TWO ESSAYS ON KEYNESIAN USER COST

by

STEPHEN MARTIN

B.A., Michigan State University

(1970)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1977

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Chairman, Committee on Graduate Studies

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ABSTRACT

Essay one, Keynesian user cost, begins with a survey of the literature engendered by Keynes' definition of user cost as the value of depreciation resulting from use of a capital stock for production. Although the early literature remained faithful to Keynes' concept, the Keynesian notion of user cost eventually became confused with the implicit rental to the services of a capital stock. A dynamic model of firm behavior which allows for Keynesian user cost is presented. The model is dynamic because the firm is assumed to accumulate capital in an optimal manner, maximizing the present discounted value of net revenue over the indefinite future. The behavior indicated by this dynamic model is incorporated in different one-sector growth models. The first remains faithful to conventional theory, assuming an endogenous discount rate and constant intertemporal barter prices. The second assumes an exogenous discount rate and allows intertemporal barter prices to vary; the result is a plausible model in which Pasinetti's paradox does not occur.

Essay two, On depletion and natural resources, deals with the

specialization of Keynesian user cost to natural resources. The concept of depletion of a natural resource stock is analytically similar to that of depreciation of a capital stock. The essay begins by clarifying the role of depletion in the standard Hotelling model of natural resource extraction. Subsequently, a model of natural resource extraction based on a three factor production function (capital, labor, and reserves) is developed. Both profit maximizing and cost minimizing behavior are considered. The analysis of cost minimizing extraction yields an intertemporal cost function which gives the discounted present value of minimum extraction cost over the indefinite future as a function of output and factor price sequences, including the shadow price of reserves, a measure of the scarcity of the resource stock. Applying Shephard's Lemma to this cost function, a relation is obtained explaining current expenditures on labor by current and expected output, current and expected factor prices, and the shadow price of reserves. As an empirical test of the underlying model, this equation is estimated for four mineral industries, using physical and market measures of depletion as proxies for the shadow price of reserves. The results confirm the importance of depletion for the extraction of natural resources.

Thesis Supervisor: Morris A. Adelman

Title: Professor of Economics

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KEYNESIAN USER COST

I. The definition of Keynesian user cost

The term "user cost" was first employed by Keynes in the General Theory to denote "the reduction in the value of the equipment due to using it as compared with not using it" [1973, p. 70]. In the following decade and a half, the concept was refined and clarified in a literature which attempted to extend well accepted static cost concepts to dynamic models. The essence of Keynes' definition was preserved. In 1949, in an influential work, Arthur Lewis extended the definition of user cost to include what would then have been called retainer cost and what would now be called the rental rate of capital services [1949, p. 10n]. Immediately subsequent papers employed the term "user cost" to indicate sometimes one, sometimes the other idea (and sometimes both). Later work has used the phrase almost entirely in the sense of the Lewis extension. The original Keynesian notion of user cost appears rarely, if at all, in current discussions of firm behavior, capital, and growth.

We will first review the literature engendered by Keynes' definition of user cost, and show that Lewis' contribution led to a movement away from the Keynesian concept. We will develop a dynamic model of firm behavior which illustrates Keynesian user cost. Finally we will incorporate dynamic firm behavior into one-sector models of growth, which will explore the implications of Keynesian user cost for the economic theory of growth and distribution.

Keynes uses the following notation to discuss user cost [1973, pp. 52-55]:

A = value of finished output sold by the firm during the period

A_1 = value of finished output purchased from other firms during the period

G = value of capital on hand at the end of the period

B' = value of expenditures which would have been made on maintenance of capital stock in the absence of production

G' = value of capital on hand at the end of the period, in the absence of production, with maintenance expenditures B'

$G' - B'$ = maximum end of period value of capital stock, in the absence of production

$G - A_1$ = value of the capital stock conserved from previous periods when production occurs

F = amount spent for the services of factors other than capital

The user cost of production is defined as

$$U = (G' - B') - (G - A_1) \tag{1}$$

which is the difference between the value of the end of period capital stock without production and the same value when production has occurred.

Keynes employed the concept so defined in the following way. He defined prime cost as the sum of factor cost and user cost:

$$F + U = F + (G' - B') - (G - A_1) \tag{2}$$

The income of the entrepreneur is defined as the excess of the value of output over prime cost:

$$A - F - U \tag{3}$$

When income is aggregated over the community, factor cost drops

out, since the cost of factors to one part of the economy is income to another part. Gross aggregate income is

(4) $$A - U$$

Net income requires the definition of supplementary cost, V, as "the depreciation of the equipment, which is involuntary but not unexpected, i.e., the excess of the expected depreciation over the user cost" [Keynes, 1973, p. 56]. Net income is defined as gross aggregate income, less supplementary cost

(5) $$A - U - V$$

which is simply the value of total output less the value of total depreciation.

Keynes' main motive for defining user cost was to precisely define income. He recognized its larger importance, however, and in an appendix [Keynes, 1973, pp. 69-70] he began the discussion which would seek to place the concept of user cost in the dynamic theory of the firm:

> User cost constitutes one of the links between the present and the future. For in deciding his scale of production an entrepreneur has to exercise a choice between using up his equipment now and preserving it to be used later on. It is the expected sacrifice of future benefit involved in present use which determines the amount of the user cost, and it is the marginal amount of this sacrifice which, together with the marginal factor cost and the expectation of the marginal proceeds, determines his scale of production.

The role of user cost is explicit [Keynes, 1973, pp. 67-68]:

> ....the short-period supply price is the sum of the marginal factor cost and the marginal user cost. ..the long-period supply price is equal to the sum of the prime cost, the supplementary cost, the risk cost and the interest cost....

This notion of user cost was immediately incorporated into discussions of firm behavior, in a way which clearly identified user cost with depreciation. Writing shortly after the publication of the General Theory, Bain [1937, p. 707], referring to Keynes' definition of user cost, discusses the argument of conventional (static) cost theory that firms characterized by a production structure involving primarily fixed factors will have relatively great fixed costs, relatively small variable costs, hence will be likely to exhibit price flexibility during depression:

> The implicit postulate crucial to the argument is that all costs associated with fixed factors are for the short run regarded as fixed. ...This logic is faulty. The costs associated with fixed factors, in particular depreciation, can be variable as well as fixed for the short period. ...Depreciation results not from a single force but from a composite of forces. Certain of these forces produce a depreciation which is a function solely of time, and which is therefore properly regarded as a short-run fixed cost. Others, just as important, cause a depreciation which is a function of output. The latter is a variable cost; it enters into the marginal cost curve, and it is price determining for the short period.

The identification of user cost with depreciation was intensified by its immediate application to the economic extraction of natural resources. Keynes [1973, pp. 72-73] wrote:

> In the case of raw materials the necessity of allowing for user cost is obvious;--if a ton of copper is used up to-day it cannot be used to-morrow, and the value which the copper would have for the purposes of to-morrow must clearly be reckoned as a part of the marginal cost. But the fact has been overlooked that copper is only an extreme case of what occurs whenever capital equipment is used to produce. The assumption that there is a sharp division between raw materials where we must allow for disinvestment due to using them and fixed capital where we can safely neglect it does not correspond to the facts....

Paish [1938, p. 385], discussing the supply of gold, used the depreciation/depletion notion of user cost:

> ...every ton of ore extracted means a ton less to be extracted at some future date; and if the deposit is a valuable one, which is expected to show a large profit over the costs of extraction, the greater proportion of marginal cost may be the user cost of the deposit.

User cost was incorporated into discussions of renewable resources [Bauer, 1945, pp. 93-94]:

> Rubber differs, however, from all other forms of agriculture in that the annual output can be varied substantially with the rate of bark removal. ..Direct cost..is the marginal cash cost, and does not include the user cost element in bark removal.

Bauer specifies his concept of user cost [1945, p. 97]:

> The user cost of a unit of output in the short period is the reduction in the discounted value of expected future quasi-rents of a piece of equipment through using it for the production of that unit of output rather than leaving it unused.

In a careful discussion of dynamic firm behavior, Neal [1942, pp. 50-52] summarized emerging cost concepts.[1]

> The total of those costs which would be incurred if there were..no production by the firm are supplementary costs. ....prime costs will be associated with positive production. Prime cost...is simply total cost minus supplementary cost for any contemplated level of production. ...marginal cost can be defined in the usual (formal) way as the rate of addition to total cost associated with increased production. ...Total cost at any level of production...can be divided into supplementary cost, S, and prime cost, P. Prime cost may be further subdivided into factor cost, F, the costs of the factors of production associated with operating as opposed to not operating; purchases from other entrepreneurs, M (Mr. Keynes's A_1), associated with operating as opposed to not operating, and user cost, U.. ...User cost consists of the change in the value of facilities due to operating as opposed to not operating..

A final clarification of the definition of user cost was made by Lerner [1943, pp. 131-132]. He begins with Keynes' definition of prime cost

$$F + U = F + (G' - B') - (G - A_1) \tag{2}$$

as the sum of factor cost and user cost. Lerner argues that B', maintenance costs in the absence of production, may be subdivided into F', the cost of factors used for maintenance in the absence of production, and A_1', the cost of finished output purchased from other entrepreneurs for maintenance in the absence of production.

$$B' = F' + A_1' \tag{6}$$

Substitution of (6) into (2) gives a new expression for prime cost:

$$P = (F - F') + (A_1 - A_1') + (G' - G) \tag{7}$$

Lerner defines (F - F') as prime (or extra) factor costs, $(A_1 - A_1')$ as prime materials cost, and (G' - G) as prime depreciation. Prime materials cost and prime depreciation cost together constitute what Lerner calls prime user cost:

$$(A_1 - A_1') + (G' - G) \tag{8}$$

which may be contrasted with Keynes' definition of user cost:

$$(G' - B') - (G - A_1) = (A_1 - A_1') + (G' - G) - F' \tag{9}$$

Thus Lerner narrows the definition of user cost to exclude factor purchases for maintenance in the absence of production [Lerner, 1943, pp. 131-132]:

> Nearly everybody comes away from the book after a first reading with the impression that user cost is that value of the extra wear and tear (including the using up of materials) that I have called prime user cost, although

> Keynes's analysis is consistently in line with the other division. ...it seems clear that the idea of extra wear and tear that is suggested by prime user cost is an easier notion for people to grasp and is capable of being used at least as effectively as Keynes's user cost.

Lerner also notes that the magnitude of prime materials cost, involving finished output from other producers, depends for any firm on its degree of aggregation. For the economy as a whole, $A_1 - A_1'$ disappears, and prime user cost is identical with prime depreciation.

Lerner's evaluation that the term user cost had come to be identified with "extra wear and tear" or "prime depreciation" seems to be completely consistent with the various discussions of user cost cited above.

Writing in 1949 on Overhead Costs, Lewis extended the definition of user cost to include a fundamentally different idea. In a footnote [Lewis, 1949, p. 10n], Lewis measures one component of user cost by the fact that

> This year's yield must also exceed the difference between the price that could be obtained for the asset this year and next year's price if used this year.

That this is an extension is clear when one notes that this difference will be positive, when yields are discounted, even if there is no depreciation at all from year to year. If depreciation occurs, its value (which Keynes would call user cost plus supplementary cost) would be included in the above definition.

Lewis later clarified his definition of user cost [1961, pp. 38-39]:

> The retainer cost is the cost of holding the assets unused. This is the interest on the sum the assets would fetch if they were sold, plus any deterioration in their value during the period for which they are

> held. ...It is a matter of the choice of words whether to define user cost so that it includes retainer cost, or is confined to wear and tear only. Retainer cost, if positive, is part of the cost of use; it enters into marginal cost, and is relevant to profit maximization. ..Retainer cost was neglected by Lord Keynes in his definition of user cost..because he assumed that the assets are retained whether they are used or not; i.e., he assumed that retainer cost is zero or negative.

Lewis acknowledges that his definition of user cost is fundamentally different from that of Keynes. However, the idea that Keynes ignored what Lewis calls retainer cost is incorrect. In his appendix on user cost [Keynes, 1973, p. 68] Keynes writes

> ..the long-period cost of the output is equal to the expected sum of the prime cost and the supplementary cost; and, furthermore, in order to yield a normal profit, the long-period supply price must exceed the long-period cost thus calculated by an amount determined by the current rate of interest on loans of comparable term and risk, reckoned as a percentage of the cost of the equipment.

Keynes' "interest reckoned as a percentage of the cost of the equipment" represents the same concept as Lewis' "interest on the sum the assets would fetch if they were sold", which Lewis identifies as retainer cost.

In any event, Lewis' choice of words had immediate impact. Later discussions of user cost took that term to indicate both retainer cost and Keynesian user cost, the value of depreciation resulting from use.

In 1953 Scott published Notes on User Cost in four sections. Three of the four notes deal with the Keynesian concept; the fourth employs the Lewis extension.

In Part I [Scott, 1953, pp. 370-371] we read:

> Lerner...has suggested that the difference between the value of the asset after production, and what it would have been without production, (G' - G), be called prime user cost. This is the concept to be used in this paper. It is a measurement of wear and tear during use...

In Part II [Scott, 1953, p. 373], it is argued that a firm will maximize profit

> ...where marginal user cost = marginal net revenue. ...It is thus worth while to add to output until the increase in net revenue is just offset by an equal fall in capital value.

In Part IV [Scott, 1953, p. 381]:

> ...even if output be zero, so that user costs are zero, the present value of possible net revenues may diminish from period to period. It would be tempting, paradoxically, to christen this diminution "fixed user costs," as opposed to the "prime user costs" so far discussed. Keynes, however, called them supplementary costs (...); other suggestions have been resting costs, (...), time depreciation and time appreciation, and Mr. Solomons's "retainer costs." The value of the asset falls as a result not of use but of retaining it over time. ...Thus, for every type of equipment, there will be a prime user cost and a retainer cost of output, the first a function of the rate of output, and the second of the age of the asset.

We must now deal with Part III. At the close of Part II, Scott writes[2] [1953, p. 374]:

> In the following part we shall ask whether user costs of current output can be more simply calculated than the complicated methods of determining current output which have been suggested elsewhere. This section will be, in fact, an excursus from our main thread, the effect of user cost on current decisions.

In Part III [1953, pp. 374-375] we find:

> A mine, for example, can choose a rate between closing down and attempting to exhaust the mineral within a single period. The manager, as periods go by, encounters

> varying outlay and revenue situations, revises his own expectations, adopts methods. Some of these changes occur independently, others depend on his own earlier actions. The latter, the consequences of his own actions, interest us particularly, for user cost is the opportunity lost when another decision is carried through (emphasis added).

Almost unnoticed, the Lewis extension has slipped in. Keynesian user cost, the value of wear and tear resulting from production, is part of the opportunity cost of the decision to produce, but it is not all of it. The opportunity cost of deciding to employ a certain amount of capital includes both the retainer cost (in the Lewis sense) of holding that capital, and the Keynesian user cost. Further on, we have [Scott, 1953, p. 376]:

> An entrepreneur...who is about to enter period 1, will wish to know how much he should produce during the period, having an eye to the effect of this decision on the net revenue of other periods, and to the ultimate effect on the present value of his enterprise. As we saw in the preceding part he needs to know his user costs in order to make a correct decision. The author best known for his answer to this implied question is Professor W. A. Lewis, in the footnote to p. 10 of Overhead Costs, which we summarize here.

Scott proceeds to summarize the Lewis extension of the definition of user cost.

The same confusion of two fundamentally distinct concepts appears elsewhere. In The Theory of Investment of the Firm, Lutz and Lutz [1951, p. 61] write of a short-run user cost which embodies the Keynesian concept:

> Thus a machine will have no short-run user cost at all whenever using it, rather than leaving it idle, in any operational period makes no difference either to the quasi rents or to the maintenance costs of the capital expense type in subsequent operational periods.

However, in the section on long-run cost, they use the term user cost in the Lewis sense [Lutz and Lutz, 1951, p. 58]:

> The "cost" of existing durable equipment with which the present value of its quasi rents must be compared is here not the initial cost, but the "user" cost in the sense of the value which the equipment would realize if sold...

In 1965, in an influential article, Jorgenson says [1965, p. 40]:

> Little attention was paid to the precise way in which the rate of interest and the price of capital goods enter the demand for capital services. Both variables enter only through the user cost of capital services, that is, through the implicit rental of the services of capital.

