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RESEARCH ARTICLE

Modified operational matrix method for second-order nonlinear ordinary differential equations with quadratic and cubic terms

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ABSTRACT

In this study, by means of the matrix relations between the Laguerre polynomials, and their derivatives, a novel matrix method based on collocation points is modified and developed for solving a class of second-order nonlinear ordinary differential equations having quadratic and cubic terms, via mixed conditions. The method reduces the solution of the nonlinear equation to the solution of a matrix equation corresponding to system of nonlinear algebraic equations with the unknown Laguerre coefficients. Also, some illustrative examples along with an error analysis based on residual function are included to demonstrate the validity and applicability of the proposed method.

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1. Introduction

Nonlinear differential equations and the related initial and boundary value problems play an important role in astrophysics, physics and engineering. In recent years, to solve these problems both analytically and numerically which have applications in various branches of pure and applied sciences, several numerical and analytical methods have been given. But it may not be possible to find the analytical solutions of such problems for all coefficient functions.

These type of mathematical models can be described by particular names such as Riccati equation, nonlinear equations of motion, Duffing's equation, Van Der Pol's equation, the equation of motion with quadratic damping, Emden's equation, Liouville's equation [1–5].

In this study, we consider the second-order nonlinear ordinary differential equations with quadratic and cubic terms:

$$\begin{aligned}&\sum_{k=0}^{2} P_k(x)y^{(k)}(x)+\sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x)y^{(p)}(x)y^{(q)}(x)\\&+\sum_{p=0}^{2}\sum_{q=0}^{p}\sum_{r=0}^{q} Q_{pqr}(x)y^{(p)}(x)y^{(q)}(x)y^{(r)}(x)\\&=g(x),\quad 0\le x\le b<\infty,\end{aligned}\tag{1}$$

with the mixed conditions

$$\sum_{k=0}^{1}(a_{kj}y^{(k)}(0)+b_{kj}y^{(k)}(b))=\lambda_j,\quad j=0,1,\tag{2}$$

where $P_k(x), Q_{pq}(x), Q_{pqr}(x)$ and $g(x)$ are functions defined on the interval $0 \le x \le b < \infty$; a_{kj}, b_{kj} and λ_j are appropriate and real constants; $y(x)$ is an unknown function to be determined [6].

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In this study, we develop a new numerical methods to find the approximate solutions of Eq. (1) in the truncated Laguerre series form

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n L_n(x), \quad 0 \le x \le b < \infty, \tag{3}$$

where $a_n, n = 0, 1, ..., N, N \ge 2$ are the unknown Laguerre coefficients and $L_n(x), n = 0, 1, ..., N$ are the Laguerre functions of first kind defined by

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} x^k, \quad 0 \le x \le b < \infty. \tag{4}$$

2. Operational matrix relations

Firstly, let us write Eq. (1) in the form

$$L[y(x)] + N_2[y(x)] + N_3[y(x)] = g(x), \tag{5}$$

where the linear ordinary differential part

$$L[y(x)] = \sum_{k=0}^{2} P_k(x) y^{(k)}(x), \tag{6}$$

the nonlinear quadratic part

$$N_2[y(x)] = \sum_{p=0}^{2} \sum_{q=0}^{p} Q_{pq}(x) y^{(p)}(x) y^{(q)}(x), \tag{7}$$

and the nonlinear cubic part

$$N_3[y(x)] = \sum_{p=0}^{2} \sum_{q=0}^{p} \sum_{r=0}^{q} Q_{pqr}(x) y^{(p)}(x) y^{(q)}(x) y^{(r)}(x). \tag{8}$$

2.1. Matrix representation of linear ordinary differential part

Now, we consider Eq.(1) and find the matrix forms of each term in the equation. So, we convert Laguerre polynomial solution (3) to the matrix form as

$$\begin{aligned} y(x) = y^{(0)}(x) &\cong \mathbf{L}(x)\mathbf{A}, \\ y^{(1)}(x) &= \mathbf{L}(x)\mathbf{C}\mathbf{A}, \\ y^{(2)}(x) &= \mathbf{L}(x)\mathbf{C}^2\mathbf{A}, \end{aligned} \tag{9}$$

where

$$\mathbf{L}(x) = \begin{bmatrix} L_0(x) & L_1(x) & \cdots & L_N(x) \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ 0 & 0 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T.$$

