and Technology
2018, 11(2), 237-242

ISSN: 1307-9085, e-ISSN: 2149-4584 DOI: 10.18185/erzifbed.402310

A Note on Ricci Solitons on Para-Sasakian Manifolds

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Geliş / Received: 06/03/2018, Kabul / Accepted: 12/07/2018

Abctract
In the present paper, in an almost para-contact metric manifold, especially on para-Sasakian manifolds, Ricci solitons were examined. Also, we consider the *generalized quasi-conformal curvature tensor* on a para-Sasakian manifold admitting Ricci soliton.

Keywords: Para-contact manifolds, Ricci soliton, Semi-symmetric manifolds

Para-Sasakian Manifoldlarda Ricci Solitonlar Üzerine Bir Not

Öz
Bu çalışmada, bir hemen hemen para-kontakt metrik manifoldda, özellikle para-Sasakian manifoldlar üzerinde, Ricci solitonlar incelenmiştir. Ayrıca bir para-Sasakian manifold üzerinde Ricci soliton şartı ile birlikte *genelleştirilmiş quasi-conformal eğrilik tensörü* ele alınmıştır.

Anahtar Kelimeler: Para-kontakt manifoldlar, Ricci soliton, Semi simetrik manifold

1. Introduction

R.S. Hamilton studied the notion of Ricci solitons as which were generalization of an Einstein metric (Hamilton, 1982). On a semi-Riemannian manifold M with semi-Riemannian metric g, a Ricci soliton is a triple (g,V,λ) such that

$$(\pounds_V g)(U,W)+2S(U,W)+2\lambda g(U,W)=0, \quad (1)$$

where W,U vector fields on M, λ is a constant and $\pounds$ denotes the Lie derivative.

The Ricci soliton is called expanding, steady or shrinking if

$$\lambda>0,\ \lambda=0 \ \text{ or } \ \lambda<0.$$

In the recent years, this subject has been introduced in some papers; on Kaehler manifolds Hodes and Fong (2013), on contact manifolds Calin and Crasmereanu (2010); Turan et al., (2012); De et al., (2012) and paracontact manifolds Yüksel Perktaş and Keleş, (2015); Adara et al., (2018).

In 1985, the concept of almost para-contact structure in a semi-Riemannian manifold was introduced in Kaneyuki and Konzai (1985) and then the almost para-complex structure on $M^{2n+1}\times$ characterized.

Later, S. Zamkovoy Zamkovoy (2009) investigated almost para-contact structure and some considerable subclasses. Especially, in the recent years many authors have pointed out the importance of para-contact geometry Erdem (2002); Cortes et al. (2006), and in particular of para-Sasakian manifold by several papers (Acet et al. 2014; Acet et al. 2016).

K. Yano and S. Sawaki defined the concept of quasi-conformal curvature tensor (Yano and Sawaki, 1968). Later, generalized quasi-conformal curvature tensor was investigated in (Baishya and Chawdhury, 2016). This tensor is given by

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$$\begin{aligned}\tilde{W}(U,Y)T &= \frac{2n-1}{2n+1}\begin{pmatrix}(2nr-s+1)\\-\{2n(r+s)+1\}t\end{pmatrix}C(U,Y)T\\&\quad+\big(2nr-s+1\big)E(U,Y)T\\&\quad+\big(2n(s-r)\big)P(U,Y)T\\&\quad\frac{2n-1}{2n+1}\begin{pmatrix}2n(t-1)(r+s)\\+(t-1)\end{pmatrix}K(U,Y)T\end{aligned}$$