And in an accompanying footnote:

> The most complete discussion of the concept of user cost is that of Lewis [38, especially pp. 33-45]; see also Keynes [30, pp. 66-73], Lutz and Lutz [39] and Scott [53].

His references are to the works of the respective authors quoted above. In fact, of course, the rate of interest and the price of capital affect the demand for capital services through the retainer cost of capital, which Jorgenson calls the implicit rental of the services of capital. The value of Keynesian user cost is simply physical depreciation, resulting from use, multiplied by the price of capital goods. The influence of the Lewis extension on Jorgenson's work is clear. Jorgenson proceeds to present a model of investment behavior which has become standard, and he does so under the assumption that the rate of depreciation is constant, independent of use [Jorgenson, 1965, p. 44].

Jorgenson's acceptance of the Lewis extension may have been

influenced by his feeling that [1965, p. 51; see also 1971]:

> It is a fundamental result of renewal theory that the distribution of replacements....approaches a constant fraction of the capital stock for (almost) any distribution of replacements over time and for any initial distribution of the capital stock. This result holds for a constant stock and for a growing stock as well.

Theoretical [Feldstein and Rothschild, 1974] and empirical studies [Feldstein and Foot, 1971; Eisner, 1972; Bitros and Kelejian, 1974; Bitros, 1976] have established that this is not a valid application of renewal theory. However, these studies follow Jorgenson in assuming away Keynesian user cost, since they typically assume [Coen, 1975, p. 60]:

> Let d_{ji} be the fraction of the original productive capacity of a capital good which is lost in the ith period after its acquisition according to the jth "mortality distribution." A mortality distribution is defined by a combination of an assumed service life and an assumed pattern of capacity depreciation. ...losses in productive capacity depend only on the time elapsed since the acquisition date (emphasis added).

Losses in productive capacity are assumed to be independent of use. From early discussions of Keynesian user cost, it is clear that depreciation which is independent of use is what Keynes would have called supplementary cost. Keynes designated by user cost an influence which worked through the variation in depreciation with use [Keynes, 1973, p. 67][3]:

> ....we still have to allow for the marginal disinvestment in the firm's own equipment in producing marginal output. Even if all production is carried on by a completely integrated firm, it is still illegitimate to suppose that the marginal user cost is zero, i.e., that the marginal disinvestment in equipment due to the production of the marginal output can generally be neglected.

Jorgenson consistently employs the term "user cost" to mean what Lewis would call the retainer cost of capital, including what Keynes would call supplementary cost, the concept being the rental rate of capital services plus the value of time depreciation.[4] The concept which Keynes designated by user cost has almost disappeared from the literature. For example, a popular intermediate textbook [Branson, 1972, p. 215] defines the user cost of a capital good as

$$C = rP_I + \delta P_I - \frac{dP_I}{dt} \tag{10}$$

where r is the interest rate, P_I the price of investment goods, and δ the constant (time) depreciation rate.

The first right hand side term of (10) is the interest cost of the capital good, the second is the value of time depreciation, and the third is the capital gain. But Keynesian user cost, in the sense of depreciation which results from use, is completely absent.[5]

The distinction between the two meanings of user cost survives with careful writers. Thus, in a footnote, Stigler [1966, p. 251] writes:

> If the future life of the instrument is shortened by using it in the present short run, this shortened life is a cost of current use, and is called the user cost of the instrument. The user cost must be covered by receipts to justify present use, but if it is covered, we must reckon the gross returns inclusive of user cost as the quasi-rent.

This indicates one of the objections to the use of the term "user cost" in the way that has become standard: the rental rate of capital plus the time depreciation rate of capital is none other than the one-period quasi-rent to a fixed stock of capital.[6] There seems

to be no obvious reason to have two names (user cost and quasi-rent) to designate the same idea.

Hicks [1965, p. 52] says:

> One of the things which Marshall left out (which he had to leave out) was what Keynes was later to call user cost. If the rate of usage of the fixed capital has any effect (in either direction) upon the measures which will have to be taken for its future replacement, there is an element in cost of production which will not fit into the static scheme. For the valuation of this usage depends on what is to happen - what is expected to happen - outside the short period. It cannot be considered at all without introducing into the model something that is more dynamic.

This goes back to Keynes' original description of user cost as "one of the links between the present and the future" [Keynes, 1973, p. 69]. The Keynesian notion of user cost is most at home in dynamic models of the firm.

A recent article by Canarella [1976] attempts to integrate Keynesian user cost into now standard models of firm behavior. The model seems inspired by a literature discussing capital utilization rates and adjustment costs of investment.[7]

II. Keynesian user cost in a dynamic model of the firm

The distinction between Keynesian user cost and the retainer cost of capital may be clearly shown in a dynamic model of optimal firm behavior. This model is an extension of Bain [1937] and Canarella [1976]. Let

$F(K_t,N_t)$ = neoclassical[8] production function

K_t = capital stock on hand in period t

N_t = labor employed in period t

I_t = investment in period t

w_t = wage rate in period t

p^k_t = purchase price of a unit of capital in period t

p_t = price of output in period t

r = rate of time preference

The rate of depreciation of the capital stock is given by[9]

$$\delta_t = \delta\left[\frac{F(K_t,N_t)}{K_t}\right] \qquad \delta'_t > 0 \tag{11}$$

The rate of depreciation is an increasing function of the output-capital ratio. Producing a greater output from a fixed capital stock works each unit of capital more intensively, so it should depreciate at a greater rate.[10] Following Bitros and Kelejian [1974, p. 271] we may argue that

> ...the intensity of usage has long been recognized as a major cause of capital deterioration. Most theories of replacement from Marx through Keynes to those of contemporary writers presume that the more intense the use of capital the more its wear and tear will be.

The firm is assumed to know the price of output in each period,

the wage rate in each period, and the cost of capital in each period. The firm acts to maximize the discounted present value of net revenues,

$$(12) \qquad \sum_{t=0}^{\infty}(1 + r)^{-t}\left[p_t F(K_t, N_t) - p_t^k I_t - w_t N_t\right]$$

The capital stock on hand in period t is

$$(13) \qquad K_t = K_0 + \sum_{\alpha=0}^{t-1}(I_\alpha - K_\alpha \delta_\alpha)$$

The capital stock available in any period equals the initial capital stock, plus all previous investment, less all previous depreciation. The firm alters the capital stock available in any period by altering the pattern of previous investment (which changes available capital) and by altering output in previous periods (which alters depreciation in those periods). There is nothing in this formulation which prevents the firm from selling capital ($I_t < 0$) if that is the most profitable action.

Since the production function is assumed to be homogeneous of the first degree, it can never be optimal to employ only a portion of the capital stock. Since a unit of capital which has survived several periods is indistinguishable from a newly purchased unit of capital, the problem of scrapping a particular piece of equipment does not arise. Although the related problems of holding capital idle and the length of economic life figure prominently in early discussions of Keynesian user cost, they arise naturally only in putty-clay models of capital, and will not be dealt with here.

First order conditions for optimal factor use are derived by maximizing the following Lagrangian with respect to K_t, N_t, and I_t:

$$L = \sum_{t=0}^{\infty}(1 + r)^{-t}\{p_t F(K_t,N_t) - p_t^k I_t - w_t N_t + \lambda_t[K_0 + \sum_{\alpha=0}^{t-1}(I_\alpha - K_\alpha\delta_\alpha) - K_t]\} \tag{14}$$

Reversing the order of summation in the constraint, (14) may be rewritten

$$L = \sum_{t=0}^{\infty}(1 + r)^{-t}\{p_t F(K_t,N_t) - [p_t^k - \sum_{\alpha=t+1}^{\infty}(1 + r)^{t-\alpha}\lambda_\alpha]I_t - w_t N_t - \lambda_t K_t - [\sum_{\alpha=t+1}^{\infty}(1 + r)^{t-\alpha}\lambda_\alpha]K_t\delta_t\} + K_0\sum_{t=0}^{\infty}(1 + r)^{-t}\lambda_t \tag{15}$$

λ_t is the Lagrangian multiplier associated with the capital constraint in period t. It is the shadow price of a unit of capital services in period t. Necessary conditions for optimal factor employment are obtained by differentiating (15) with respect to I_t, K_t, and N_t.

$$I_t: \qquad p_t^k = \sum_{\alpha=t+1}^{\infty}(1 + r)^{t-\alpha}\lambda_\alpha \tag{16}$$

Investment will be optimal when the price paid for a unit of capital in each period equals the discounted sum of the shadow prices of capital services in future periods. The cost of a unit of capital, p_t^k, is given to the firm; investment and use (and resulting depreciation) determine the shadow prices λ_t. This recalls the statement of Bauer [1945, p. 97] that the value of a piece of equipment is "the discounted value of expected future quasi-rents of

a piece of equipment". Comparing (16) for successive time periods implies that in equilibrium

$$\lambda_t = (1 + r)p^k_{t-1} - p^k_t \tag{17}$$

The shadow price of a unit of capital services in period t, in equilibrium, is the difference between the cost of purchasing one unit of capital stock in period t-1 for use in period t ("discounted" at rate r) and the price for which that unit may be sold at the end of period t. This takes a more familiar form if written

$$\frac{\lambda_t}{p^k_{t-1}} = r - \frac{p^k_t - p^k_{t-1}}{p^k_{t-1}} \tag{18}$$

The shadow rental per dollar's worth of capital services equals the interest rate less the capital gain per dollar. This is none other than the retainer cost of utilizing a dollar's worth of capital stock.

Using (16) to eliminate the infinite summation, the following first order conditions are obtained for the employment of labor and capital.

$$K_t: \quad p_t F_1(K_t, N_t) = \lambda_t + p^k_t \frac{d}{dK_t}(K_t \delta_t) \tag{19}$$

$$N_t: \quad p_t F_2(K_t, N_t) = w_t + p^k_t \frac{d}{dN_t}(K_t \delta_t) \tag{20}$$

The left hand side of (19) is the value of the product of the last unit of capital employed; the right hand side is the marginal cost of employing that unit of capital. The first component of marginal cost is the rental rate of capital services, defined by (17).

The second component is the value of the change in total depreciation which results from employment of the marginal unit of capital. In Keynesian terms, we might define supplementary cost as

$$p_t^k K_t \delta(0) \tag{21}$$

which is the value of the depreciation which will occur in the absence of production. User cost is given by

$$p_t^k K_t \{\delta[\frac{F(K_t,N_t)}{K_t}] - \delta(0)\} \tag{22}$$

Keynesian user cost is the value of total depreciation, minus supplementary cost. Thus (19) states that capital will be employed until the value of the marginal product of capital equals the rental rate of capital services plus the partial derivative of user cost with respect to capital. This partial derivative will combine two distinct influences. Employment of the last unit of capital will increase output, which will tend to increase the output-capital ratio, increasing the depreciation rate. On the other hand, increasing the capital stock will increase the stock with which output is produced, tending to decrease the output-capital ratio and the rate of depreciation. These opposing influences will be separated below.

Employment of labor will be optimal when the value of the marginal product of labor equals the marginal cost of labor. From (20), the marginal cost of labor equals the wage rate plus the change in user cost which results from employment of the last unit of labor. The effect of greater employment on user cost is unambiguous: employment of labor tends to increase output without altering the

capital stock; the output-capital ratio will rise, and with it the rate of depreciation.

The derivatives on the right hand sides of (19) and (20) may be written

$$\frac{d}{dK_t}(K_t\delta_t) = \delta_t + K_t\delta_t'\frac{K_tF_{1t} - F_t}{K_t^2} = \delta_t'F_{1t} + (\delta_t - \frac{F_t}{K_t}\delta_t') \tag{23}$$

$$= \frac{d}{dF_t}(K_t\delta_t)\Big|_{K_t=\bar{K}_t} F_{1t} + \frac{d}{dK_t}(K_t\delta_t)\Big|_{F_t=\bar{F}_t}$$

$$\frac{d}{dN_t}(K_t\delta_t) = \delta_t'F_{2t} = \frac{d}{dF_t}(K_t\delta_t)\Big|_{K_t=\bar{K}_t} F_{2t} \tag{24}$$

(23) breaks the change in total depreciation resulting from employment of the last unit of capital into two terms. The first is the marginal product of capital, times the change in total depreciation which results from increasing total output by one unit, with a constant capital stock. The second term is the change in depreciation that results from employing an additional unit of capital to produce a given output.

(24) shows that the change in total depreciation which results from employment of the last unit of labor hired is the marginal product of that unit of labor, times the change in total depreciation which results from producing an additional unit of output with a constant capital stock. Employment of labor affects depreciation only insofar as it affects output.

Substitution of (23) and (24) into (19) and (20) respectively

gives

$$(25) \qquad \left[p_t - p_t^k \frac{d}{dF_t}(K_t\delta_t)\Big|_{K_t=\bar{K}_t}\right]F_{1t} = \lambda_t + p_t^k \frac{d}{dK_t}(K_t\delta_t)\Big|_{F_t=\bar{F}_t}$$

$$(26) \qquad \left[p_t - p_t^k \frac{d}{dF_t}(K_t\delta_t)\Big|_{K_t=\bar{K}_t}\right]F_{2t} = w_t$$

From the left hand side of (25) and (26) we may define

$$(27) \qquad p_t^n = p_t - p_t^k \frac{d}{dF_t}(K_t\delta_t)\Big|_{K_t=\bar{K}_t}$$

as the net price received for the marginal unit of output in period t. We may regard

$$(28) \qquad p_t^k \frac{d}{dF_t}(K_t\delta_t)\Big|_{K_t=\bar{K}_t}$$

as the marginal user cost of output in period t (see (22)). It is the incremental depreciation which results from the production of an additional unit of output, from a constant capital stock, valued at the price of capital, which, considering (16), is the discounted sum of the quasi-rents which the depreciated capital stock would have earned. Substituting (27) in (25) and (26) gives

$$(29) \qquad p_t^n F_{1t} = \lambda_t + p_t^k \frac{d}{dK_t}(K_t\delta_t)\Big|_{F_t=\bar{F}_t}$$

$$(30) \qquad p_t^n F_{2t} = w_t$$

Employment of capital is optimal when the net value of the marginal product of capital equals the rental rate (retainer cost) of capital plus the value of the change in depreciation which results from employment of an additional unit of capital to produce a constant output. Employment of labor is optimal when the net value of the

marginal product of labor equals the cost of that labor.

Marginal user cost enters the employment conditions of both labor and capital, since both labor and capital contribute to the production of the marginal unit of output.

The entire right hand side of (29) should be regarded as the quasi-rent to a unit of capital, since it is a return to a man-made productive instrument.[11]

Some insight into the implications of optimal behavior may be gained by substituting

$$I_t = K_{t+1} - K_t + \delta_t K_t \tag{31}$$

and (17) into (12) and rearranging terms to write the discounted net revenue of the firm as

$$\sum_{t=1}^{\infty}(1+r)^{-t}\{p_t F(K_t,N_t) - \lambda_t K_t - w_t N_t - p_t^k \delta_t K_t\} + p_0 F(K_0,N_0) - w_0 N_0 - p_0^k \delta_0 K_0 + p_0^k K_0 \tag{32}$$

(recall that λ_t is defined for t=1,2,3,...; see (17)). When the firm behaves optimally the discounted present value of net revenue is the discounted sum of the difference within each period between revenue and what Keynes would call factor cost

$$w_t N_t \tag{33}$$

interest cost (Lewis' retainer cost)

$$\lambda_t K_t \tag{34}$$

and user cost plus supplementary cost (total depreciation cost)

$$p_t^k \delta_t K_t \tag{35}$$

Since the initial capital stock, K_0, is given, interest cost appears only after period zero. Substituting from (29) and (30) into (32) to eliminate λ_t and w_t gives the following expression for discounted net revenue:

$$(36) \quad \sum_{t=1}^{\infty}(1+r)^{-t}p_t^n[F(K_t,N_t) - F_1(K_t,N_t)K_t - F_2(K_t,N_t)N_t] + [p_0F(K_0,N_0) - p_0^nF_2(K_0,N_0)N_0 - p_0^k\delta_0K_0] + p_0^kK_0$$

In periods after zero, it is as if the firm pays each factor its marginal physical product; the surplus remaining to the firm is evaluated at the net price of output, which is the gross price of output minus the marginal user cost of capital. The assumption that the production function displays first degree homogeneity in factor inputs means that the surplus remaining to the firm in these periods is zero.