2.2. Matrix representation of nonlinear quadratic part

Now, we consider matrix representation of nonlinear quadratic part. So, we define the matrices with related to (7) and (9)

$$\begin{aligned} (y^{(0)}(x))^2 &= \mathbf{L}(x)\overline{\mathbf{L}}(x)\overline{\mathbf{A}}, \\ y^{(1)}(x)y^{(0)}(x) &= \mathbf{L}(x)\mathbf{C}\overline{\mathbf{L}}(x)\overline{\mathbf{A}}, \\ (y^{(1)}(x))^2 &= \mathbf{L}(x)\mathbf{C}\overline{\mathbf{L}}(x)\overline{\mathbf{C}\mathbf{A}}, \\ y^{(2)}(x)y^{(1)}(x) &= \mathbf{L}(x)\mathbf{C}^2\overline{\mathbf{L}}(x)\overline{\mathbf{C}\mathbf{A}}, \\ y^{(2)}(x)y^{(0)}(x) &= \mathbf{L}(x)\mathbf{C}^2\overline{\mathbf{L}}(x)\overline{\mathbf{A}}, \\ (y^{(2)}(x))^2 &= \mathbf{L}(x)\mathbf{C}^2\overline{\mathbf{L}}(x)\overline{\mathbf{C}^2\mathbf{A}}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \overline{\mathbf{L}}(x) &= diag\begin{bmatrix} \mathbf{L}(x) & \mathbf{L}(x) & \cdots & \mathbf{L}(x) \end{bmatrix}, \\ \overline{\mathbf{C}} &= diag\begin{bmatrix} \mathbf{C} & \mathbf{C} & \cdots & \mathbf{C} \end{bmatrix}, \\ \overline{\mathbf{A}} &= \begin{bmatrix} a_0\mathbf{A} & a_1\mathbf{A} & \cdots & a_N\mathbf{A} \end{bmatrix}^T. \end{aligned}$$

2.3. Matrix representation of nonlinear cubic part

Let us consider (8) as

$$\begin{aligned} N_3[y(x)] = &\sum_{p=0}^{2} \sum_{q=0}^{p} \sum_{r=0}^{q} Q_{pqr}(x) y^{(p)}(x) y^{(q)}(x) y^{(r)}(x) \\ &+ Q_{000}(x)y^{(0)}(x)y^{(0)}(x)y^{(0)}(x) \\ &+ Q_{100}(x)y^{(1)}(x)y^{(0)}(x)y^{(0)}(x) \\ &+ Q_{110}(x)y^{(1)}(x)y^{(1)}(x)y^{(0)}(x) \\ &+ Q_{111}(x)y^{(1)}(x)y^{(1)}(x)y^{(1)}(x) \\ &+ Q_{200}(x)y^{(2)}(x)y^{(0)}(x)y^{(0)}(x) \\ &+ Q_{210}(x)y^{(2)}(x)y^{(1)}(x)y^{(0)}(x) \\ &+ Q_{211}(x)y^{(2)}(x)y^{(1)}(x)y^{(1)}(x) \\ &+ Q_{220}(x)y^{(2)}(x)y^{(2)}(x)y^{(0)}(x) \\ &+ Q_{221}(x)y^{(2)}(x)y^{(2)}(x)y^{(1)}(x) \\ &+ Q_{222}(x)y^{(2)}(x)y^{(2)}(x)y^{(2)}(x). \end{aligned}$$

So, we define the matrices as

$$\begin{aligned}
(y^{(0)}(x))^3 &= \mathbf{L}(x)\overline{\mathbf{L}}(x)\overline{\overline{\mathbf{L}}}(x)\overline{\overline{\mathbf{A}}},\\
y^{(1)}(x)(y^{(0)}(x))^2 &= \mathbf{L}(x)\mathbf{C}\overline{\mathbf{L}}(x)\overline{\overline{\mathbf{L}}}(x)\overline{\overline{\mathbf{A}}},\\
(y^{(1)}(x))^2y^{(0)}(x) &= \mathbf{L}(x)\mathbf{C}\overline{\mathbf{L}}(x)\overline{\mathbf{C}\overline{\mathbf{L}}}(x)\overline{\overline{\mathbf{A}}},\\
(y^{(1)}(x))^3 &= \mathbf{L}(x)\mathbf{C}\overline{\mathbf{L}}(x)\overline{\mathbf{C}\overline{\mathbf{L}}}(x)\overline{\overline{\mathbf{C}\mathbf{A}}},\\
y^{(2)}(x)(y^{(0)}(x))^2 &= \mathbf{L}(x)\mathbf{C}^2\overline{\mathbf{L}}(x)\overline{\overline{\mathbf{L}}}(x)\overline{\overline{\mathbf{A}}},\\
y^{(2)}(x)y^{(1)}(x)y^{(0)}(x) &= \mathbf{L}(x)\mathbf{C}^2\overline{\mathbf{L}}(x)\overline{\mathbf{C}\overline{\mathbf{L}}}(x)\overline{\overline{\mathbf{A}}},
\end{aligned} \tag{11}$$