(2)

for any $Y,U,T\in TM$, where $r,s,t\in\mathbb{R}$ and curvature tensors are given by

$$\begin{aligned}C(U,Y)T &= R(U,Y)T-\frac{1}{2n-1}\begin{pmatrix}S(Y,T)U-S(U,T)Y\\+g(Y,T)QU-g(U,T)QY\end{pmatrix}\\&\quad+\frac{r}{(2n-1)2n}\big(g(Y,T)U-g(U,T)Y\big)\end{aligned}$$

which is known conformal curvature tensor,

$$E(U,Y)T = R(U,Y)T-\frac{\tau}{2n(2n-1)}\big(g(Y,T)U-g(U,T)Y\big)$$

which is known concircular curvature tensor,

$$K(U,Y)T = R(U,Y)T-\frac{1}{2n-1}\begin{pmatrix}S(Y,T)U-S(U,T)Y\\-g(U,T)QY+g(Y,T)QU\end{pmatrix}$$

which is known conharmonical curvature tensor

$$P(U,Y)T = R(U,Y)T-\frac{1}{2n}\big(S(Y,T)U-S(U,T)Y\big)$$

which is known projective curvature tensor.

This generalized quasi-conformal curvature tensor is reduced to be

i) *Riemannian curvature tensor* if

$$r=s=0,\quad t=0,$$

ii) *Conformal curvature tensor* (Eisenhart, 1949) if

$$r=-\frac{1}{2n-1}=s,\ t=1,$$

iii) *Concircular curvature tensor* (Ishii, 1957) if

$$r=s=0,\ t=1,$$

iv) *Conharmonic curvature tensor* (Yano and Bochner, 1953) if

$$r=-\frac{1}{2n-1}=s,\ t=0,$$

v) *Projective curvature tensor* (Ishii, 1957) if

$$r=-\frac{1}{2n},\ s=0=t,$$

vi) *m-Projective curvature tensor* (Pokhariyal and Mishra, 1970) if

$$r=-\frac{1}{4n}=s,\ t=0.$$

Preliminaries

A para-contact manifold is a differentiable manifold equipped with a vector field ξ, a 1-form η and a tensor field ϕ of type (1,1) satisfying (Kaneyuki and Konzai, 1985):

$$\phi^2=I-\eta\otimes\xi,$$

(3)

$$\eta(\xi)=1,\ \phi\xi=0.$$

(4)

As a consequence of (4), we get

$$\eta\circ\phi=0,\ rank\phi=2n.$$

Moreover if the manifold M is equipped with a semi Riemannian metric g which is known compatible metric satisfying (Zamkovoy, 2009)

$$g(\phi W,\phi X)=-g(W,X)+\eta(W)\eta(X),\quad (5)$$

then one can say that M is an almost para-contact metric manifold.

From the definition, we get (Zamkovoy, 2009),

$$\eta(W)=g(\xi,W).$$

(6)

and

$$g(\phi W,X)=-g(W,\phi X),$$

(7)

The fundamental 2-form Φ of an almost para-contact structure (ϕ,ξ,η,g) is defined by

$$\Phi(W,X)=g(W,\phi X),$$

(8)

If

$$g(W,\phi X)=d\eta(W,X)$$

where

$$d\eta(W,X)=(1/2)(W\eta(X)-X\eta(W)-\eta([W,X])),$$

an almost para-contact metric structure is called a para-contact metric structure.

For (M,ϕ,ξ,η,g) an almost para-contact metric manifold one can find a local orthonormal basis which is known $\phi-$basis $(X_i,\phi X_i,\xi)$ $(i=1,2,...,n)$ (Zamkovoy, 2009).

An almost para-contact metric manifold M is a para-Sasakian manifold iff (Zamkovoy, 2009)

$$(\nabla_X\phi)W=-g(X,W)\xi+\eta(W)X. \tag{9}$$

In view of (9), one arrive at

$$\nabla_W\xi=-\phi W. \tag{10}$$

Moreover, in para-Sasakian manifolds, we get

$$R(\xi,W)U=-g(U,W)\xi+\eta(U)W, \tag{11}$$

$$g(R(W,Y)X,\xi)=g(W,X)\eta(Y)-g(Y,X)\eta(W), \tag{12}$$

$$R(\xi,W)\xi=W-\eta(W)\xi, \tag{13}$$

$$R(W,X)\xi=\eta(W)X-\eta(X)W, \tag{14}$$

$$S(W,\xi)=-2n\eta(W). \tag{15}$$

Assume that M^{2n+1} is a para-Sasakian manifold and (g,V,λ) is a Ricci soliton on M. From (1), we get

$$2S(W,X)=-(\pounds_V g)(W,X)-2\lambda g(W,X).$$

Using (6) with (10), we arrive at

$$S(W,X)=-\lambda g(W,X). \tag{16}$$

So, we can give:

Theorem 2.1 A para-Sasakian manifold admitting a Ricci soliton is an Einstein manifold.