The bracketed term in period zero revenue is a rental to the fixed initial capital stock of the firm. The final term is the value of the initial capital stock of the firm. Thus the discounted present value of the firm's net revenue is in the end the value of the initial capital stock plus the rental to that stock.[12]

The optimality conditions (29) and (30) may be used to investigate the response of the firm to changes in various parameters. rewritten in intensive form,[13] those equations are

$$(37) \quad p_t^nf'(k_t) = (1+r)p_{t-1}^k - p_t^k + p_t^k(\delta_t - \frac{f(k_t)}{k_t}\delta_t')$$

$$(38) \quad p_t^n[f(k_t) - k_tf'(k_t)] = w_t$$

Divide (38) by (37) to eliminate p_t^n:

$$(39) \qquad \frac{f(k_t)}{f'(k_t)} - k_t = \frac{w_t}{\lambda_t + p_t^k(\delta_t - [f(k_t)/k_t]\delta_t')}$$

This is the factor price frontier, which relates the capital-labor ratio in period t to the wage-capital rental ratio in that period.[14] When Keynesian user cost is present, the rental rate of capital is not independent of the capital labor ratio.

For convenience write

$$(40) \qquad D_t = f_t\{\frac{f_t''}{f_t'}[\lambda_t + p_t^k(\delta_t - \frac{f_t}{k_t}\delta_t')] - k_t p_t^k \delta_t''[\frac{d}{dk_t}(f_t/k_t)]^2\} < 0$$

The sign of D_t depends on the assumption

$$(41) \qquad \delta_t'' \geq 0$$

which is made below during stability analysis of growth models, and which will also be made here.

Straightforward but tedious differentiation yields

$$(42) \qquad \frac{d}{dw_t}(k_t) = -\frac{f'(k_t)}{D_t} > 0$$

$$(43) \qquad \frac{d}{dp_t^k}(k_t) = \frac{1}{D_t}[k_t^2 \frac{d}{dk_t}(f_t/k_t)](1 - \delta_t + \frac{f_t}{k_t}\delta_t') > 0$$

$$(44) \qquad \frac{d}{dp_{t-1}^k}(k_t) = -\frac{1}{D_t} k_t^2 \frac{d}{dk_t}(f_t/k_t)(1 + r) < 0$$

$$(45) \qquad \frac{d}{dr}(k_t) = -\frac{1}{D_t} k_t^2 \frac{d}{dk_t}(f_t/k_t)\, p_{t-1}^k < 0$$

These derivatives establish the comparative dynamics of the firm: the relative magnitude of the capital-labor ratios chosen by two firms which face identical values of all parameters except the one with

respect to which differentiation is taken.

A higher wage in period t means a higher capital-labor ratio in period t. A larger price of investment goods in period t means that it will be more expensive to purchase capital in period t with the intent to pass it on to period t + 1; some purchase of capital will be shifted from period t to period t - 1, maintaining a larger capital stock in period t, and reducing the purchase of capital in period t (see below). A higher cost of investment goods in period t - 1 makes the purchase of capital for period t more expensive; the period t capital-labor ratio is reduced. A greater rate of time preference reduces the optimal capital-labor ratio in each period.

To investigate the effects of changes in various parameters on investment, we must fix the scale at which the firm operates, which for simplicity we will do by assuming that the firm employs a constant labor force ($N_t = N$ for all t). Then (see (31))

$$(46) \qquad i_t = \frac{I_t}{N} = k_{t+1} - k_t + \delta_t k_t$$

Differentiating (46) and substituting from (42) - (45) gives

$$(47) \qquad \frac{d}{dp_t^k} i_{t-1} = \frac{d}{dp_t^k} k_t > 0$$

$$(48) \qquad \frac{d}{dp_t^k} i_t = \frac{d}{dp_t^k} k_{t+1} - \left[1 - \delta_t - k_t \delta_t' \frac{d}{dk_t}\left(\frac{f_t}{k_t}\right)\right] \frac{d}{dp_t^k} k_t < 0$$

$$(49) \qquad \frac{d}{dp_t^k} i_{t+1} = -\left[1 - \delta_{t+1} - k_{t+1} \frac{d}{dk_{t+1}}\left(\frac{f_{t+1}}{k_{t+1}}\right)\right] \frac{d}{dp_t^k} k_{t+1} > 0$$

An increase in the price of investment goods in period t causes a substitution of investment away from period t and toward both period t - 1 and period t + 1. There is no change in optimal investment in other periods. Similarly,

$$(50) \qquad \frac{d}{dw_t} i_{t-1} = \frac{d}{dw_t} k_t > 0$$

$$(51) \qquad \frac{d}{dw_t} i_t = -\left[1 - \delta_t - k_t \delta_t' \frac{d}{dk_t}\left(\frac{f_t}{k_t}\right)\right] \frac{d}{dw_t} k_t < 0$$

An increased wage in period t increases the optimal capital-labor ratio in period t; investment in period t - 1 is increased so that the necessary capital stock is on hand in period t. The optimal capital-labor ratio in period t + 1 is unaffected, so investment in period t is reduced to compensate for the earlier increase in the capital stock. Altered wages in period t affect investment only in periods t - 1 and t.

A greater rate of time preference reduces the capital-labor ratio in each period; investment per capita may rise or fall in any one period.

We may distinguish three special cases of the depreciation function. The rate of depreciation may be constant:

$$(52) \qquad \delta\left[F(K_t, N_t)/K_t\right] = \delta$$

In this case there is supplementary cost and retainer cost, but no Keynesian user cost, since the rate of depreciation is independent of the intensity of use. The optimality conditions for the employment of capital and labor are:

(53) K_t: $p_t F_1(K_t,N_t) = \lambda_t + p_t^k \delta$

(54) L_t: $p_t F_2(K_t,N_t) = w_t$

Alternatively, depreciation per unit of capital may be proportional to output per unit of capital:

(55) $$\delta[F(K_t,N_t)/K_t] = \frac{F(K_t,N_t)}{K_t}\,\delta$$

The first order conditions for optimal employment of labor and capital become:

(56) K_t: $(p_t - p_t^k \delta)F_1(K_t,N_t) = \lambda_t$

(57) L_t: $(p_t - p_t^k \delta)F_2(K_t,N_t) = w_t$

In this case, only the marginal user cost term appears as a result of depreciation. Since total depreciation is a constant fraction of total output (from (55))

(58) $$K_t\delta[F(K_t,N_t)/K_t] = \delta F(K_t,N_t)$$

employment of an additional unit of capital to produce a given output will not alter total depreciation. The quasi-rent to the capital stock consists of the retainer cost of capital alone. Production of an additional unit of output does increase total depreciation, so that there is a marginal user cost associated with the decision to produce the marginal unit of output.

A depreciation function which is the sum of depreciation functions of these types, such as

(59) $$\delta[F(K_t,N_t)/N_t] = \delta_1 + \frac{F(K_t,N_t)}{K_t}\,\delta_2$$

provides a first order Taylor series approximation to any depreciation function.

Bain [1937] discusses a total depreciation function which combines elements of both time and use depreciation, but not in an additive way. He assumes a certain minimum rate of time depreciation, which is independent of use. Low levels of output do not increase the rate of depreciation above this minimum, but output above a critical level will raise the depreciation rate above the minimum. This may be modeled with the following rate of depreciation function:

$$(60) \qquad \delta_t = \max \left\{ \delta_0, \delta_0 - \delta_1 \frac{F(K_t,N_t)}{K_t} + \delta_2 \left[\frac{F(K_t,N_t)}{K_t}\right]^2 \right\}$$

The critical output-capital ratio is given by

$$(61) \qquad \frac{F(K_t,N_t)}{K_t} = \frac{\delta_1}{\delta_2} = \frac{f(k_t)}{k_t}$$

If the output-capital ratio is below the critical level δ_1/δ_2, capital will depreciate at the rate δ_0. When the output-capital ratio exceeds the critical level, the depreciation rate will rise with the output-capital ratio, so that Keynesian user cost will appear. For a fixed capital stock, there will be a range of output below which user cost will not arise. Supposing the capital stock to be fixed, the depreciation rate will relate to output as in Figure 1. The rate of depreciation will be constant as output varies from zero to $(\delta_1/\delta_2)K$. The rate will rise when output exceeds that level. The range of constant depreciation rate will be greater as δ_1/δ_2 is greater. Bain distinguishes between obsolescence and use as causes of depreciation

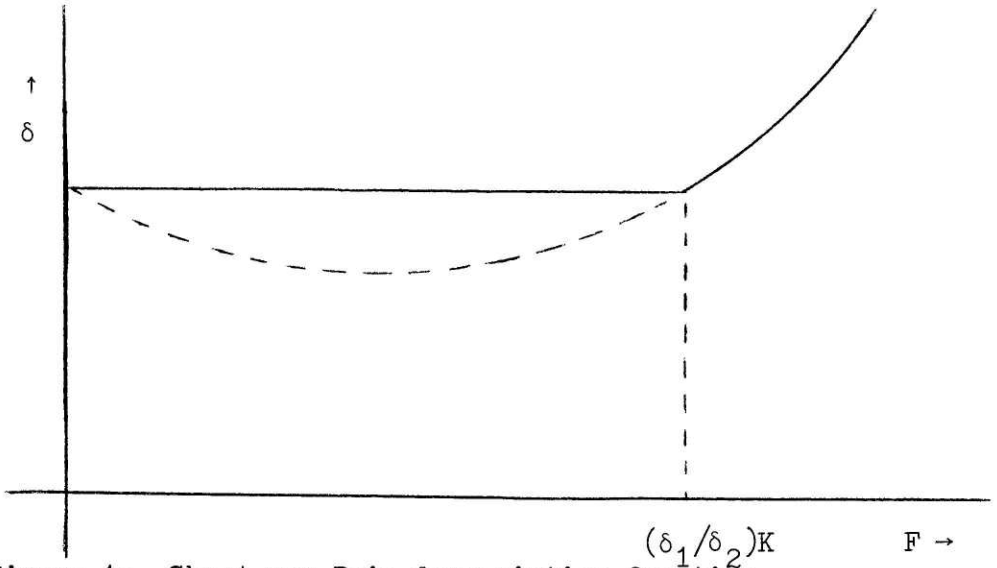


Figure 1: Short run Bain depreciation function

[1937, pp. 709-710]:

> ...obsolescence and physical deterioration, are incurred with the passage of time regardless of the rate of output. Equipment may become obsolete as the result of (1) technological development of more efficient substitute equipment, (2) a decline in the demand for the products it can produce, or (3) a change in the rate of interest. Since all of these forces reduce valuations quite independently of the rate at which the equipment is used, obsolescence may be regarded as a depreciation which is solely a function of time. That deterioration of fixed equipment which results solely from the action of the elements is by definition also independent of output. It may therefore be grouped with obsolescence as a depreciation solely a function of time.

The larger of the rate of obsolescence and the rate of use depreciation will predominate. If δ_1/δ_2 is large, low levels of output will reduce the rate of use depreciation (since δ_1 enters with a negative coefficient) and output will reach large levels before the second order term (δ_2) overwhelms the first order term. Obsolescence will determine depreciation until output reaches relatively high levels. On the other hand, if δ_1/δ_2 is small, at relatively low

output levels the second order term in use depreciation will raise the depreciation rate above the minimum rate of obsolescence.

Bain argues that firms with a production structure involving primarily obsolescence will exhibit price flexibility during depression. We may illustrate this by adapting our model of the firm to a short-run situation, in which the capital stock is fixed.

Suppose that in the short run the firm takes the price of output and the wage rate as given, and chooses employment to maximize

$$(62) \qquad p_t F(\bar{K}_t, N_t) - w_t N_t + p_t^k \{\bar{K}_t - \bar{K}_t \delta [\frac{F(\bar{K}_t, N_t)}{\bar{K}_t}]\}$$

which is the value of within period net revenue plus the value of the end of period capital stock. The firm will employ labor so that [15]

$$(63) \qquad p_t^n F_2(\bar{K}_t, N_t) = w_t$$

where

$$(64) \qquad p_t^n = p_t - p_t^k \delta_t'$$

Substituting (64) into (63) and rearranging terms gives

$$(65) \qquad p_t = \frac{w_t}{F_2(\bar{K}_t, N_t)} + p_t^k \delta_t'$$

The first right hand side term is marginal factor cost. The second is marginal user cost. With Keynes [1973, p. 67] we may say that "the short-period supply price is the sum of the marginal factor cost and the marginal user cost".

If the depreciation rate is given by (60), the critical output level is given by

$$(66) \qquad F(\bar{K}_t, N_t) = \frac{\delta_1}{\delta_2} \bar{K}_t$$

and the critical employment level associated with the critical output level is found by solving (66) for N^*.

From (60) and (65), below the critical output level the supply price will be given by

$$p_t = \frac{w_t}{F_2(\bar{K}_t, N_t)} \tag{67}$$

since $\delta_t' = 0$ below the critical output level. Above that level, the supply price is given by

$$p_t = \frac{w_t}{F_2(\bar{K}_t, N_t)} + p_t^k\left[- \delta_1 + 2\delta_2 \frac{F(\bar{K}_t, N_t)}{\bar{K}_t} \right] \tag{68}$$

The critical output level is determined so that

$$- \delta_1 + \delta_2 \frac{F(\bar{K}_t, N^*)}{\bar{K}_t} = 0 \tag{69}$$

so that

$$- \delta_1 + 2\delta_2 \frac{F(\bar{K}_t, N^*)}{\bar{K}_t} = \delta_2 \frac{F(\bar{K}_t, N^*)}{\bar{K}_t} = \delta_1 \tag{70}$$

Thus there is a discontinuity in the supply curve at the critical output level, as shown in Figure 2.

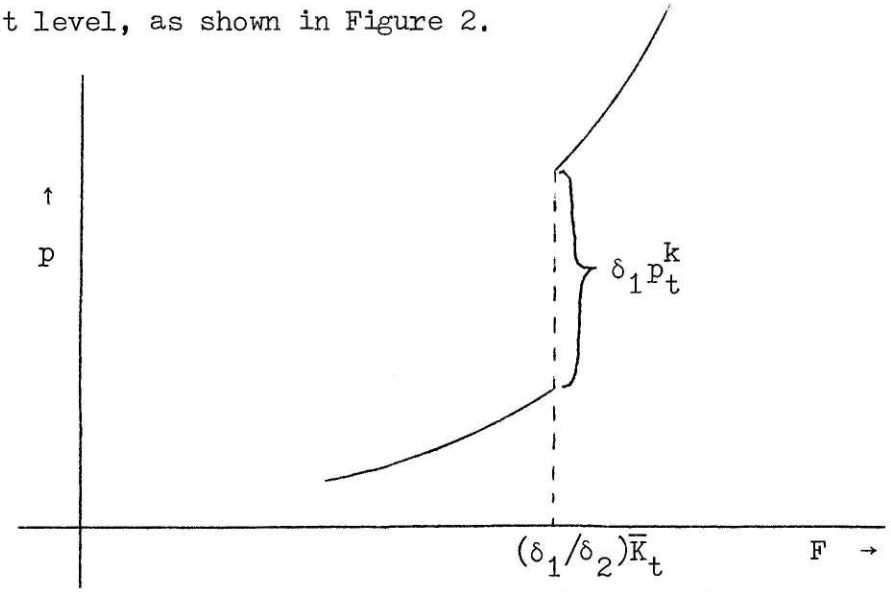


Figure 2: Supply curve, Bain depreciation function

Differentiating (67) with respect to p_t gives

$$(70) \qquad \frac{dN}{dp_t} = -\frac{1}{w_t}\frac{[F_2(\bar{K}_t,N)]^2}{F_{22}(\bar{K}_t,N)}$$

Inverting

$$(71) \qquad \frac{dF}{dp_t} = F_2(\bar{K}_t,N)\frac{dN}{dp_t}$$

and substituting gives the slope of the supply curve below the critical output level:

$$(72) \qquad \frac{dp}{dF} = -\frac{w_t F_{22}(\bar{K}_t,N)}{[F_2(\bar{K}_t,N)]^3} > 0$$

Differentiating (68) with respect to p_t gives

$$(73) \qquad \frac{dN}{dp_t} = 1/\{\, 2p_t^k\delta_2 \frac{F_2(\bar{K}_t,N)}{\bar{K}_t} - w_t \frac{F_{22}(\bar{K}_t,N)}{[F_2(\bar{K}_t,N)]^3} \,\}$$

Inverting (71) and substituting (73) gives

$$(74) \qquad \frac{dp}{dF} = 2\frac{p_t^k\delta_2}{\bar{K}_t} - w_t\frac{F_{22}(\bar{K}_t,N)}{[F_2(\bar{K}_t,N)]^3} > -\frac{w_t F_{22}(\bar{K}_t,N)}{[F_2(\bar{K}_t,N)]^3}$$

as the slope of the supply curve above the critical output level. Comparing (72) and (74), the supply curve is seen to rise more steeply above the critical output level. As demand falls below the critical output level, firms operating along their short-run supply curve will undergo a one time discontinuous fall in price, after which supply price will fall less quickly in response to further declines in quantity demanded.