$$\begin{aligned}
y^{(2)}(x)(y^{(1)}(x))^2 &= \mathbf{L}(x)\mathbf{C}^2\overline{\mathbf{L}}(x)\overline{\mathbf{C}\overline{\mathbf{L}}}(x)\overline{\overline{\mathbf{C}\mathbf{A}}},\\
(y^{(2)}(x))^2y^{(0)}(x) &= \mathbf{L}(x)\mathbf{C}^2\overline{\mathbf{L}}(x)\overline{\mathbf{C}^2\overline{\mathbf{L}}}(x)\overline{\overline{\mathbf{A}}},\\
(y^{(2)}(x))^2y^{(1)}(x) &= \mathbf{L}(x)\mathbf{C}^2\overline{\mathbf{L}}(x)\overline{\mathbf{C}^2\overline{\mathbf{L}}}(x)\overline{\mathbf{C}\mathbf{A}},\\
(y^{(2)}(x))^3 &= \mathbf{L}(x)\mathbf{C}^2\overline{\mathbf{L}}(x)\overline{\mathbf{C}^2\overline{\mathbf{L}}}(x)\overline{\mathbf{C}^2\overline{\mathbf{A}}},
\end{aligned}$$

where

$$\begin{aligned}
\overline{\overline{\mathbf{L}}}(x) &= diag\left[\begin{array}{cccc}\overline{\mathbf{L}}(x) & \overline{\mathbf{L}}(x) & \cdots & \overline{\mathbf{L}}(x)\end{array}\right],\\
\overline{\overline{\mathbf{C}}} &= diag\left[\begin{array}{cccc}\overline{\mathbf{C}} & \overline{\mathbf{C}} & \cdots & \overline{\mathbf{C}}\end{array}\right],\\
\overline{\overline{\mathbf{A}}} &= \left[\begin{array}{cccc}a_0\overline{\mathbf{A}} & a_1\overline{\mathbf{A}} & \cdots & a_N\overline{\mathbf{A}}\end{array}\right]^T.
\end{aligned}$$

3. Method of solution

Now, we define the collocation points as

$$x_i = \frac{b}{N}i,\ \ i = 0, 1, N;\ \ 0 \le x_0 < x_1 < ... < x_N = b. \tag{12}$$

We substitute the collocation points (12) into Eq. (1), we have the system of matrix equations for $i = 0, 1, ..., N$,

$$\begin{aligned}
&\sum_{k=0}^{2} P_k(x_i)y^{(k)}(x_i) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x_i)y^{(p)}(x_i)y^{(q)}(x_i)\\
&+\sum_{p=0}^{2}\sum_{q=0}^{p}\sum_{r=0}^{q} Q_{pqr}(x_i)y^{(p)}(x_i)y^{(q)}(x_i)y^{(r)}(x_i)\\
&=g(x_i), \quad 0 \le x \le b < \infty,
\end{aligned}$$

or briefly,

$$\begin{aligned}
&\sum_{k=0}^{2} \mathbf{P}_k\mathbf{Y}^{(k)} + \sum_{p=0}^{2}\sum_{q=0}^{p} \mathbf{Q}_{pq}\mathbf{Y}^{(p,q)}\\
&+\sum_{p=0}^{2}\sum_{q=0}^{p}\sum_{r=0}^{q} \mathbf{Q}_{pqr}\mathbf{Y}^{(p,q,r)}\\
&= \mathbf{G}, \quad 0 \le x \le b < \infty,
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
\mathbf{P}_k &= diag\left[\begin{array}{cccc}P_k(x_0) & P_k(x_1) & \cdots & P_k(x_N)\end{array}\right],\\
\mathbf{Q}_{pq} &= diag\left[\begin{array}{cccc}Q_{pq}(x_0) & Q_{pq}(x_1) & \cdots & Q_{pq}(x_N)\end{array}\right],\\
\mathbf{Q}_{pqr} &= diag\left[\begin{array}{cccc}Q_{pqr}(x_0) & Q_{pqr}(x_1) & \cdots & Q_{pqr}(x_N)\end{array}\right],
\end{aligned}$$

and

$$\mathbf{Y}^{(k)} = \begin{bmatrix} y^{(k)}(x_0)\\ y^{(k)}(x_1)\\ \vdots\\ y^{(k)}(x_N)\end{bmatrix},\ \mathbf{G} = \begin{bmatrix} g(x_0)\\ g(x_1)\\ \vdots\\ g(x_N)\end{bmatrix},$$