Moreover, putting $X=\xi$ in (16) and using (15), we get

$$\lambda=2n. \tag{17}$$

So, we get the following.

Theorem 2.2 A Ricci soliton in para-Sasakian manifold M^{2n+1} is expanding.

Main Theorems

Definition 3.1 A para-contact manifold is called semi-symmetric type if the curvature tensor $\bar{W}$ fulfills the equation

$$R(U,V)\cdot\bar{W}=0 \tag{18}$$

for all $U,V\in TM$.

We suppose that the equation (18) fulfills on M. So we get

$$\begin{aligned}&R(\xi,U)\bar{W}(Z,Y)X-\bar{W}(R(\xi,U)Z,Y)X\\&-\bar{W}(Z,R(\xi,U)Y)X-\bar{W}(Z,Y)R(\xi,U)X\\&=0.\end{aligned} \tag{19}$$

Using the equation (13) in (19), we have

$$\begin{aligned}&\eta(\bar{W}(Z,Y)X)U-g(U,\bar{W}(Z,Y)X)\xi\\&+g(U,Z)\bar{W}(\xi,Y)X-\eta(Z)\bar{W}(U,Y)X\\&+g(U,Y)\bar{W}(Z,\xi)X-\eta(Y)\bar{W}(Z,U)X\\&+g(U,X)\bar{W}(Z,Y)\xi-\eta(X)\bar{W}(Z,Y)U\\&=0.\end{aligned} \tag{20}$$

Taking an inner product with ξ, we obtain

$$\begin{aligned}&\eta(\bar{W}(Z,Y)X)\eta(U)-g(U,\bar{W}(Z,Y)X)\\&+g(U,Z)\eta(\bar{W}(\xi,Y)X)\\&-\eta(Z)\eta(\bar{W}(U,Y)X)\\&+g(U,Y)\eta(\bar{W}(Z,\xi)X)-\eta(Y)\eta(\bar{W}(Z,U)X)\end{aligned} \tag{21}$$

$= 0.$

Using (2) in (21) and after some calculations, we get

$$\left[\begin{array}{c} 1+\lambda(r+s) \\ +\frac{t\tau}{2n+1}\left(\frac{1}{2n}+r+s\right) \end{array}\right]\left(\begin{array}{c} g(U,Y)g(Z,X) \\ -g(U,Z)g(Y,X) \end{array}\right)$$

$$-g(U,\bar{W}(Z,Y)X)$$

(22)

$$= 0.$$

In view of (2) with (16) in (22), we find

$$-g(\bar{W}(Z,Y)X,U)+g(Z,X)g(U,Y)$$
$$-g(U,Z)g(Y,X) = 0.$$

(23)

Since $\{e_i, \phi e_i, \xi\}$ is a basis of M . from (23), we have

$$S(Y,X)+2ng(Y,X) = 0.$$

(24)

Now, one can state:

Theorem 3.1 A para-Sasakian manifold satisfying $R(\xi,X)\cdot\bar{W} = 0$ admitting a Ricci soliton (g,V,λ) is an Einstein manifold.

Taking $X = \xi = Y$ in (24) and in view of (16), we arrive at

$$\lambda = 2n,$$

which yields λ is positive. Thus, we have :

Theorem 3.2 Let (g,V,λ) be a Ricci soliton on para-Sasakian manifold. If the semi-symmetric condition $R(\xi,X)\cdot\bar{W} = 0$ satisfies on M, then the Ricci soliton is expanding.