Firms for which obsolescence is the primary element of depreciation will have a relatively high critical output. In depression, in the short run, such firms are likely to operate

below the critical output level, on the lower segment of the short-run supply curve. With Bain [1937, p. 715] we may say "Obsolescence of industrial equipment is thus a force tending to produce flexibility of supply price." In contrast, firms for which use depreciation is dominant will have a low critical output and will tend to operate on the upper segment of the short-run supply curve.

III. The dynamic model of the firm in one-sector growth models

We will begin by presenting a standard one-sector growth model which incorporates Keynesian user cost. The first purpose of this presentation is to demonstrate that the inclusion of Keynesian user cost requires no unusual assumptions to obtain the usual results of such models. Although the results of this section are standard, the story that is told about the process by which capital is accumulated is not standard. The conventional one-sector growth model assumes an accumulation of capital via saving from income (either by workers or capitalists), with the resulting stock being rented to the productive sector for the appropriate return. We will assume the purchase of capital by the productive sector, with possible dividends to capitalists or interest payments to bondholders. This is the natural extension of the dynamic model of the firm presented above. The development of a conventional one-sector growth model based on dynamic firm behavior paves the way for the second model presented, which presents a distinctive one-sector growth model involving unconventional assumptions which are suggested by the dynamic model of the firm. Although this model incorporates user cost, its primary result (the failure of the Pasinetti theorem) depends on the assumptions which distinguish the model, not on the presence of user cost.

III-A. Constant intertemporal barter prices

We will show that Keynesian user cost may be incorporated into a conventional one-sector growth model. We will examine an economy which produces one good, which is either purchased as an investment and incorporated into the capital stock, or purchased for consumption. The labor force is assumed to grow exponentially at an exogenously given rate g, and full employment of the labor force is assumed. The productive sector is assumed to invest and employ labor and capital in an optimal manner, according to the dynamic model of firm behavior presented above.

We will begin with the standard assumption that the price of output is the same in all periods, and we will normalize this price at one. We thus assume that a unit of output in any period will exchange for output in any other period on a one to one basis. The rate of time preference will vary between periods.

It will be convenient to work with a continuous model. Thus the firm acts to maximize

$$(75) \qquad \int_{t=0}^{\infty} \exp\left[-\int_{\alpha=0}^{t} r(\alpha)d\alpha\right]\{p(t)F[K(t),N(t)] - p_t^k(t)I(t) - w(t)N(t)\}dt$$

where the capital stock at time t is given by

$$(76) \qquad K(t) = K(0) + \int_{\alpha=0}^{t} \{I(\alpha) - K(\alpha)\delta[\frac{F(K(\alpha),N(\alpha))}{K(\alpha)}]\}d\alpha$$

Notation is strictly analogous to the discrete model presented above. Suppose that $\lambda(t)$ is the Lagrangian function associated with the capital constraint (76). We may form a Lagrangian for constrained

maximization with respect to I(t), K(t), and N(t). The results are

$$(77) \qquad I(t): \qquad p^k(t) = \int_{\alpha=t}^{\infty} \exp\left[-\int_{\beta=t}^{\alpha} r(\beta)d\beta\right]\lambda(\alpha)d\alpha$$

which implies

$$(78) \qquad \lambda(t) = r(t)p^k(t) - \dot{p}^k(t)$$

(where " · " signifies the time derivative).

$$(79) \qquad K(t): \qquad pF_1(K,N) = \lambda + p^k \frac{dK\delta}{dK}$$

$$(80) \qquad N(t): \qquad pF_2(K,N) = w + p^k \frac{dK\delta}{dN}$$

(Time variable suppressed for simplicity.)

(77) - (80) are analogous to (16) - (20). Written in intensive form, (79) and (80) become

$$(81) \qquad pf'(k) = \lambda + p^k \frac{dk\delta}{dk}$$

$$(82) \qquad [p - p^k\delta'][f(k) - kf'(k)] = w$$

In the one-sector model, there is one good; assuming constant intertemporal barter prices, we have

$$(83) \qquad p(t) = p^k(t) = 1 \qquad \dot{p}^k(t) = 1$$

Substituting (78) and (83) in (81), the optimality condition for the employment of capital becomes

$$(84) \qquad r = f'(k) - \frac{dk\delta}{dk}$$

where

$$(85) \qquad \delta = \delta[f(k)/k]$$

The optimality condition for employment of labor becomes

$$(86) \qquad w = (1 - \delta')[f(k) - kf'(k)]$$

The optimality condition for investment reduces to

$$\lambda = r \tag{87}$$

The available labor force grows according to the rule

$$N(t) = N(0)e^{gt} \tag{88}$$

and full employment is assumed. From (76), investment is

$$I(t) = \dot{K}(t) + K(t)\delta(t) \tag{89}$$

From (88) and (89), investment per capita is

$$i(t) = \dot{k}(t) + [g + \delta(t)]k(t) \tag{90}$$

From (90), independent of what follows, we conclude that in a steady state, investment per unit of capital will equal the growth rate plus the depreciation rate:

$$\frac{i}{k} = g + \delta\left[\frac{f(k)}{k}\right] \tag{91}$$

The simplest complete model is obtained by assuming the following market clearing equation:

$$f(k) = i + w \tag{92}$$

The flow of output at each instant is either consumed as wages or purchased for addition to the capital stock. Substituting from (86) and (90) into (92) and rearranging terms determines the time path of the capital stock in this model:

$$\frac{\dot{k}}{k} = f'(k) - \frac{dk\delta}{dk} - g \tag{93}$$

Given the initial capital stock, (93) completely determines the time path of the capital stock in this simple economy. Then from (84) we have

$$r(t) = \frac{\dot{k}(t)}{k(t)} + g = f'(k) - \frac{dk\delta}{dk} \tag{94}$$

Thus the rate of time preference must be endogenous in this model. (93) gives the rate of growth of the capital stock; the rate of time preference is, from (94), the net marginal product of capital.

Given the time path of the capital-labor ratio, (86) gives the per capita wage, and (90) gives per capita investment.

Considering Figure 3,[16] we see that the capital-labor ratio will converge to a steady-state value if

(95) $$\frac{d}{dk}\left(\frac{\dot{k}}{k}\right) < 0$$

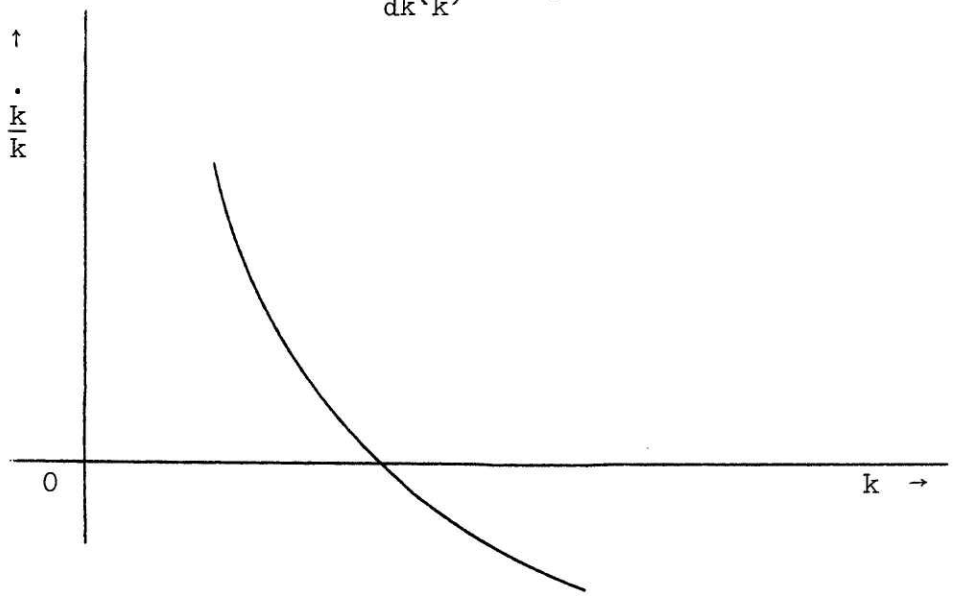


Figure 3: Graph of (93)

or, from (93), if

(96) $$f''(k) - \frac{d^2 k\delta}{dk^2} < 0$$

The left hand side of (96) may be written

(97) $$\frac{d}{dk}\left(\frac{\dot{k}}{k}\right) = f''(k)(1 - \delta') - k\delta''\left[\frac{d}{dk}\left(\frac{f}{k}\right)\right]^2$$

It is sufficient (but not necessary) for the right hand side of

(97) to be negative that

(98) $$\delta'' > 0$$

which will henceforth be assumed. This assumption is satisfied for the special case depreciation functions (52), (55), and (60).

From (93) and (94), the value to which the system converges is given by

(99) $$r(k) = f'(k) - \frac{dk\delta}{dk} = g$$

This will be recognized as the defining condition of a growth path which obeys Phelps' golden rule [Phelps, 1965, p. 793]:

> ...a golden age path on which the social net rate of return to investment equals the rate of growth ...produces a path of consumption which is uniformly higher than the consumption path associated with any other golden age.

The rate of "saving" - steady state investment per unit of output, from (91), is

(100) $$\frac{i}{f(k)} = (g + \delta)\frac{k}{f(k)}$$

Substituting from (93) (setting $\dot{k} = 0$) to eliminate g and rearranging terms gives

(101) $$\frac{i}{f(k)} = (1 - \delta')\frac{k}{f(k)}f'(k) + \delta'$$

In the steady state the rate of social saving is a weighted average of the elasticity of per capita output with respect to the capital-labor ratio and one; the weights are given by the marginal rate of depreciation.[17] Then since (from (92))

(102) $$\frac{w}{f(k)} = 1 - \frac{i}{f(k)}$$

the wage rate as a fraction of per capita output is given by

$$\frac{w}{f(k)} = (1 - \delta')\left[1 - \frac{k}{f(k)} f'(k)\right] \tag{103}$$

Consumption per capita (the wage rate) in the steady state is per capita output times one minus marginal depreciation times one minus the elasticity of per capita output with respect to the capital-labor ratio.

The importance of this simple model as a benchmark will become clear when we extend the model to allow for consumption out of profit.

Suppose that dividends at a rate R per unit of capital are distributed to the owners of capital in each period[18]. In that case, total dividends must be given by

$$RK = F(K,N) - I - wN \tag{104}$$

as a residual output after the purchase of investment goods by industry and of consumption goods by workers. Rewriting (104) in intensive form and rearranging gives

$$f(k) = i + w + Rk \tag{105}$$

which is the revised market clearing equation of the extended model.

The complete system of equations which describes this model is (84), (86), (87), (90), and (105). Substituting (86) and (90) into (105) gives

$$\frac{\dot{k}}{k} = f'(k) - \frac{dk\delta}{dk} - g - R \tag{106}$$

or, substituting (84)

$$\frac{\dot{k}}{k} = r - g - R \tag{107}$$

The capital-labor ratio which obtains in a steady-state is given by

$$f'(k) - \frac{dk\delta}{dk} = g + R \tag{108}$$

(98) remains sufficient to ensure convergence to the steady-state capital-labor ratio. Differentiating (108) with respect to R shows that a greater dividend rate reduces the steady-state capital-labor ratio:

$$\frac{dk}{dR} = \frac{1}{f''(k) - \frac{d^2k\delta}{dk^2}} < 0 \tag{109}$$

(by (96)). Substituting (108) into (100) to eliminate g shows that the steady-state share of investment in output is

$$\frac{i}{f(k)} = (1 - \delta')\frac{k}{f(k)} f'(k) + \delta' - \frac{Rk}{f(k)} \tag{110}$$

The share of dividends in output is a direct reduction of the share of investment in output, because all dividends are consumed by shareholders. We may write, from (100),

$$\frac{d}{dR}\left[\frac{i}{f(k)}\right] = -\frac{g + \delta(1 - \epsilon_{\delta,f/k})}{(f/k)^2} \frac{d}{dk}\left(\frac{f}{k}\right) \frac{dk}{dR} \tag{111}$$

where $\epsilon_{\delta,f/k}$ is the elasticity of the depreciation rate with respect to the output-capital ratio. Then if $1 > \epsilon_{\delta,f/k}$,

$$\frac{d}{dR}\left[\frac{i}{f(k)}\right] < 0 \tag{112}$$

so that the steady-state share of social saving in output will decline as the dividend rate rises.

Regarding the steady-state share of dividends in output, we may

write

$$\frac{d}{dR}\left[\frac{Rk}{f(k)}\right] = \frac{k}{f(k)}\left[1 - \epsilon_{f/k,R}\right] \tag{113}$$

$\epsilon_{f/k,R}$, the elasticity of the output-capital ratio with respect to the dividend rate, is positive; if it is less then one, we have

$$\frac{d}{dR}\left[\frac{Rk}{f(k)}\right] > 0 \tag{114}$$

so that an increase in the dividend rate will increase the share of dividends in steady-state output. If the dividend rate is set to maximize the share of output going to the owners of capital, (113) indicates that it will be chosen so that $\epsilon_{f/k,R}$ equals one. [19]

From (105), the steady-state share of wages in output is

$$\frac{w}{f(k)} = 1 - \frac{i}{f(k)} - \frac{Rk}{f(k)} \tag{115}$$

Substituting from (110) in (115) and rearranging gives, as before (see (103))

$$\frac{w}{f(k)} = (1 - \delta')(1 - \epsilon_{f,k}) \tag{116}$$

where $\epsilon_{f,k}$ is the elasticity of output with respect to capital. Differentiating (116) with respect to R gives

$$\frac{d}{dR}\left[\frac{w}{f(k)}\right] = -\left[\delta''\frac{df/k}{dk}(1 - \epsilon_{f,k}) + (1 - \delta')\frac{d}{dk}(\epsilon_{f,k})\right]\frac{dk}{dR} \tag{117}$$

The sign of the right hand side of (117) will be that of the bracketed right hand side term. The first term within brackets is negative. Since

$$\frac{d}{dk}(\epsilon_{f,k}) = \frac{kf''}{f}(1 - \sigma) \tag{118}$$

where σ is the elasticity of substitution, the second term within brackets will be nonpositive if

(119) $$1 \geq \sigma$$

that is, if the elasticity of substitution is not greater than one. In this case

(120) $$\frac{d}{dR}\left[\frac{w}{f(k)}\right] < 0$$

so that the share of wages in steady-state output will decline as the dividend rate rises.

However, the steady-state wage level falls unambiguously:

(121) $$w = f(k) - i - Rk = f(k) - (g + \delta)k - Rk$$

so that

(122) $$\frac{dw}{dR} = \left[f'(k) - g - \frac{dk\delta}{dk} - R\right]\frac{dk}{dR} - k = -k < 0$$

Both the wage rate and per capita output fall as the dividend rate rises; the ratio of the two, (116), may rise or fall.

The dividend rate R is exogenous, and the model will converge to the steady-state equilibrium given by (108) for any R. Restrictions on R may be derived by comparing this method for modeling consumption out of the income of capital with the customary formulation, which assumes a saving rate s_c out of the income generated by capital. The parameters R and s_c are related as follows:

(123) $$s_c = \frac{r - R}{r}$$

so that the restriction

(124) $$0 \leq s_c \leq 1$$

translates into

$$R \leq r \leq r + R \tag{125}$$

Note from (123) that a constant R will generate a varying s_c, and vice versa (since r will be constant only in a steady-state).