$$\mathbf{Y}^{(p,q,r)} = \begin{bmatrix} y^{(p)}(x_0)y^{(q)}(x_0)y^{(r)}(x_0)\\ y^{(p)}(x_1)y^{(q)}(x_1)y^{(r)}(x_1)\\ \vdots\\ y^{(p)}(x_N)y^{(q)}(x_N)y^{(r)}(x_N)\end{bmatrix},$$

$$\mathbf{Y}^{(p,q)} = \begin{bmatrix} y^{(p)}(x_0)y^{(q)}(x_0)\\ y^{(p)}(x_1)y^{(q)}(x_1)\\ \vdots\\ y^{(p)}(x_N)y^{(q)}(x_N)\end{bmatrix}.$$

By the other hand, we can write following matrix forms of the nonlinear quadratic and nonlinear cubic parts from (8) and (9) for $p, q, r = 0, 1, 2$

$$\begin{aligned}
\mathbf{Y}^{(0,0)} &= \mathbf{L}^*_{(0,0)}\overline{\mathbf{A}}, \quad \mathbf{Y}^{(1,0)} = \mathbf{L}^*_{(1,0)}\overline{\mathbf{A}},\\
\mathbf{Y}^{(1,1)} &= \mathbf{L}^*_{(1,1)}\overline{\mathbf{A}}, \quad \mathbf{Y}^{(2,0)} = \mathbf{L}^*_{(2,0)}\overline{\mathbf{A}},\\
\mathbf{Y}^{(2,1)} &= \mathbf{L}^*_{(2,1)}\overline{\mathbf{A}}, \quad \mathbf{Y}^{(2,2)} = \mathbf{L}^*_{(2,2)}\overline{\mathbf{A}},
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{Y}^{(0,0,0)} &= \mathbf{L}^*_{(0,0,0)}\overline{\overline{\mathbf{A}}}, \ \mathbf{Y}^{(1,0,0)} = \mathbf{L}^*_{(1,0,0)}\overline{\overline{\mathbf{A}}},\\
\mathbf{Y}^{(1,1,0)} &= \mathbf{L}^*_{(1,1,0)}\overline{\overline{\mathbf{A}}}, \ \mathbf{Y}^{(1,1,1)} = \mathbf{L}^*_{(1,1,1)}\overline{\overline{\mathbf{A}}},\\
\mathbf{Y}^{(2,0,0)} &= \mathbf{L}^*_{(2,0,0)}\overline{\overline{\mathbf{A}}}, \ \mathbf{Y}^{(2,1,0)} = \mathbf{L}^*_{(2,1,0)}\overline{\overline{\mathbf{A}}},\\
\mathbf{Y}^{(2,1,1)} &= \mathbf{L}^*_{(2,1,1)}\overline{\overline{\mathbf{A}}}, \ \mathbf{Y}^{(2,2,0)} = \mathbf{L}^*_{(2,2,0)}\overline{\overline{\mathbf{A}}},\\
\mathbf{Y}^{(2,2,1)} &= \mathbf{L}^*_{(2,2,1)}\overline{\overline{\mathbf{A}}}, \ \mathbf{Y}^{(2,2,2)} = \mathbf{L}^*_{(2,2,2)}\overline{\overline{\mathbf{A}}},
\end{aligned}$$

where

$$\mathbf{L}^*_{(0,0)} = \begin{bmatrix} \mathbf{L}(x_0)\overline{\mathbf{L}}(x_0) \\ \mathbf{L}(x_1)\overline{\mathbf{L}}(x_1) \\ \vdots \\ \mathbf{L}(x_N)\overline{\mathbf{L}}(x_N) \end{bmatrix},$$

$$\mathbf{L}^*_{(1,0)} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}\overline{\mathbf{L}}(x_0) \\ \mathbf{L}(x_1)\mathbf{C}\overline{\mathbf{L}}(x_1) \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}\overline{\mathbf{L}}(x_N) \end{bmatrix},$$

$$\mathbf{L}^*_{(1,1)} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}\overline{\mathbf{L}}(x_0)\overline{\mathbf{C}} \\ \mathbf{L}(x_1)\mathbf{C}\overline{\mathbf{L}}(x_1)\overline{\mathbf{C}} \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}\overline{\mathbf{L}}(x_N)\overline{\mathbf{C}} \end{bmatrix},$$

$$\mathbf{L}^*_{(2,0)} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}^2\overline{\mathbf{L}}(x_0) \\ \mathbf{L}(x_1)\mathbf{C}^2\overline{\mathbf{L}}(x_1) \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}^2\overline{\mathbf{L}}(x_N) \end{bmatrix},$$