Now, assume that (g,V,λ) is a Ricci soliton on a para-Sasakian manifold M and the equation $S(\xi,X)\cdot\bar{W} = 0$ satisfies on M . By use of the following equations

$$(S(\xi,X)\cdot\bar{W})(U,Y)Z = ((X\wedge_S\xi)\bar{W})(U,Y)Z$$

$$= (X\wedge_S\xi)\bar{W}(U,Y)Z+\bar{W}((X\wedge_S\xi)U,Y)Z$$

$$+\bar{W}(U,(X\wedge_S\xi)Y)Z+\bar{W}(U,Y)(X\wedge_S\xi)Z,$$

where the endomorphisim

$$(U\wedge_S Y)Z = S(Y,Z)U-S(U,Z)Y,$$

we obtain

$$(S(X,\xi)\cdot\bar{W})(U,Y)Z = S(\xi,\bar{W}(U,Y)Z)X$$
$$-S(X,\bar{W}(U,Y)Z)\xi+S(\xi,U)\bar{W}(X,Y)Z$$

$$-S(X,U)\bar{W}(\xi,Y)Z+S(\xi,Y)\bar{W}(U,X)Z$$

(25) $$-S(X,Y)\bar{W}(U,\xi)Z$$

$$+S(\xi,Z)\bar{W}(U,Y)X$$

$$-S(X,Z)\bar{W}(U,Y)\xi.$$

Now, if we consider the equation (16) in (25), then we find

$$\lambda\left[\begin{array}{l} -\eta(\bar{W}(U,Y)Z)X+g(X,\bar{W}(U,Y)Z)\xi \\ +g(X,U)\bar{W}(\xi,Y)Z-\eta(U)\bar{W}(X,Y)Z \\ +g(X,Y)\bar{W}(U,\xi)Z-\eta(Y)\bar{W}(U,X)Z \\ +g(X,Z)\bar{W}(U,Y)\xi-\eta(Z)\bar{W}(U,Y)X \end{array}\right] = 0.$$

(26)

Taking inner product with ξ and in view of (11) - (14) with (2), one can find

$$\lambda\begin{bmatrix} -\eta(\tilde{W}(U,Y)Z)\eta(X)+g(X,\tilde{W}(U,Y)Z) \\ +g(X,U)\eta(\tilde{W}(\xi,Y)Z)-\eta(U)\eta(\tilde{W}(X,Y)Z) \\ +g(X,Y)\eta(\tilde{W}(U,\xi)Z)-\eta(Y)\eta(\tilde{W}(U,X)Z) \\ +g(X,Z)\eta(\tilde{W}(U,Y)\xi)-\eta(Z)\eta(\tilde{W}(U,Y)X) \end{bmatrix}=0,$$

which yields

$$\lambda\begin{bmatrix} g(X,R(U,Y)Z)-g(Y,Z)g(U,X)+g(X,Y)g(U,Z) \\ -2\begin{pmatrix} 1+\lambda(r+s) \\ +\frac{t\tau}{2n+1}\left(\frac{1}{2n}+r+s\right) \end{pmatrix}\begin{pmatrix} g(U,Z)\eta(X)\eta(Y) \\ -g(Y,Z)\eta(X)\eta(U) \end{pmatrix} \end{bmatrix}=0.$$

(27)

Let $\{e_i,\phi e_i,\xi\}_{(i=1,\ldots,n)}$ is a basis of M. Thus by a contraction of (27), we get

$$\lambda(S(Y,Z)-2ng(Y,Z))=0.$$

(28)

Theorem 3.3 A para-Sasakian manifold M satisfying $S(\xi,X)\cdot\tilde{W}=0$ admitting a non-steady Ricci soliton (g,V,λ) is an Einstein manifold.