Implicit in the market clearing equation (104) is a story of firm behavior in which the employment condition for capital is

$$pF_1(K,N) = \lambda^* + p^k \frac{dK\delta}{dK} + Rp^k \tag{126}$$

(compare with (79)), where

$$\lambda^* = r^* p^k - \dot{p}^k = (r - R)p^k - \dot{p}^k \tag{127}$$

The first portion of the shadow rental of capital services, λ^*, must be had if the firm is to maintain an optimal capital stock and remain in long-run equilibrium. The second portion, Rp^k, will be distributed to shareholders. We may rewrite (126) as

$$p = \frac{\lambda^*}{F_1} + \frac{p^k}{F_1} \frac{dK\delta}{dK}\bigg|_{F=\bar{F}} + p^k \frac{dK\delta}{dF}\bigg|_{K=\bar{K}} + \frac{Rp^k}{F_1} \tag{128}$$

(128) divides the supply price of the firm into four components. The first right hand side term is the rental of capital services per unit of output, net of dividends. The second is the depreciation rental. Together these terms constitute marginal factor cost (net of dividends to the owners of capital). The third term is marginal user cost. The fourth term represents dividends paid by the marginal unit of output. Thus price is divided into net marginal factor cost, marginal user cost, and dividends.[20]

It is important to emphasize that this same division of price

is implicit in any conventional one-sector growth model which assumes a rate of saving out of the income of capital less than one, as may be seen by substituting

$$R = r(1 - s_c) \tag{129}$$

into (128) (compare (123)).

Although we have not included a fixed natural factor ("land") in out model, (128) recalls the classical division of price (income) into wages (factor cost), rent, and profit. Further, it suggests the Marxian division of value into constant capital, variable capital, and surplus value.

If we suppose that the productive sector consists of a great many owners of small bits of capital, then we would expect them to compete, and we would expect R to go to zero, at least in a world without uncertainty. The attempt of any one owner of capital to maintain a dividend rate $R > 0$ would be frustrated as rivals undercut his dividend rate with the intention of undercutting his supply price and capturing his share of the market. Thus, in contrast to Pasinetti [1974, p. 143], who states

> Of particular interest to our purposes is, however, the case in which the crucial propensity to save s_c is equal to unity. This cannot, of course, happen in a capitalist economy, but can be obtained in a socialist system....

we may argue that the case $s_c = 1$ ($R = 0$) is very much to be expected in simple models which abstract from uncertainty. Note too that this reasoning applies whether firms depend solely on internal finance or not, so long as we abstract from risk and uncertainty.

This in turn leads to the Knightian notion of risk and uncertainty as the source of profit. In this case, competition by the owners of capital would drive R to the minimum value needed to compensate for the risk of enterprise. Again, competition would insure that internal and external funds would earn the same return.

A final extension incorporates saving by wage earners in the dynamic one-sector model. Workers save by loaning to the productive sector. Each bond represents the transfer of one unit of output to the productive sector. The total stock of bonds outstanding at time t is B(t), and this stock pays a return r(t)B(t) to bondholders. At time t, bondholders transfer $\dot{B}(t)$ units of output to the productive sector, and acquire a corresponding number of bonds. The dividend rate R is applied to the stock of capital not owed to bondholders, K(t) - B(t). The market clearing equation becomes

$$F(K,N) = I + wN + rB - \dot{B} + R(K - B) \tag{130}$$

which in intensive form is

$$f(k) = i + w + Rk + (r - g - R)b - \dot{b} \tag{131}$$

The system of equations which describes this model is (84), (86), (87), (90), and (131). Substituting (84), (86) and (90) in (131) gives the equation which describes the time path of the capital-labor ratio:

$$\frac{\dot{k}}{k} = \left[f'(k) - \frac{dk\delta}{dk} - g - R\right]\left[1 - \frac{b}{k}\right] + \frac{\dot{b}}{k} \tag{132}$$

Maintenance of a steady-state requires that $\dot{b} = 0$. Taking this as given, the first steady-state capital-labor ratio is given by

(133) $$r = f'(k) - \frac{dk\delta}{dk} = g + R$$

which is the steady-state value of the model without private saving. This is a result of Pasinetti [1962; 1974]. Suppose that per capita bond holdings are constant; then from (131) output per capita is divided according to

(134) $$f(k) = i + w + R(k - b) + \left[f'(k) - \frac{dk\delta}{dk} - g\right]b$$

The per capita return to owners of capital is $R(k - b)$. The per capita return to bondholders is $(r - g)b$. In a steady state, $r - g = R$, and the net return to bondholders is Rb, so that the total of dividends to the owners of capital and interest paid to bondholders (net of purchases of bonds), per capita, is Rk, which is just the return to owners of capital when private saving does not occur.

The second steady-state capital-labor ratio is given by [21]

(135) $$k = b$$

The steady-state capital-labor ratio is absolutely arbitrary, so long as the entire capital stock is owned by bondholders. To determine the capital-labor ratio in the second case, we must specify the process by which private savers accumulate bonds. To this end, suppose that all bonds are held by workers, and that workers save a constant fraction s_w of their income by converting it into bonds. On these assumptions, workers' income is

(136) $$Y(t) = w(t)N(t) + r(t)B(t)$$

and private saving is

(137) $$S(t) = s_w\left[w(t)N(t) + r(t)B(t)\right] = \dot{B}(t)$$

Written in intensive form, (137) becomes

$$\dot{b} + bg = s_w w + s_w r b \tag{138}$$

from which

$$b = \frac{s_w w - \dot{b}}{g - s_w r} \tag{139}$$

If there is to be a positive and finite steady-state level of per capita bond holdings, we must have

$$g > s_w r \tag{140}$$

If not, from

$$\dot{b} = s_w w + b(s_w r - g) \tag{141}$$

the per capita level of bond holdings will rise continuously. From (132), in that case, since $\dot{b} \neq 0$, there will be no convergence of the capital-labor ratio to a steady-state value.

To investigate the particular steady-state capital-labor ratio implied by (135) and the particular saving assumption (137), in (139) set $b = k$ and $\dot{b} = 0$. The steady-state capital-labor ratio is then given by

$$k = \frac{s_w w}{g - s_w r} \tag{142}$$

Substituting (84) and (86) into (142) to eliminate r and w gives the determining condition of the steady-state:

$$\frac{f(k) - k\delta}{k} = \frac{g}{s_w} \tag{143}$$

This will be recognized as the defining condition of what Samuelson and Modigliani term the anti-Pasinetti golden age [Samuelson and Modigliani, 1966a, p. 270], extended to a model in

which depreciation occurs.

To determine the share of bondholders in the capital stock of the Pasinetti steady-state, a rule must be specified by which bonds are accumulated. Assuming that (137) applies, the share of bondholders in the capital stock is

$$(144) \qquad \frac{b}{k} = \frac{s_w}{g - s_w r}\left[\frac{f - k\delta}{k} - r\right] = \frac{s_w}{g - s_w r}\frac{w}{k}$$

The share of capitalists in the capital stock is then

$$(145) \qquad \frac{k - b}{k} = \frac{s_w}{g - s_w r}\left[\frac{g}{s_w} - \frac{f - k\delta}{k}\right]$$

From (144), for the steady-state level of per capita bond holdings to be positive, s_w must satisfy

$$(146) \qquad g > s_w r$$

From (145), for the capitalists' share of the capital stock to be positive in the steady-state, s_w must satisfy

$$(147) \qquad g > s_w \frac{f - k\delta}{k}$$

Since

$$(148) \qquad \frac{f - k\delta}{k} > r$$

if (147) is satisfied, then (146) is satisfied as well.[22]

If the Pasinetti steady-state exists, then in that steady-state workers' per capita consumption is

$$(149) \quad c = (1 - s_w)(w + rb) = (1 - s_w)w\left(1 + \frac{rs_w}{g - s_w r}\right) = gw\frac{1 - s_w}{g - s_w r}$$

where use is made of (139) with $\dot{b} = 0$. Comparing the steady-state consumption levels corresponding to different saving rates (and noting that the wage rate w is the same in different Pasinetti steady states),

$$(150) \qquad \frac{dc}{ds_w} = gw \frac{r - g}{(g - s_w r)^2} = \frac{gwR}{(g - s_w r)^2} > 0$$

so that a greater private saving rate always increases the steady-state per capita consumption level of workers.

Under simple saving rules (i.e., b constant or b/k constant), previous assumptions will ensure convergence to the Pasinetti steady state. For more complex saving rules, the discussion of stability is more complex.[23] Since the level of the capital-labor ratio in the anti-Pasinetti steady state (in which per capita bond holdings are constant and all capital is owned by bondholders) is essentially arbitrary, discussions of stability depend crucially on savings assumptions.

III-B. Endogenous intertemporal barter prices

The dynamic one-sector growth model developed above made two assumptions which are standard in static[24] one-sector growth models. The first assumption is that the value placed on a unit of output in different periods is the same. The second is that the discount rate is endogenous. Under these assumptions, the rate of time preference must equal the net marginal product of capital, which is determined elsewhere in the model (see (84)).

It is not obvious that the discount rate, which is a rate of time preference, should be endogenous. Nor is it obvious that the value placed on a unit of output at different times will be the same. We will develop a one-sector growth model which does not make these assumptions, showing that they are unnecessary.

The optimality conditions of this model are derived from the continuous model of the firm developed in (75) - (80) above. Suppose in (75) that

$$r(t) = r \tag{151}$$

so that the rate of time preference is exogenous and constant. In a one sector model, set

$$p^k(t) = p(t) \tag{152}$$

We normalize prices so that $p(0) = 1$. There is no particular reason to suppose that physically identical output at different times represents the same economic good; being different goods, such outputs will have different relative prices. Our normalization assumption means that $p(t)$ will be measured in units of time zero output per unit

of time t output. These are barter prices which reflect the relative market value placed on different goods, goods which are distinguished by time.

The optimality condition for investment, (75), becomes

$$p(t) = \int_{\alpha=t}^{\infty} e^{r(t-\alpha)}\lambda(\alpha)d\alpha \tag{153}$$

from which (compare with (78))

$$\lambda(t) = rp(t) - \dot{p}(t) \tag{154}$$

Written in intensive form, and substituting (154), the optimality conditions for employment of capital and labor are:

$$K(t): \quad pf'(k) = rp - \dot{p} + p\frac{dk\delta}{dk} \tag{155}$$

$$N(t): \quad p(1-\delta')[f(k) - kf'(k)] = w \tag{156}$$

As before, per capita investment is

$$i = \dot{k} + (g+\delta)k \tag{90}$$

To close the system, a market clearing equation is needed. We will begin with the simplest model, which allows neither distributed dividends nor private saving:

$$pf(k) = pi + w \tag{157}$$

Substitute from (156) and (90) into (157) to eliminate i and w; the price term cancels, and we obtain

$$\frac{\dot{k}}{k} = f'(k) - \frac{dk\delta}{dk} - g \tag{158}$$

Comparing (158) with (93), we see that the time path of the capital-labor ratio, given the same initial value, will be the same

whether intertemporal prices are constant or endogenous. From (155)

$$\frac{\dot{p}}{p} = r - \left[f'(k) - \frac{dk\delta}{dk}\right] \tag{159}$$

This is subject to a natural interpretation. If the rate of time preference exceeds the net return to investment, so that

$$r > f'(k) - \frac{dk\delta}{dk} \tag{160}$$

then the price of the capital good is rising. Since the net return to investment is insufficient to match the rate of time preference, a capital gain to the owners of capital is required to increase the total return to the investment of a unit of capital. The opposite occurs if the rate of time preference is less than the net return to investment (owners of capital suffer a capital loss), and if

$$r = f'(k) - \frac{dk\delta}{dk} \tag{161}$$

the price does not change. From (84), this is a condition which holds continuously in the constant barter prices model.

Substituting from (159) into (154) to eliminate $\dot{p}$ gives

$$\lambda = \left[f'(k) - \frac{dk\delta}{dk}\right]p \tag{162}$$

Comparing (162) with (84) and (87), and recalling that the conventional model sets $p(t) = 1$, we see that in both models the shadow value of a unit of capital services is the value of the net return to the services of a unit of capital.

Since the time path of the capital-labor ratio is the same in

both models, the same condition, (98), is sufficient to ensure convergence to a steady-state value, which, from (158), will be given by

$$f'(k) - \frac{dk\delta}{dk} = g \tag{163}$$

Suppose that such a steady-state capital-labor ratio has been reached. Substituting in (159) gives

$$\frac{\dot{p}}{p} = r - g \tag{164}$$

from which

$$p(t) = e^{(r - g)t} \tag{165}$$

(since we have normalized $p(0) = 1$). If the rate of time preference exceeds the growth rate, in a steady-state the owners of capital will enjoy continual capital gains. However, the value of a unit of output, discounted to time zero, falls continuously:

$$e^{-rt}p(t) = e^{-gt} \tag{166}$$

It is as though each of the initial $N(0)$ individuals in the society founds a dynasty, which grows at rate g, and places a constant discounted value on the share of output going to the dynasty:

$$e^{-rt}p(t)\,\frac{F[K(t),N(t)]}{N(0)} = f(k) \tag{167}$$

But the portion of the dynasty's share which goes to each member falls exponentially at rate g, since that is the rate at which the dynasty grows.

The introduction of dividends to the endogenous intertemporal barter

prices model is straightforward, and produces a model similar to the constant barter prices model with dividends. We will proceed directly to the complete model, which allows both dividends and private saving.

There is a stock of bonds $B(t)$ outstanding at time t. Each bond represents the transfer of one unit of output to the productive sector. Outstanding bonds have value $p(t)B(t)$, and earn a return $rp(t)B(t)$. At time t, individuals acquire bonds $\dot{B}(t)$ of value $p(t)\dot{B}(t)$. On these assumptions, the market clearing equation is

$$pF(K,N) = pI + wN + Rp(K - B) + rpB - p\dot{B} \tag{168}$$

Written in intensive form, (168) becomes

$$pf(k) = pi + w + Rpk + (r - R - g)pb - p\dot{b} \tag{169}$$

The complete model is described by (90), (153), (154), (156), and (169). Substituting from (90) and (156) into (169) gives

$$\frac{\dot{k}}{k} = f'(k) - \frac{dk\delta}{dk} - r + (r - g - R)\left(1 - \frac{b}{k}\right) + \frac{\dot{b}}{k} \tag{170}$$

Suppose we are in a steady state, so that $\dot{k} = 0 = \dot{b}$, and suppose that

$$\frac{b}{k} = \beta \tag{171}$$

The steady-state capital-labor ratio is determined by

$$f'(k) - \frac{dk\delta}{dk} - r + (r - g - R)(1 - \beta) = 0 \tag{172}$$

Differentiating (172) with respect to β, we obtain

$$\frac{dk}{d\beta} = \frac{r - g - R}{f''(k) - \frac{d^2k\delta}{dk^2}} \tag{173}$$

According to our stability assumptions (see (96)), the denominator of the right hand side fraction is negative. Thus

$$\frac{dk}{d\beta} \gtreqless 0 \text{ as } r \lesseqgtr R + g \tag{174}$$

The steady-state capital-labor ratio is not, in general, invariant to the amount of private saving. Pasinetti's Theorem [1962;1974] does not generalize to a model in which intertemporal barter prices are endogenous. But if $r = g + R$, as in the steady-state of the constant barter prices model, we obtain the result of that model.

Some insight into the reason for the failure of the Pasinetti theorem may be gained by comparing the per capita, steady-state market clearing equations for the two models:

$$\text{CBP:} \qquad f(k) = i + w + R(k - b) + (r - g)b \tag{175}$$

$$\text{EBP:} \qquad pf(k) = pi + w + pR(k - b) + (r - g)pb \tag{176}$$

(Compare with (134) and (169)). In the constant barter prices model, income spent by stockholders is $R(k - b)$, which is the dividend rate applied to that portion of capital which has not been borrowed from bondholders. The income consumed by bondholders (as bondholders, not as workers) is $(r - g)b$, which is the total return to bonds less the purchase of new bonds necessary to maintain constant per capita bond holdings in the face of an increasing population. Thus the total income consumed from capital, the leakage from investment, is

$$R(k - b) + (r - g)b = Rk + b(r - g - R) = Rk \tag{177}$$

since in the steady state of CBP model $r = g + R$. The income consumed from capital, per capita, Rk, is just that consumed in the absence of private saving, so the steady state capital-labor ratio is invariant to the occurrence of private saving.