$$\mathbf{L}^*_{(2,1)} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}^2\overline{\mathbf{L}}(x_0)\overline{\mathbf{C}} \\ \mathbf{L}(x_1)\mathbf{C}^2\overline{\mathbf{L}}(x_1)\overline{\mathbf{C}} \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}^2\overline{\mathbf{L}}(x_N)\overline{\mathbf{C}} \end{bmatrix},$$

$$\mathbf{L}^*_{(2,2)} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}^2\overline{\mathbf{L}}(x_0)\overline{\mathbf{C}^2} \\ \mathbf{L}(x_1)\mathbf{C}^2\overline{\mathbf{L}}(x_1)\overline{\mathbf{C}^2} \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}^2\overline{\mathbf{L}}(x_N)\overline{\mathbf{C}^2} \end{bmatrix};$$

$$\mathbf{L}^*_{(0,0,0)} = \begin{bmatrix} \mathbf{L}(x_0)\overline{\mathbf{L}}(x_0)\overline{\overline{\mathbf{L}}}(x_0) \\ \mathbf{L}(x_1)\overline{\mathbf{L}}(x_1)\overline{\overline{\mathbf{L}}}(x_1) \\ \vdots \\ \mathbf{L}(x_N)\overline{\mathbf{L}}(x_N)\overline{\overline{\mathbf{L}}}(x_N) \end{bmatrix},$$

$$\mathbf{L}^*_{(1,0,0)} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}\overline{\mathbf{L}}(x_0)\overline{\overline{\mathbf{L}}}(x_0) \\ \mathbf{L}(x_1)\mathbf{C}\overline{\mathbf{L}}(x_1)\overline{\overline{\mathbf{L}}}(x_1) \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}\overline{\mathbf{L}}(x_N)\overline{\overline{\mathbf{L}}}(x_N) \end{bmatrix},$$

$$\mathbf{L}^*_{(1,1,0)} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}\overline{\mathbf{L}}(x_0)\overline{\mathbf{C}\overline{\mathbf{L}}}(x_0) \\ \mathbf{L}(x_1)\mathbf{C}\overline{\mathbf{L}}(x_1)\overline{\mathbf{C}\overline{\mathbf{L}}}(x_1) \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}\overline{\mathbf{L}}(x_N)\overline{\mathbf{C}\overline{\mathbf{L}}}(x_N) \end{bmatrix},$$

$$\mathbf{L}^*_{(1,1,1)} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}\overline{\mathbf{L}}(x_0)\overline{\mathbf{C}\overline{\mathbf{L}}}(x_0)\overline{\overline{\mathbf{C}}} \\ \mathbf{L}(x_1)\mathbf{C}\overline{\mathbf{L}}(x_1)\overline{\mathbf{C}\overline{\mathbf{L}}}(x_1)\overline{\overline{\mathbf{C}}} \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}\overline{\mathbf{L}}(x_N)\overline{\mathbf{C}\overline{\mathbf{L}}}(x_N)\overline{\overline{\mathbf{C}}} \end{bmatrix},$$

$$\mathbf{L}^*_{(2,0,0)} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}^2\overline{\mathbf{L}}(x_0)\overline{\overline{\mathbf{L}}}(x_0) \\ \mathbf{L}(x_1)\mathbf{C}^2\overline{\mathbf{L}}(x_1)\overline{\overline{\mathbf{L}}}(x_1) \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}^2\overline{\mathbf{L}}(x_N)\overline{\overline{\mathbf{L}}}(x_N) \end{bmatrix},$$

$$\mathbf{L}^*_{(2,1,0)} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}^2\overline{\mathbf{L}}(x_0)\overline{\mathbf{C}\overline{\mathbf{L}}}(x_0) \\ \mathbf{L}(x_1)\mathbf{C}^2\overline{\mathbf{L}}(x_1)\overline{\mathbf{C}\overline{\mathbf{L}}}(x_1) \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}^2\overline{\mathbf{L}}(x_N)\overline{\mathbf{C}\overline{\mathbf{L}}}(x_N) \end{bmatrix},$$

$$\mathbf{L}^*_{(2,1,1)} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}^2\overline{\mathbf{L}}(x_0)\overline{\mathbf{C}\overline{\mathbf{L}}}(x_0)\overline{\overline{\mathbf{C}}} \\ \mathbf{L}(x_1)\mathbf{C}^2\overline{\mathbf{L}}(x_1)\overline{\mathbf{C}\overline{\mathbf{L}}}(x_1)\overline{\overline{\mathbf{C}}} \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}^2\overline{\mathbf{L}}(x_N)\overline{\mathbf{C}\overline{\mathbf{L}}}(x_N)\overline{\overline{\mathbf{C}}} \end{bmatrix},$$