Now, taking $Y=\xi=Z$ in (28) and using (16), we obtain

$$\lambda(\lambda+2n)=0,$$

which implies $\lambda=0$ or $\lambda=-2n$. Thus, we get:

Theorem 3.4 Let (g,V,λ) be a Ricci soliton on a para-Sasakian manifold. If the equation $S(\xi,X)\cdot\tilde{W}=0$ satisfies on M, the Ricci soliton is either steady or shrinking.

Take a para-Sasakian manifold satisfying the condition $S(\xi,X)\cdot R=0$. By definition we have

$$(S(\xi,X)\cdot R)(U,Y)Z=((X\wedge_S\xi)R)(U,Y)Z$$

$$=(X\wedge_S\xi)R(U,Y)Z+R((X\wedge_S\xi)U,Y)Z$$

$$+R(U,(X\wedge_S\xi)Y)Z+R(U,Y)(X\wedge_S\xi)Z,$$

where the endomorphisim

$$(U\wedge_S Y)Z=S(Y,Z)U-S(U,Z)Y,$$

we obtain

$$\begin{aligned}(S(X,\xi)\cdot R)(U,Y)Z &= S(\xi,R(U,Y)Z)X\\ &-S(X,R(U,Y)Z)\xi+S(\xi,U)R(X,Y)Z\end{aligned}$$

$$\begin{aligned}&-S(X,U)R(\xi,Y)Z+S(\xi,Y)R(U,X)Z\\ &-S(X,Y)R(U,\xi)Z\\ &+S(\xi,Z)R(U,Y)X\end{aligned}$$

(29)

$$-S(X,Z)R(U,Y)\xi.$$

Now, if we consider the equation (16) in (29), then we get

$$\lambda\begin{bmatrix} -\eta(R(U,Y)Z)X+g(X,R(U,Y)Z)\xi \\ +g(X,U)R(\xi,Y)Z-\eta(U)R(X,Y)Z \\ +g(X,Y)R(U,\xi)Z-\eta(Y)R(U,X)Z \\ +g(X,Z)R(U,Y)\xi-\eta(Z)R(U,Y)X \end{bmatrix}=0.$$

(30)

By taking inner product with ξ and by use of (11)-(14) with (2), we find

$$\lambda\begin{bmatrix} -\eta(R(U,Y)Z)\eta(X)+g(X,R(U,Y)Z) \\ +g(X,U)\eta(R(\xi,Y)Z)-\eta(U)\eta(R(X,Y)Z) \\ +g(X,Y)\eta(R(U,\xi)Z)-\eta(Y)\eta(R(U,X)Z) \\ +g(X,Z)\eta(R(U,Y)\xi)-\eta(Z)\eta(R(U,Y)X) \end{bmatrix}=0,$$

(31)

from which

$$\lambda\begin{bmatrix} g(X,R(U,Y)Z)-g(Y,Z)g(U,X) \\ +g(X,Y)g(U,Z) \\ -2\begin{pmatrix} g(U,Z)\eta(X)\eta(Y) \\ -g(Y,Z)\eta(X)\eta(U) \end{pmatrix} \end{bmatrix}=0.$$

(32)

Let $\{e_i, \phi e_i, \xi\}_{(i=1,\ldots,n)}$ is an orthonormal basis of M. Thus by a contraction of (32), we have

$$\lambda\begin{bmatrix} S(Y,Z)-(2n-2)g(Y,Z) \\ -2\eta(Y)\eta(Z) \end{bmatrix} = 0. \quad (33)$$

Theorem 3.5 A para-Sasakian manifold satisfying $S(\xi, X)\cdot R = 0$ admitting a non-steady Ricci soliton (g, V, λ) is a $\eta-$ Einstein manifold.

Also, putting $Y = \xi = Z$ in (33) and in view of (16), we obtain

$\lambda(-\lambda - 2n) = 0,$

which implies $\lambda = 0$ or $\lambda = -2n$. Thus, we get:

Theorem 3.6 Let (g, V, λ) be a Ricci soliton on a para-Sasakian manifold M. If the equation $S(\xi, X)\cdot R = 0$ satisfies on M, then the Ricci soliton is either steady or shrinking.

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