On the other hand, in the endogenous barter prices model the income consumed by stockholders is $Rp(k - b)$, and the income consumed by bondholders is $(r - g)pb$, so that the income consumed from capital, per capita, is

$$p\left[R(k - b) + (r - g)b\right] = p\left[Rk + b(r - g - R)\right] \tag{178}$$

However, in the EBP model there is no reason to expect $r = g + R$, so that the second term within the right hand side brackets in (178) does not vanish in the steady state. Then if $r > g + R$, the income consumed from capital rises with the level of per capita bond holdings; one would expect the steady-state capital-labor ratio to fall. This is just the message of (174). Similarly, if $r < g + R$, the income consumed from capital falls as private saving rises, so the steady-state capital-labor ratio rises.

The failure of the Pasinetti theorem in the EBP model is a result of the fact that in the EBP model the rate of time preference is exogenous, so that it does not converge to the particular value $r = R + g$.

Suppose that $\beta \neq 1$, so that we are in a Pasinetti world in which capitalists and bondholders share the ownership of capital. Then from (172), the steady-state capital-labor ratio is given by

$$r - \left[f'(k) - \frac{dk\delta}{dk}\right] = (r - g - R)(1 - \beta) \tag{179}$$

which may be rewritten

$$f'(k) - \frac{dk\delta}{dk} = \beta r + (1 - \beta)(g + R) \tag{180}$$

The steady-state capital-labor ratio is such that the net marginal product of capital is a weighted average of the rate of time preference and the sum of the growth rate and the dividend rate; the weights are given by the share of bondholders and capitalists in the capital stock.

If workers save a constant fraction of all income, steady-state bond holdings are (see (139)

$$b = \frac{s_w}{g - s_w r} \frac{W}{p} \tag{181}$$

from which

$$\frac{b}{k} = - \frac{s_w}{g - s_w r} (1 - \delta')k\frac{df/k}{dk} \tag{182}$$

Substitution of (182) for β in (179) gives

$$f'(k) - \delta - k\frac{df/k}{dk}\left\{\frac{(r - g - R)s_w}{g - s_w r} - \delta'\left[\frac{g(1 - s_w) - Rs_w}{g - s_w r}\right]\right\} = g + R \tag{183}$$

as the equation which determines the steady-state capital-labor ratio in the Pasinetti world for this particular saving assumption.

The anti-Pasinetti world exhibits complete ownership of the capital stock by bondholders. Setting $\beta = 1$ in (180) gives the determining equation of the steady-state capital-labor ratio:

$$f'(k) - \frac{dk\delta}{dk} = r \tag{184}$$

In the anti-Pasinetti steady-state, the net marginal product of

capital will equal the rate of time preference. If we assume that a constant fraction of income is saved, we may substitute $k = b$ in (181) and combine with (184) to obtain

$$\frac{f(k) - k\delta}{k} = \frac{g}{s_w} \tag{185}$$

Comparing with (143), we conclude that the anti-Pasinetti steady state capital-labor ratio is the same in the CBP and the EBP models.

The assumption that the second derivative of the depreciation rate function is nonnegative remains sufficient to ensure convergence to the steady state in the Pasinetti world with a constant share of bondholders in the capital stock, and in the anti-Pasinetti world.

Substituting from (172) into (159) shows that in the Pasinetti steady state the rate of change of price is

$$\frac{\dot{p}}{p} = (r - g - R)(1 - \beta) \tag{186}$$

Price rises over time if $r > g + R$, and falls if $r < g + R$. Differentiating (186) with respect to β gives

$$\frac{d}{d\beta}\left(\frac{\dot{p}}{p}\right) = -(r - g - R) \tag{187}$$

Increased private saving will lower the rate of change of price if $r > g + R$, and raise it if $r < g + R$.

In the anti-Pasinetti steady state, price is constant, which may be seen by substituting $\beta = 1$ in (186) or by substituting (184) into (159).

In the Pasinetti steady state, suppose that all bonds are held

by workers. Consumption per worker is

(188) $$c = f(k) - i - Rk(1 - \beta)$$

or, substituting $i = (g + \delta)k$,

(189) $$c = f(k) - (g + \delta)k - Rk(1 - \beta)$$

Differentiating (189) with respect to the rate of time preference r gives

(190) $$\frac{\partial c}{\partial r} = \left[f'(k) - \frac{dk\delta}{dk} - g - R(1 - \beta)\right] \frac{dk}{dr}$$

Substituting from (180) gives[25]

(191) $$\frac{\partial c}{\partial r} = \beta(r - g)\frac{dk}{dr}$$

Since differentiating (180) shows that $\frac{dk}{dr} < 0$, the rate of time preference which maximizes workers' consumption in the Pasinetti steady state is

(192) $$r = g$$

so that Phelps' golden rule (see p. 35 above) generalizes to the endogenous intertemporal barter price model.

The same result holds in the anti-Pasinetti world, which may be seen by setting $\beta = 1$ in (191).

If the golden rule rate of time preference is chosen, then from (171) we have

(193) $$\frac{\dot{p}}{p} = - R(1 - \beta)$$

so that the price level declines in the Pasinetti world. The rate of time preference is less than the net marginal product of capital (see (179), (180)), so capitalists suffer capital losses.

III-C. Optimal one-sector growth

Some insight into the fundamental difference between the assumptions which characterize the two models discussed above may be gained by investigating optimal economic growth in a dynamic context.

Suppose that a society seeks to maximize the discounted present value of the utility of consumption

$$\int_{t=0}^{\infty} e^{-rt} U[C(t)]dt \tag{194}$$

where

$$C(t) + I(t) = F[K(t),N(t)] \tag{195}$$

$$K(t) = K(0) + \int_{\alpha=0}^{t} [I(\alpha) - K(\alpha)\delta(\alpha)]d\alpha \tag{196}$$

$$N(t) = N(0)e^{gt} \tag{197}$$

To discern the path of investment which the society should choose to meet this goal, form the following Lagrangian and maximize it with respect to I, K, and N (time variables suppressed for simplicity):

$$L = \int_{t=0}^{\infty} e^{-rt}\{U[F(K,N) - T] + \lambda_1 [K(0) + \int_{\alpha=0}^{t} [I - K\delta]d\alpha - K] + \lambda_2 [e^{gt}N(0) - N]\}dt \tag{198}$$

which may be written

$$L = \int_{t=0}^{\infty} e^{-rt}\{U[F(K,N) - I] + [I - K\delta]\int_{\alpha=t}^{\infty} e^{r(t-\alpha)}\lambda_1 d\alpha - \lambda_1 K - \lambda_2 N\}dt + K(0)\int_{t=0}^{\infty} e^{-rt}\lambda_1 dt + N(0)\int_{t=0}^{\infty} e^{(g-r)t}\lambda_2 dt \tag{199}$$

The first order condition for investment is

$$U'(C) = \int_{\alpha=t}^{\infty} e^{r(t-\alpha)}\lambda_1 d\alpha \tag{200}$$

from which

$$\lambda_1 = rU'(C) - \dot{U'(C)} \tag{201}$$

The first order conditions for capital and labor are:

$$U'(C)F_1(K,N) = \lambda_1 + U'(C)\frac{dK\delta}{dK} \tag{202}$$ K:

$$U'(C)F_2(K,N) = \lambda_2 + U'(C)\frac{dK\delta}{dN} \tag{203}$$ N:

(201), (202), and (203) may be compared with the first order conditions of the endogenous intertemporal barter price model (from (78), (79), and (80)):

$$\lambda = rp - \dot{p} \tag{204}$$

$$pF_1(K,N) = \lambda + p\frac{dK\delta}{dK} \tag{205}$$

$$pF_2(K,N) = w + p\frac{dK\delta}{dN} \tag{206}$$

The identification of the market conditions with the optimality conditions is immediate: from the same initial conditions, the market model will evolve so that

$$p = U'(C) \tag{207}$$

the barter price at time t equals the marginal utility of consumption at time t. The shadow price of capital will then be the same in both models, and the wage paid to laborers in the market model will equal

the shadow wage (λ_2) in the optimal growth model.

In our first group of models, we made the conventional assumption that the barter price of output is the same at different times. Inspection of the optimal growth model suggests that this involves the implicit assumption that the marginal utility of output is constant, so that the utility function takes the rather special form

(208) $$U(C) = a + bC$$

If the first order conditions for the constant barter price model are compared with (201) - (203), it will be seen that the model can be directly shown to be optimal only if $U'(C) = 0$.

We may also investigate optimal economic growth when there is a class of capitalists who receive dividends RK(t) at each instant. Suppose that such a society seeks to maximize

(209) $$\int_{t=0}^{\infty} e^{-rt}\{\theta(t)U^W[C(t)] + [1 - \theta(t)]U^C[RK(t)]\}dt$$

where $\theta(t)$ is the weight given to the utility of workers and $1 - \theta(t)$ the weight given to the utility of capitalists, at time t, in the social welfare function. Maximization proceeds subject to

(210) $$C = F(K,N) - I - RK$$

and subject to (196) and (197). We may proceed as before: form a Lagrangian by substituting from (210) into (209) to eliminate C; let $\lambda_1(t)$ be the Lagrangian multiplier associated with the capital stock constraint (196) and let $\lambda_2(t)$ be the Lagrangian multiplier associated with the labor constraint (197). This Lagrangian will be maximized with respect to I(t), K(t), and N(t).

The resulting first order conditions are:

(211) I: $$\theta U^{W}{}'(C) = \int_{\alpha=t}^{\infty} e^{r(t-\alpha)}\lambda_1 d\alpha$$

from which

(212) $$\lambda_1 = r\theta U^{W}{}'(C) - \overline{[\theta U^{W}{}'(C)]}$$

(213) K: $$\theta U^{W}{}'(C)F_1(K,N) = \lambda_1 + \theta U^{W}{}'(C)\frac{dK\delta}{dK} + R\{\theta U^{W}{}'(C) - (1 - \theta)U^{C}{}'(RK)\}$$

(214) N: $$\theta U^{W}{}'(C)F_2(K,N) = \lambda_2 + \theta U^{W}{}'(C)\frac{dK\delta}{dN}$$

Comparison of (212) - (214) with (204) - (206) shows that the endogenous barter price model will produce the optimal result only if the bracketed term on the right hand side of (213) is zero, which will occur only if the weights given the utility of the different classes in the social welfare function are

(215) $$\theta = \frac{U^{C}{}'(RK)}{U^{W}{}'(C) + U^{C}{}'(RK)}$$

(216) $$1 - \theta = \frac{U^{W}{}'(C)}{U^{W}{}'(C) + U^{C}{}'(RK)}$$

The market will produce the optimal solution only if the weight given the utility of workers falls as the consumption of capitalists rises (on the assumption of declining marginal utility).

III-D. The assumptions of the two models combined

So far we have contrasted the assumption of a constant intertemporal barter price and a variable, endogenous rate of time preference with the assumption of endogenous intertemporal barter prices and a fixed, exogenous rate of time preference. We may combine these assumptions and investigate the consequences of fixing both the price of output and the rate of time preference.[26] The first order conditions for investment, capital, and labor are (in intensive form):

(217) I: $\lambda = r$

(218) K: $f'(k) - \frac{dk\delta}{dk} = r$

(219) N: $(1 - \delta')[f(k) - kf'(k)] = w$

The market clearing equation is (see (130), (131))

(220) $f(k) = i + w + rk + (r - R - g)b - \dot{b}$

where

(221) $i = \dot{k} + (g + \delta)k$

(218) gives the constant value of the capital-labor ratio, which is a function of the rate of time preference r: k is determined so that the net marginal product of capital equals the rate of time preference. Substituting this value of k in (219) gives the wage rate, which is also constant. Since $\dot{k} = 0$, (221) becomes

(222) $i = (g + \delta)k$

Substituting (219) and (222) in (220) and rearranging gives

(223) $(g + R - r)(1 - \frac{b}{k}) - \frac{\dot{b}}{k} = 0$

from which

$$R(t) = r - g + \frac{\dot{b}}{k - b} \tag{224}$$

which determines the time path of the dividend rate over time. If the relative price of output at different times is constant, and the rate of time preference is constant, then the dividend rate is determined by private saving.

Suppose that $\frac{b}{k} = \beta$, a constant, so that per capita bond holdings are constant. Then (223) becomes

$$(g + R - r)(1 - \beta) = 0 \tag{225}$$

which has two solutions. If $\beta < 1$, then

$$R = r - g \tag{226}$$

is the dividend rate. If $\beta = 1$, the dividend rate is indeterminate, but that is a matter of little consequence, since the share of the capital stock to which the dividend rate is applied is then zero.

IV. Summary and conclusion

We began by tracing the concept of Keynesian user cost from its definition through a discussion which attempted to implant it in a dynamic model of firm behavior, but which concluded with its obliteration by the distinct concept of retainer cost.

Identifying Keynesian user cost with the value of depreciation resulting from use, we developed a dynamic model of firm behavior which includes Keynesian user cost. Such a firm purchases investment goods to maximize the discounted present value of net revenue, and will modify output if production increases depreciation. Marginal user cost enters the employment conditions of both factors, and output is adjusted so that the price received for output just equals marginal factor cost plus marginal user cost. We derived a dynamic factor price frontier and studied the comparative dynamic response of the firm to changes in factor prices and in the rate of time preference. We discussed a particular depreciation function due to Bain [1937], and showed that obsolescence in depreciation promotes short-run price flexibility.

The introduction of Keynesian user cost in a model of firm behavior requires no unusual assumptions to obtain reasonable results.

We presented three sorts of dynamic one-sector growth models. Capital accumulation in these models results from optimal investment behavior, not from arbitrary saving rules.

The first model makes conventional assumptions and obtains conventional results. The incorporation of Keynesian user cost

in a conventional one sector growth model requires only the additional assumption that the marginal depreciation rate is nondecreasing to guarantee convergence to a steady state. The dynamic model casts doubt on Pasinetti's assertion [1974, p. 131n]

> that marginal productivity theory was successful mainly because it could be used to support the legitimacy of a rate of profit in a capitalist system.

since it suggests that in a world of perfect knowledge and competitive capital, the dividend rate would be driven to zero, and all the income of capital would be invested. In this case the return to capital serves strictly to allocate output between investment and consumption.

We then pursue two alternative one sector growth models, the distinctive assumptions of which are suggested by the dynamic model of the firm. By fixing the rate of time preference and allowing the intertemporal barter price of output to be determined within the model, we obtain a growth model to which the Pasinetti paradox of the independence of the steady-state capital-labor ratio from private saving does not apply. It is seen that the conventional model, which makes the relative price of output at different times equal, involves the implicit assumption that the marginal social value of output at different times is the same, while the endogenous barter price model does not make this assumption.

By combining the assumption of an equal barter price of output with the assumption of an exogenous rate of time preference, we obtain a model which determines the dividend rate as a function of private

saving. If per capita bond holdings are constant, we develop a model in which the dividend rate, the capital-labor ratio, the capital-output ratio, the depreciation rate, and the share of capital and labor in output are all constant over time.

Two conclusions may be drawn from the discussion of one sector growth models. The consideration of Keynesian user cost does not obviate conventional results. The consideration of optimal dynamic (rather than static) behavior by the productive sector offers insight into the process of growth and distribution which the assumptions of conventional (static) models do not suggest.

The range for future research involving Keynesian user cost is wide. I suggest two such areas. The model of firm behavior developed above, in which competitive price equals marginal factor cost plus marginal user cost, suggests a role for user cost in cross-sectional studies which attempt to explain the discrepancy between price and marginal cost.[27] The discussion of growth models indicates that it may be of interest to replace arbitrary investment functions by optimal investment behavior in conventional macroeconomic models.

Footnotes

1. See also MacDougall [1936].

2. One of those suggesting "complicated methods of determining output" was Harold Hotelling [1925, 1931], to whom Scott refers.

3. The other writers cited above also emphasize the importance of the depreciation which results from use; see Bain [1937, p. 707]; Paish [1938, pp. 384-385]; Neal [1942, p. 58]; Scott [1953, p. 381]; Lutz & Lutz [1951, pp. 61-62].

4. See Jorgenson [1963, p. 249], Jorgenson & Stephenson [1967, p. 17n], Jorgenson & Siebert [1968, pp. 1126-7] for models in which the rate of depreciation is independent of use.