$$\mathbf{L}^*_{(2,2,0)} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}^2\overline{\mathbf{L}}(x_0)\overline{\mathbf{C}^2\overline{\mathbf{L}}}(x_0) \\ \mathbf{L}(x_1)\mathbf{C}^2\overline{\mathbf{L}}(x_1)\overline{\mathbf{C}^2\overline{\mathbf{L}}}(x_1) \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}^2\overline{\mathbf{L}}(x_N)\overline{\mathbf{C}^2\overline{\mathbf{L}}}(x_N) \end{bmatrix},$$

$$\mathbf{L}^*_{(2,2,1)} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}^2\overline{\mathbf{L}}(x_0)\overline{\mathbf{C}^2\overline{\mathbf{L}}}(x_0)\overline{\overline{\mathbf{C}}} \\ \mathbf{L}(x_1)\mathbf{C}^2\overline{\mathbf{L}}(x_1)\overline{\mathbf{C}^2\overline{\mathbf{L}}}(x_1)\overline{\overline{\mathbf{C}}} \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}^2\overline{\mathbf{L}}(x_N)\overline{\mathbf{C}^2\overline{\mathbf{L}}}(x_N)\overline{\overline{\mathbf{C}}} \end{bmatrix},$$

$$\mathbf{L}^*_{(2,2,2)} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}^2\overline{\mathbf{L}}(x_0)\overline{\mathbf{C}^2\overline{\mathbf{L}}}(x_0)\overline{\overline{\mathbf{C}^2}} \\ \mathbf{L}(x_1)\mathbf{C}^2\overline{\mathbf{L}}(x_1)\overline{\mathbf{C}^2\overline{\mathbf{L}}}(x_1)\overline{\overline{\mathbf{C}^2}} \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}^2\overline{\mathbf{L}}(x_N)\overline{\mathbf{C}^2\overline{\mathbf{L}}}(x_N)\overline{\overline{\mathbf{C}^2}} \end{bmatrix}.$$

Then the fundamental matrix equation is gained from (5)-(13)

$$\begin{aligned} \sum_{k=0}^{2} \mathbf{P}_k\mathbf{L}\mathbf{A} + \sum_{p=0}^{2}\sum_{q=0}^{p} \mathbf{Q}_{pq}\mathbf{L}^*_{(p,q)}\overline{\mathbf{A}} \\ + \sum_{p=0}^{2}\sum_{q=0}^{p}\sum_{r=0}^{q} \mathbf{Q}_{pqr}\mathbf{L}^*_{(p,q,r)}\overline{\overline{\mathbf{A}}} \\ = \mathbf{G}, \quad 0 \le x \le b < \infty, \end{aligned} \tag{14}$$

Briefly, we can write Eq.(14) as

$$\mathbf{W}\mathbf{A} + \mathbf{V}\overline{\mathbf{A}} + \mathbf{Z}\overline{\overline{\mathbf{A}}} = \mathbf{G}, \tag{15}$$

where

$$\mathbf{W}=\sum_{k=0}^{2}\mathbf{P}_k\mathbf{L}=[w_{i,j}];\ i,j=0,1,...,N,$$

$$\mathbf{V}=\sum_{p=0}^{2}\sum_{q=0}^{p}\mathbf{Q}_{pq}\mathbf{L}^*_{(p,q)}=[v_{i,j}]_{(N+1)\times(N+1)^2},$$

$$\mathbf{Z}=\sum_{p=0}^{2}\sum_{q=0}^{p}\sum_{r=0}^{q}\mathbf{Q}_{pqr}\mathbf{L}^*_{(p,q,r)}=[z_{i,j}]_{(N+1)\times(N+1)^3},$$

Moreover, fundamental matrix equation (15) can be written in the augmented matrix form

$$[\mathbf{W};\mathbf{V};\mathbf{Z}:\mathbf{G}]. \tag{16}$$

3.1. Matrix representation of the conditions

Let us define the matrix form of the conditions given by (2) can be written as

$$\text{for}\ \ j=0,\ \ \mathbf{U}_0=[y^{(0)}(0)]=\mathbf{L}(0),$$
$$\text{for}\ \ j=1,\ \ \mathbf{U}_1=[y^{(1)}(0)]=\mathbf{L}(0)\mathbf{C}.$$

Then, we have

$$\mathbf{U}=\begin{bmatrix}\mathbf{U}_0\\ \mathbf{U}_1\end{bmatrix}_{2\times(N+1)},$$

$$\mathbf{O}_2=\begin{bmatrix}0&0&\cdots&0\\0&0&\cdots&0\end{bmatrix}_{2\times(N+1)^2},$$

$$\mathbf{O}_3=\begin{bmatrix}0&0&\cdots&0\\0&0&\cdots&0\end{bmatrix}_{2\times(N+1)^3},$$

or briefly,

$$[\mathbf{U};\mathbf{O}_2;\mathbf{O}_3:\lambda_j]. \tag{17}$$

Consequently, in order to find the Laguerre coefficients $a_n,(n=0,1,...,N)$ related with the approximate solution (3) of the problem (1)-(2), by replacing the 2 row matrices (17) by the last 2 rows (or any 2 rows) of the augmented matrix (16), we obtain new augmented matrix

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{V}};\widetilde{\mathbf{Z}}:\widetilde{\mathbf{G}}]. \tag{18}$$

Thence the unknown Laguerre coefficients are calculated by solving (18) [7]- [8]. Therefore, the Laguerre polynomial solution can be acquired as

$$y_N(x)=\sum_{n=0}^{N}a_nL_n(x).$$

4. Error analysis

Definition 1 (Residual function)**.** *We define the residual function $R_N(x_\alpha)$ for $x=x_\alpha\in[0,b]$*

$$\begin{aligned}R_N(x_\alpha)=&\sum_{k=0}^{2}P_k(x_\alpha)y^{(k)}(x_\alpha)\\&+\sum_{p=0}^{2}\sum_{q=0}^{p}Q_{pq}(x_\alpha)y^{(p)}(x_\alpha)y^{(q)}(x_\alpha)\\&+\sum_{p=0}^{2}\sum_{q=0}^{p}\sum_{r=0}^{q}Q_{pqr}(x_\alpha)y^{(p)}(x_\alpha)y^{(q)}(x_\alpha)y^{(r)}(x_\alpha)\\&-g(x_\alpha)\cong 0\end{aligned}$$

or

$$R_N(x_\alpha)\le 10^{-k_\alpha},\qquad \text{for } k_\alpha\in\mathbb{Z}^+.$$

Then $|R_N(x_\alpha)|$ is called as the residual function *on the interval $[0,b]$.*

Theorem 1. *$|R_N(x_\alpha)|$ is the residual function on the interval $[0,b]$. Then*

$$\left|\int_0^b R_N(x)dx\right|\le\int_0^b|R_N(x)|\,dx$$

So, that the upper bound of the mean error $\overline{R}_n$ is

$$|R_N(x)|\le\frac{\int_0^b|R_N|\,(x)dx}{b}=\overline{R}_n.$$

Proof. In order to see the proof briefly, we consider the Mean Value Theorem and the definition below. Then

$$\left|\int_0^b R_N(x)dx\right|\le\int_0^b|R_N(x)|\,dx$$
$$\left|\int_0^b R_N(x)dx\right|\le b\,|R_N(c)|,\quad 0\le c\le b$$
$$\left|\int_0^b R_N(x)dx\right|\le b\,|R_N(c)|\le\int_a^b|R_N(x)|\,dx$$
$$|R_N(x)|\le\frac{\int_0^b|R_N|\,(x)dx}{b}=\overline{R}_n$$

□

4.1. Algorithm

- **Step 0.** Input initial data: $P_k(x), Q_{pq}(x), Q_{pqr}(x)$ and $g(x)$. Determine the mixed conditions.
- **Step 1.** Set N where $N \in \mathbb{N}$.
- **Step 2.** Construct the matrices $\mathbf{L}(x), \mathbf{C}, \overline{\mathbf{L}}(x), \overline{\mathbf{C}}, \overline{\overline{\mathbf{L}}}(x), \overline{\overline{\mathbf{C}}}$ and $\mathbf{G}$ then $\mathbf{W}, \mathbf{V}, \mathbf{Z}$.
- **Step 3.** Define the collocation points $x_i = \frac{b}{N} i,\ i = 0, 1, ..., N$.
- **Step 4.** Compute $[\mathbf{W}; \mathbf{V}; \mathbf{Z} : \mathbf{G}]$.
- **Step 5.** Compute $[\mathbf{U}; \mathbf{O}_2; \mathbf{O}_3 : \lambda_j]$.
- **Step 6.** Construct the augmented matrix $[\tilde{\mathbf{W}}; \tilde{\mathbf{V}}; \tilde{\mathbf{Z}} : \tilde{\mathbf{G}}]$.
- **Step 7.** Input: the augmented matrix arguments, forward elimination, back substitution. Output: $\mathbf{A}$ (Solve the system by Gaussian elimination method).
- **Step 8.** Put arguments a_n in the truncated Laguerre series form.
- **Step 9.** Output data: the approximate solution $y_N(x)$.
- **Step 10.** Construct $y(x)$ is the exact solution of (1).
- **Step 11.** Stop when $R_N(x) \leq 10^{-k}$ where $k \in \mathbb{Z}^+$. Otherwise, increase N and return to Step 1.