5. See Latham & Peel [1975] for an article on user cost which has absolutely nothing to do with Keynesian user cost.

6. See Marshall [1930, p. 74, p. 421], Meriam [1936, p. 317], Stigler [1966, p. 250]. In the model of the firm developed below, marginal user cost is (28), and the quasi-rent to the capital stock is the right hand side of (29).

7. Lucas [1967] assumes a constant time depreciation but obtains a term which he identifies with user cost by including investment in the production function. Gould [1968] and Treadway [1969, 1970] assume a constant depreciation rate. Taubman and Wilkinson [1970a,1970b] make depreciation a function of the utilization rate of the capital stock, and make that utilization rate a distinct factor in the production function; their formulation is the only one which could be identified with Keynesian user cost as here interpreted. Canarella interprets Keynesian user cost just as we do, but he makes his depreciation function a separable, additive function of output and capital stock.

$$G(Q,K) = M(Q) + N(K)$$

and the first derivatives of the functions M, N are assumed to be constant. Further, he includes in his model a cost-of-adjustment feature by making the cost of investment goods an increasing function of investment - C(I) - rather than a constant price of capital $p^k(t)$.

8. See Burmeister & Dobell [1970, pp. 9 - 10] for a discussion of the properties of a neoclassical production function.

9. For the depreciation function assumed by Canarella [1976], see footnote 7. Bain [1937] makes depreciation a function of obsolescence, physical deterioration, and use, and assumes a non-

additive functional form. The depreciation rate function (11) may be thought of as coming from a total depreciation function

$$(11)' \qquad D_t = D[F(K_t,N_t),K_t]$$

which is homogeneous of the first degree.

10. However, see the discussion of the Bain depreciation function, below.

11. See footnote 6.

12. Alternatively, the period zero term may be written

$$p_0^n F_1(K_0,N_0)K_0 + p_0^k \frac{d}{dF_0}(K_0\delta_0)\Big|_{K_0=\bar{K}_0} F(K_0,N_0) + p_0^k(1-\delta_0)K_0$$

which is the net value of the marginal product of capital times the initial capital stock, plus the marginal user cost times initial output, plus the value of the capital stock on hand after the production of initial output. In a continuous model, the discounted present value of net revenue is simply the value of the initial capital stock.

13. Thus $F(K,N) = NF(k,1) = Nf(k)$, where $k = K/N$.

14. Actually, the size of the firm is indeterminate, because of the first degree homogeneity assumption made for the production function. The size may be determined by compelling the firm to pay for investment purchases from current receipts

$$p_t^k I_t = p_t F(K_t,N_t) - w_t N_t$$

which would have to be added to the maximization problem as a separate constraint. In effect, this is the market clearing equation of the simple growth models discussed below.

15. Of course, the firm will produce only if the revenue (62), when labor is optimally employed (according to (63)) exceeds

$$(15.1) \qquad p_t^k \bar{K}_t(1 - \delta_0)$$

which is the value of the capital stock at the end of the period in the absence of production. Thus the firm will produce if

$$(15.2) \qquad p_t F(\bar{K}_t,N_t) - w_t N_t \geq p_t^k \{\delta[\frac{F(\bar{K}_t,N_t)}{\bar{K}_t}] - \delta_0\}$$

where N_t is to be regarded as a function of p_t, defined by (63).

But for low levels of output, from (60)

$$\delta\left[\frac{F(\bar{K}_t, N_t)}{\bar{K}_t}\right] = \delta_0 \tag{15.3}$$

so that the right hand side of (15.2) is zero. Substituting (63) in (15.2) and converting to intensive notation, the firm will produce if

$$k_t f'(k_t) \geq 0 \tag{15.4}$$

which is true for all levels of output. Thus the firm will always choose to produce in the short run. Because of the assumptions that capital is malleable and the production function is first degree homogeneous, it will never be optimal for the firm to leave part of the capital stock idle, while the rest is used. In a putty-putty model of capital, the problems of holding capital idle or of scrapping capital cannot be properly studied; such problems are at home in putty-clay models.

16. Formally, define a Lyapunov function, a la Burmeister & Dobell [1970, p. 28].

17. Compare (101) with Burmeister & Dobell [1970, p. 50].

18. R need not be constant, although this is necessary for convergence to a steady state; the important thing is that R be exogenous.

19. However, if as might be expected R is chosen to maximize Rk, R will be chosen so that

$$1 - \epsilon_{f/k,R} + \epsilon_{f,R} = 0$$

in which case (114) is satisfied.

20. Were we considering the employment condition for labor, marginal factor cost in terms of labor would equal the marginal factor cost of capital, net of dividends, plus the dividend term. The restriction of employment of capital, which generates the greater return out of which dividends are paid, distorts the employment condition for labor.

21. The discrete version of (132) is

$$(1+g)\frac{k_{t+1} - k_t}{k_t} = \left[f'(k_t) - \frac{d}{dk_t}(k_t\delta_t) - g - R\right]\left[1 - \frac{1}{1+g}\frac{b_{t-1}}{k_t}\right] + \frac{b_t - b_{t-1}}{k_t} \tag{132}'$$

so that the anti-Pasinetti steady state does not occur in the discrete model.

22. Substituting (129) in (108) translates $r = g + R$ into $g = s_c r$; (146) translates into $s_c > s_w$.

23. See Samuelson & Modigliani [1966a, p. 275-280; 1966b].

24. In the conventional one sector growth model, capital is accumulated according to a variety of saving assumptions; the accumulated capital stock and labor force are fully employed according to static optimality relations; see Burmeister & Dobell [1970, p. 15. p. 22]. The dynamic one sector growth model contains the static optimality conditions as a subset, but includes additional conditions to ensure optimal accumulation of capital.

25. The equation in the discrete model which corresponds to (179) is

$$(179)' \qquad r - \left[f'(k) - \frac{dk\delta}{dk}\right] = (r - g - R)\left(1 - \frac{\beta}{1 + g}\right)$$

Equations (189) and (190) are the same in both the continuous and the discrete model; substituting (179)' into (190) gives the equivalent to (192) in the discrete model:

$$(192)' \qquad r = g(1 - R)$$

This is the generalization of Phelps' golden rule to the case when $R > 0$ in the discrete model.

26. The rate of time preference need not be constant; it can vary with time, so long as it does so exogenously.

27. See Neal [1942, p. 63, p. 64n].

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ON DEPLETION AND NATURAL RESOURCES

I. The received theory of nonrenewable resources

The economic theory of nonrenewable resources emphasizes two characteristics which distinguish the extraction of a nonrenewable resource by application of labor and capital to a finite resource stock from conventional production of output by combination of labor and capital.

First, a nonrenewable resource is exhaustible, in the sense that production requires an irreplaceable factor which is itself reduced by production. If prices are such that extraction is profitable, it is possible that the resource stock will be completely exhausted, and production will cease. This does not mean that exhaustion will occur. As Gordon remarks [1967, pp. 275-76]:

> The automatic assumption that the pure theory of exhaustion is applicable to natural resource commodities involves a complete misunderstanding. ...The pure theory only provides a test of whether exhaustion is profitable.

Second, and perhaps more important, a nonrenewable resource is depletable. As defined by Herfindahl [1967, p. 63]:

> The essence of the concept of depletion is not a sudden running out of something but rather a progressive change in the quality of a resource of in the current costs (...) of producing the resource product. Sudden running out is included in this more general concept as the special case in which current cost is constant up to true exhaustion.

The idea of depletion of the resource stock is conceptually the same as the user cost of production from a capital stock. As defined by Keynes [1973, p. 53] user cost is "the sacrifice in value" (of

equipment inherited from the past) "involved in production". In his discussion of the U. S. coal industry, Zimmerman [1975, p. 196A] states:

> Prices will reflect, in addition to the mining and processing costs, the rent earned by the stock--the so-called user cost. User cost arises because the stock of coal is depletable. ...the exploitation of a ton of coal in a thick seam today hastens the time when mining will proceed to thinner and, ceteris paribus, more expensive seams. This movement to more expensive seams is reflected in today's price. This reflection is simply the present value of the future increments in cost.

The implications of the potential exhaustibility of a nonrenewable resource may be given an attractive mathematical form, due originally to Hotelling [1931] and refined by Gordon [1967]. Consider a competitive (price taking) producer of a natural resource, with certain knowledge of the initial resource stock and of the output price which will hold at any time during the production period. Suppose that

$p(t)$ = price of the resource at time t

$q(t)$ = rate of output at time t

$C(t) = C[q(t)]$ = cost of extracting at rate $q(t)$; $C'(q) \geq 0$

R = initial resource stock

r = discount rate

The competitive producer acts to maximize the discounted present value of profits between time zero and time T, subject to the constraint that cumulative production over that period cannot exceed the initial resource stock. The producer acts to

$$(1) \qquad \text{maximize} \int_{t=0}^{T} e^{-rt}\{p(t)q(t) - C[q(t)]\}dt$$

$$(2) \qquad \text{such that} \quad R \geq \int_{t=0}^{T} q(t)dt$$

A necessary condition which characterizes the optimal production path may be derived by forming the following Lagrangian expression

$$(3) \qquad L = \int_{t=0}^{T} e^{-rt}\{p(t)q(t) - C[q(t)]\}dt + \lambda[R - \int_{t=0}^{T} q(t)dt]$$

$$(4) \qquad L = \int_{t=0}^{T} e^{-rt}\{[p(t) - \lambda e^{rt}]q(t) - C[q(t)]\}dt + \lambda R$$

(where λ is the Lagrangian multiplier associated with the exhaustion constraint) and applying to it the Euler equation of the calculus of variations

$$(5) \qquad \frac{\partial I}{\partial q} = \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{q}}\right)$$

for

$$(6) \qquad I[q(t),\dot{q}(t),t] = e^{-rt}\{[p(t) - \lambda e^{rt}]q(t) - C[q(t)]\}$$

Note that $\dot{q}(t)$ does not appear on the right hand side of (6).[1] Substituting from (6) into (5) gives

$$(7) \qquad p(t) = C'[q(t)] + \lambda e^{rt}$$

The Lagrangian multiplier λ satisfies

$$(8) \qquad \lambda[R - \int_{t=0}^{T} q(t)dt] = 0$$

$$(9) \qquad \lambda = e^{-rT}\{p(T) - C'[q(T)]\}$$

(8) is the Kuhn-Tucker condition for constrained optimization. (9) is derived from (7) by substituting $t = T$.

If exhaustion does not occur, then $R > \int_{t=0}^{T} q(t)dt$. (8) then implies that $\lambda = 0$, so the profit maximization condition (7) becomes

$$p(t) = C'[q(t)] \tag{10}$$

which is the conventional "price equals marginal cost" condition of standard production theory. However, if exhaustion occurs, $R = \int_{t=0}^{T} q(t)dt$, and (8) no longer constrains λ to equal zero. If exhaustion occurs, (9) gives the value of $\lambda > 0$. Considering (4), from

$$\frac{\partial L}{\partial R} = \lambda \tag{11}$$

we see that λ is the shadow value (at time zero) of an additional unit of the resource stock. If exhaustion occurs, (9) states that the value at time zero of an additional unit of reserves is simply the excess of price over marginal cost at time T, discounted to time zero.[2]

Substituting $t = 0$ in (7) gives

$$\lambda = p(0) - C'[q(0)] \tag{12}$$

Substituting (12) into (7) to eliminate λ gives

$$p(t) - C'[q(t)] = e^{rt}\{p(0) - C'[q(0)]\} \tag{13}$$

which is the "r per cent growth rule" for natural resources. The gap between price and marginal cost will grow over time at the rate of interest in competitive, profit maximizing equilibrium. As discussed by Hotelling [1931, pp. 140-141]:

> Since it is a matter of indifference to the owner of a mine whether he receives for a unit of his product a price p_0 now or a price $p_0 e^{\gamma t}$ after time t, it is not unreasonable to expect that the price p will be a function of time of the form $p = p_0 e^{\gamma t}$. ...Here p is to be interpreted as the net price received after paying the cost of extraction and placing upon the market...

Hence, if exhaustion occurs, output at any time will be restrained, so that price exceeds marginal cost by a term which grows over time at the rate of interest. Gordon [1967] suggests that this term (λe^{rt}) be identified with user cost. It seems more appropriate to think of λe^{rt} as a rent, since it is a return associated with a fixed natural stock.

This model may easily be adapted to the case of a monopolist who extracts an exhaustible resource, setting p(t) with due regard to the consequences for the quantity demanded $q[p(t)]$. Such a monopolist will act to

$$\text{(14)} \qquad \text{maximize} \int_{t=0}^{T} e^{-rt}\{p(t)q[p(t)] - C\{q[p(t)]\}\}dt$$

subject to the exhaustion constraint (2). The Lagrangian for this constrained maximization problem is

$$\text{(15)} \qquad L = \int_{t=0}^{T} e^{-rt}\{p(t)q[p(t)] - C\{q[p(t)]\}\}dt + \lambda\{R - \int_{t=0}^{T} q[p(t)]dt\}$$

Applying the Euler equation to (15) gives as a necessary condition for profit maximization

$$\text{(16)} \qquad p(t) + \frac{q[p(t)]}{q'[p(t)]} = C'\{q[p(t)]\} + \lambda e^{rt}$$

The monopolist's optimal production path equates marginal revenue (the left hand side of (16)) to marginal production cost plus the exhaustion rent. The interpretation of λ parallels the competitive case.

These results illustrate the implications of exhaustibility for the extraction of a nonrenewable resource. However, they do not say much about depletability. The reason is not hard to find. The assumed cost function, $C[q(t)]$, implies that depletion does not occur. The cost of extracting at rate q is the same whether production has just begun or exhaustion is about to occur. The cost of extraction is independent of previous extraction. The essence of depletion is that cumulative extraction increases the cost of current extraction. To study depletion we need a new cost function

$$C(t) = C[X(t),q(t)] \quad \text{with } C_1(t) \geq 0,\ C_2(t) \geq 0 \tag{17}$$

where

$$X(t) = \int_{\alpha=0}^{t} q(\alpha)d\alpha \qquad X(0) = 0 \tag{18}$$

Suppose that a competitive natural resource industry acts to maximize the discounted present value of profits, with the cost function (17) and subject, as before, to the exhaustion constraint (2). The Lagrangian for this constrained maximization problem is

$$L = \int_{t=0}^{T} e^{-rt}\{p(t)q(t) - C[X(t),q(t)]\}dt + \lambda\left[R - \int_{t=0}^{T} q(t)dt\right] \tag{19}$$

$$L = \int_{t=0}^{T} e^{-rt}\{[p(t) - \lambda e^{rt}]q(t) - C[X(t),q(t)]\}dt + \lambda R \tag{20}$$

Applying the Euler equation

$$\frac{\partial I}{\partial X} = \frac{d}{dt}\left(\frac{\partial I}{\partial q}\right) \tag{21}$$

to

$$I[X(t),q(t),t] = e^{-rt}\{[p(t) - \lambda e^{rt}]q(t) - C[X(t),q(t)]\} \tag{22}$$

gives

$$-e^{-rt}C_1[X(t),q(t)] = \frac{d}{dt}\{e^{-rt}[p(t) - \lambda e^{rt} - C_2[X(t),q(t)]]\} \tag{23}$$

With varying notation, both Hotelling [1931, pp. 152-157] and Gordon [1967, pp. 278-279] consider this case. Gordon carries out the differentiation on the right hand side of (23) and obtains a second order differential equation in $X(t)$ which, as he indicates, is not very instructive. A more illuminating version of the necessary condition (23) is obtained by applying the integral version of the Euler equation

$$\int_{\alpha=t}^{T} I_1(\alpha)d\alpha + I_2(t) = 0 \tag{24}$$

(derived in Appendix I) to (22). This gives the necessary condition for constrained profit maximization when depletion occurs:

$$p(t) = C_2[X(t),q(t)] + \int_{\alpha=t}^{T} e^{-r(\alpha-t)}C_1[X(\alpha),q(\alpha)]d\alpha + \lambda e^{rt} \tag{25}$$

Discounted profit will be maximized when price equals marginal cost (the first right hand side of (25)) plus two other terms. The third right hand side term is the exhaustion rent. As before, it appears only if exhaustion occurs. The second term is the user cost

of current production. It is the discounted integral of the marginal increase in production cost imposed on future production by current production. Production of the marginal unit at time t increases cumulative extraction at each future instant, and this increase in cumulative extraction increases the cost of extraction at each future instant. These increases in cost must be discounted and set against the price received for the marginal unit at time t. Profit will be maximized when the price received for the marginal unit sold equals the marginal cost of production, plus the user cost of production, plus the exhaustion rent.[3]

Substituting t = T in (25) indicates that (9) holds for the depletion model. The shadow value of a unit of reserves is the excess of price over marginal cost at time T, discounted to time zero. Substituting t = 0 in (25) gives

$$(26) \qquad \lambda = p(0) - C_2[0,q(0)] - \int_{\alpha=0}^{T} e^{-r\alpha} C_1[X(\alpha),q(\alpha)]d\alpha$$

Substituting (26) into (25) to eliminate λ gives the modified "r per cent growth rule" for the depletion model:

$$(27) \qquad p(t) - C_2(t) = e^{rt}[p(0) - C_2(0)] - \int_{\alpha=0}^{t} e^{-r(\alpha-t)} C_1(\alpha)d\alpha$$

The price-marginal cost gap rises at less than the rate of interest.[4] The integral which is subtracted from the right hand side of (27) may be interpreted as the saving which results from holding the marginal unit which is produced at time t as part of the resource stock from time zero to time t. Substituting (9) into (26) and

rearranging gives

$$(28) \qquad p(0) - C_2(0) = e^{-rT}[p(T) - C_2(T)] + \int_{\alpha=0}^{T} e^{-r\alpha} C_1(\alpha)\,d\alpha$$

The initial gap between price and marginal cost exceeds λ by a depletion term. The gap rises with time, but less rapidly than the rate of interest, so that at time T the price-marginal cost gap just equals λ.