5. Illustrative examples

In this section, some examples will be given to show applicability of our method. All the problems have been calculated and plotted by using Maple18 and MatlabR2014b.

Example 1. *First, we consider the second-order nonlinear ordinary differential equation with quadratic terms*

$$y''(x) + 2y'(x) + y(x) + y^2(x) - y''(x)y'(x) = 12\exp(x) + 2 \tag{19}$$

with the initial conditions

$$y(0) = 3, \quad y'(0) = 2. \tag{20}$$

The exact solution of (19)-(20) is $y(x) = 1 + 2\exp(x)$.

Table 1. $|R_N|$ comparison of Example 1. for different N values.

x	$\lvert R_2\rvert$	$\lvert R_4\rvert$	$\lvert R_5\rvert$
(0.0)	*0.000000*	*0.000000*	*0.000000*
(0.1)	*0.341836E-4*	*0.530766E-5*	*0.450128E-6*
(0.2)	*0.280551E-3*	*0.281048E-4*	*0.194988E-5*
(0.3)	*0.971761E-3*	*0.563571E-4*	*0.318105E-5*
(0.4)	*0.236493E-3*	*0.671672E-4*	*0.339969E-5*
(0.5)	*0.474425E-2*	*0.525094E-4*	*0.365512E-5*
(0.6)	*0.842376E-2*	*0.476701E-4*	*0.530256E-5*
(0.7)	*0.137505E-2*	*0.162679E-3*	*0.552450E-5*
(0.8)	*0.211081E-1*	*0.617047E-3*	*0.104534E-4*
(0.9)	*0.309206E-1*	*0.177815E-2*	*0.808003E-4*
(1.0)	*0.436563E-1*	*0.420369E-2*	*0.282554E-3*

Example 2. *Now, we consider the second-order nonlinear ordinary differential equation with cubic terms,*

$$y''(x) - y'(x)(1 - y^2(x)) + y(x) = (2 + \sin(x))\cos(x)\sin(x) + 1 \tag{21}$$

with the initial conditions

$$y(0) = y'(0) = 1. \tag{22}$$

The exact solution of (21)-(22) is $y(x) = 1 + \sin(x)$.

Table 2. $|R_N|$ comparison of Example 2. for different N values.

x	$\lvert R_2\rvert$	$\lvert R_4\rvert$	$\lvert R_6\rvert$
(0.0)	*0.000000*	*0.000000*	*0.000000*
(0.1)	*0.516658E-7*	*0.554530E-9*	*0.551083E-12*
(0.2)	*0.213306E-8*	*0.243668E-10*	*0.241721E-11*
(0.3)	*0.494797E-8*	*0.597486E-10*	*0.593143E-11*
(0.4)	*0.905816E-8*	*0.114974E-10*	*0.114344E-11*
(0.5)	*0.145574E-8*	*0.193316E-10*	*0.192602E-11*
(0.6)	*0.215357E-8*	*0.298030E-9*	*0.297203E-11*
(0.7)	*0.300782E-7*	*0.432339E-9*	*0.430863E-10*
(0.8)	*0.402643E-7*	*0.599433E-9*	*0.595716E-10*
(0.9)	*0.521673E-7*	*0.802455E-9*	*0.793115E-10*
(1.0)	*0.658529E-7*	*0.444952E-8*	*0.234134E-9*

6. Conclusion

In this study, we introduce a matrix method depending on Laguerre polynomials in order to solve a class of second-order nonlinear ordinary differential equations having quadratic and cubic terms numerically. Furthermore, the error analysis is given to show the accuracy of the method. The present method and its error analysis are applied

on some illustrative examples which have been shown by the tables.

The method has some significant advantages such as;

- The present method has short and concise computing procedure by writing the algorithm in Maple18.
- The technique gives an alternative way of solution to the second-order nonlinear ordinary differential equations which varies the other methods in literature.
- The present method has sufficient results when N is chosen large enough.

The method also can be developed and applied to differential functional integral equations, nonlinear functional integral equations and functional systems but some modifications are required [9]-[10].

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