When depletion occurs the gap between price and marginal cost is greater than when depletion does not occur, at any time, all else equal. Since the producer faces a given market price function, the gap between price and marginal cost is made larger by restricting output, which reduces marginal cost. While some restriction is present if exhaustion occurs, this is important only if one believes exhaustion is a real possibility. But the restriction which is due to depletion occurs whether exhaustion occurs or not.

We may obtain a similar necessary condition for the profit maximizing monopolist of a depletable, nonrenewable resource. The Lagrangian for this problem is

$$(29) \qquad L = \int_{t=0}^{T} e^{-rt}\{p(t)q[p(t)] - C\{\int_{\alpha=0}^{t} q[p(\alpha)]\,d\alpha, q[p(t)]\}\}dt + \lambda[R - \int_{t=0}^{T} q[p(t)]dt]$$

and the corresponding necessary condition for profit maximization is:

$$p(t) + \frac{q[p(t)]}{q'[p(t)]} = C_2\{\int_{\alpha=0}^{t} q[p(\alpha)]d\alpha, q[p(t)]\}$$

$$+ \int_{\alpha=t}^{T} e^{-r(\alpha-t)} C_1\{\int_{\beta=0}^{\alpha} q[p(\beta)d\beta, q[p(a)]\}d\alpha \tag{30}$$

$$+ \lambda e^{rt}$$

The optimal output sequence for the monopolist is that which equates marginal revenue to marginal cost plus user cost plus the exhaustion rent. This is the obvious generalization of (16) to the depletion model.

II. The received theory extended to renewable resources

The thing that makes the analysis of depletion for a nonrenewable resource more interesting and more difficult than the analysis of exhaustion for a nonrenewable resource is the role of the resource stock, as well as the output flow, in the cost function. Essentially, this guarantees that the Euler equation will be a second order differential equation. This has in the past led to the use of phase diagrams to illustrate the nature of the solution. The alternative form of the Euler equation (24) provides another way of looking at the same solution.

This alternative form is of obvious interest for the problem of a renewable resource. The stock of a renewable resource affects its harvesting in two ways. It is reasonable to suppose that the cost of harvesting at a given rate will be less, the larger is the current resource stock. Thus resource stock appears in the cost function, as it implicitly does in the Hotelling analysis of depletion.[6] In addition, the rate of renewal of the resource will depend on the stock of the resource. If the population is either very low or very great, the renewal rate will be small, while for intermediate population levels the renewal rate will be larger. In particular, a depletable resource is simply the polar case of a renewable resource for which the renewal function is identically zero. Let

$R(t)$ = resource stock at time t

$B[R(t)]$ = renewal function

$C[R(t),q(t)]$ = cost of harvesting at rate $q(t)$ from stock $R(t)$

with $C_1(t) \leq 0$ and $C_2(t) \geq 0$, and let other notation be as in the model of a nonrenewable resource. The competitive producer takes price as given and acts to

$$\text{(31)} \qquad \text{maximize} \int_{t=0}^{T} e^{-rt}\{p(t)q(t) - C[R(t),q(t)]\}dt$$

such that

$$\text{(32)} \qquad \dot{R}(t) = B[R(t)] - q(t)$$

$$\text{(33)} \qquad R(0) + \int_{t=0}^{T} B[R(t)]dt \geq \int_{t=0}^{T} q(t)dt$$

(32) is the renewal constraint: the time rate of change of the resource stock equals the renewal rate minus the harvest rate. (33) is the exhaustion constraint: cumulative harvesting cannot exceed the initial resource stock plus cumulative reproduction.

Necessary conditions for a constrained maximum are obtained from

$$\text{(34)} \qquad L = \int_{t=0}^{T} e^{-rt}\{p(t)q(t) - C[R(t),q(t)]\}dt + \lambda\left[R(0) + \int_{t=0}^{T} B[R(t)]dt - \int_{t=0}^{T} q(t)dt\right]$$

Substituting $q(t) = B[R(t)] - \dot{R}(t)$ (see (32)) in (34) to eliminate $q(t)$ yields an expression which is a function of $R(t)$, $\dot{R}(t)$, and t:

$$\text{(35)} \qquad L = \int_{t=0}^{T} e^{-rt}\{pB(R) - [p - \lambda e^{rt}]\dot{R} - C[R,B(R) - \dot{R}]\}dt + \lambda R(0)$$

(suppressing time variables for notational convenience). Applying (24) to (35) gives the necessary condition:

$$\text{(36)} \qquad p = C_2 - \int_{\alpha=t}^{T} e^{-r(\alpha-t)}C_1 d\alpha + \lambda e^{rt} + \int_{\alpha=t}^{T} e^{-r(\alpha-t)}(p - C_2)B'(R)d\alpha$$

The first right hand side term of (36) is the marginal cost of production. The second term is the user cost of production: harvesting the marginal unit at time t reduces the stock at each future instant, and the stock reduction increases the cost of harvesting at each future instant. The discounted sum of these increases in cost is an expense associated with production of the marginal unit at time t, and must be set against the revenue received for the sale of that unit. The third term is the rent associated with the exhaustion constraint. As usual, $\lambda > 0$ only if exhaustion occurs. The fourth right hand side term of (36) reflects the distinctive feature of a renewable resource. Harvesting the marginal unit at time t reduces the resource stock at each future instant. The renewal rate at any time is a function of the resource stock at that time. Hence production of the marginal unit now entails a slight change in the renewal rate at each future instant. As noted above, the sign of this change, $B'[R(t)]$, is indeterminate. If the population is very large, a reduction in population may increase the renewal rate. For lower population levels, a reduction in population will reduce the renewal rate. In any event, harvesting now changes the renewal rate at future times. What is the proper value at which these changes should be evaluated? The value given up if the renewal rate is slightly reduced is the difference between the price at which the foregone units would be sold and the marginal cost at which they would be extraced. The same standard measures the value received if the renewal rate is increased slightly. Harvesting the marginal unit now slightly alters the resource stock at each future

instant; the discounted value of these changes is an expense or receipt associated with the harvesting of the marginal unit, and must be compared with the income earned by the sale of the marginal unit.

Discounted total profit will be maximized when the price received for sale of the marginal unit at any time equals the marginal cost of producing that unit, plus the user cost of production, plus the exhaustion rent, plus the discounted value of population changes which result from current production. The influence of user cost and the exhaustion rent is to restrict harvesting. The population effect is ambiguous. If the population is balanced in respect to the environment, harvesting now will reduce future resource stocks, hence lower future renewal rates. This expense will induce a restriction of current harvesting. But if the population is crowding the environment, a large harvest may sufficiently thin out the population to increase the renewal rate; in this case the population effect would tend to increase current harvesting.

As expected, the necessary condition for profit maximization on the part of the monopolist of a renewable resource is that the marginal revenue resulting from sale of the marginal unit at time t equal the four terms on the right hand side of (36).

III. Profit maximization and cost minimization for nonrenewable resources

The Hotelling analysis makes extreme assumptions in order to clearly show the distinctive economic characteristics of a nonrenewable resource. The model assumes certain knowledge of future resource prices and of the extent of the resource stock. The possibility of discovery of additional reserves is not allowed for. The role of factors of production other than reserves is not specifically considered. It is not surprising that empirical attempts to verify the implications of this theory have met with little success. The predicted discrepancy between price and marginal cost, and the rise of this discrepancy over time at the rate of interest, have not been observed.[7]

An analysis which incorporates some of the features specifically excluded by Hotelling may yield results more susceptible to empirical verification. I will present an analysis of the extraction of a nonrenewable resource, explicitly considering three factors of production: capital, labor, and reserves. Initially, certain knowledge of output and factor prices, and of the extent of the reserve base, will be assumed. Uncertainty will be introduced at a later stage. Let

L_t = labor input in period t

K_t = capital input in period t

R_t = reserve base at the start of period t

R_0 = initial reserve base

w_t = wage rate in period t

p_t^k = rental rate of capital in period t

r = discount rate

$\delta X(L_t, K_t, R_t)$ = production function for the nonrenewable resource

Output δX_t is a function of labor L_t and capital K_t applied to the reserve base R_t. Labor and capital are purchased in competitive factor markets; reserves decline over time with extraction. In particular

$$R_t = R_{t-1} - \delta X_{t-1} \tag{37}$$

The reserve base bequeathed to the next period equals the reserve base inherited from the previous period, less the depletion resulting from production. Extraction of one unit of output depletes reserves by exactly one unit.[8] Although the reserve base today is given, future reserves are determined by current output. Current input of labor and capital will be altered according to the scarcity placed on future reserves. Solving (37) recursively to time zero gives

$$R_t = R_0 - \sum_{\alpha=0}^{t-1} \delta X(L_\alpha, K_\alpha, R_\alpha) \tag{38}$$

The marginal product of each factor is positive, but depletion reduces the reserve base over time. More of either labor or capital or both will have to be used in the future to extract a given amount, to substitute for the depleted reserve base. It is the depletion of the reserve base which ties together production, and input choices, over time.

In addition to the depletion constraint (38), the industry is subject to the exhaustion constraint:

$$R_0 \geq \sum_{t=0}^{\infty} \delta X(L_t, K_t, R_t) \tag{39}$$

which states that cumulative extraction cannot exceed the initial reserve base.

The industry will act to maximize

$$\sum_{t=0}^{\infty} (1 + r)^{-t}[p_t \delta X(L_t, K_t, R_t) - w_t L_t - p_t^k K_t] \tag{40}$$

subject to the depletion constraint (38) and the exhaustion constraint (39). The industry can vary the input of labor, capital, and reserves over time, but the variation is constrained by (38). Necessary conditions for constrained profit maximization may be obtained from the Lagrangian

$$\begin{aligned} L = \sum_{t=0}^{\infty} (1 + r)^{-t}\{&p_t \delta X(L_t, K_t, R_t) - w_t L_t - p_t^k K_t \\ &+ \mu_{2t}[R_0 - \sum_{\alpha=0}^{t-1} \delta X(L_\alpha, K_\alpha, R_\alpha) - R_t]\} \\ &+ \mu_3[R_0 - \sum_{t=0}^{\infty} \delta X(L_t, K_t, R_t)] \end{aligned} \tag{41}$$

which may be written

$$\begin{aligned} L = \sum_{t=0}^{\infty}(1 + r)^{-t}\{&[p_t - \sum_{\alpha=t+1}^{\infty}(1 + r)^{t-\alpha}\mu_{2\alpha} - (1 + r)^t \mu_3]\delta X(L_t, K_t, R_t) \\ &- w_t L_t - p_t^k K_t - \mu_{2t} R_t\} \\ &+ R_0[\mu_3 + \sum_{t=0}^{\infty}(1 + r)^{-t}\mu_{2t}] \end{aligned} \tag{42}$$

The transition from (41) to (42) involves reversing the order of

summation in the depletion constraint.

μ_{2t}, the Lagrangian multiplier associated with the depletion constraint, is the shadow price of a unit of reserves in period t. μ_3 is the shadow rent associated with the exhaustion constraint.

First order conditions for profit maximization are obtained by differentiating (42) with respect to L_t, K_t, and R_t. The resulting necessary conditions are

$$(43) \quad (1 + r)^{-t} w_t = \left[(1 + r)^{-t} p_t - \sum_{\alpha=t+1}^{\infty} (1 + r)^{-\alpha} \mu_{2\alpha} - \mu_3\right] \delta X_1(L_t, K_t, R_t)$$

$$(44) \quad (1 + r)^{-t} p_t^k = \left[(1 + r)^{-t} p_t - \sum_{\alpha=t+1}^{\infty} (1 + r)^{-\alpha} \mu_{2\alpha} - \mu_3\right] \delta X_2(L_t, K_t, R_t)$$

$$(45) \quad (1 + r)^{-t} \mu_{2t} = \left[(1 + r)^{-t} p_t - \sum_{\alpha=t+1}^{\infty} (1 + r)^{-\alpha} \mu_{2\alpha} - \mu_3\right] \delta X_3(L_t, K_t, R_t)$$

In addition, a solution must obey the Kuhn-Tucker conditions

$$(46) \quad \mu_{2t}\left[R_0 - \sum_{\alpha=0}^{t-1} \delta X(L_\alpha, K_\alpha, R_\alpha) - R_t\right] = 0$$

$$(47) \quad \mu_3\left[R_0 - \sum_{t=0}^{\infty} \delta X(L_t, K_t, R_t)\right] = 0$$

(43) - (46) hold for all t. The left hand side of (43) is the discounted wage rate in period t. The right hand side of (43) is a product of price/shadow price terms and the marginal product of labor. The price factor on the right hand side of (43) has three parts. The first, $(1 + r)^{-t} p_t$, is the discounted value of the price of output in period t. Subtracted from that is the discounted sum of the shadow values of a unit of reserves in each future period. Also subtracted

is the exhaustion rent per unit of reserves.

We may interpret (43) as follows. Employment of the marginal unit of labor involves (discounted) expense $(1 + r)^{-t}w_t$. The marginal product of that unit is sold for a discounted price of $(1 + r)^{-t}p_t$. But production of the marginal unit reduces the reserve base in each future period. This is a real cost to the producer, since the smaller stock of reserves will compel the employment of more cooperating factors to produce a given output. The marginal reduction in reserves in period α is valued at the discounted shadow value of a unit of reserves, $(1 + r)^{-\alpha}\mu_{2\alpha}$ ($\alpha = t+1, t+2, \ldots$) and subtracted from the value received in period t for the sale of the marginal unit. Finally, if exhaustion occurs, production of the marginal unit in period t contributes to that exhaustion, and the exhaustion rent is subtracted from the value received for the sale of the marginal product of labor. (43) states that employment of labor will be optimal when the cost of labor equals the net value of the marginal product of labor; the net value of the marginal product is the gross value of the marginal product, less the shadow value of resulting depletion and exhaustion.

(44) is subject to similar interpretation: employment of capital will be optimal when the cost of capital equals the net value of the marginal product of capital.

Dividing (43) by (44) gives

$$\frac{w_t}{p_t^k} = \frac{\delta X_1(L_t, K_t, R_t)}{\delta X_2(L_t, K_t, R_t)} \tag{48}$$

In equilibrium, the ratio of the marginal products of labor and capital will equal the ratio of the prices of labor and capital. The effect of depletability and exhaustibility is to restrict output within each period, but the restricted output is produced with that combination of labor and capital which minimizes the current cost of production.

Equation (45) defines the shadow value of a unit of reserves in period t as the net value of the marginal product of a unit of reserves in period t.[9]

The first order conditions for profit maximization reflect the depletable and exhaustible nature of a nonrenewable resource, much like the key relation of the Hotelling model, (25). The connection between the two models will be clear if we consider the dual problem for a nonrenewable resource. What relations will hold if the cost of producing a specified sequence of outputs is to be minimized?

With notation as before, suppose that a competitive industry extracting a nonrenewable resource acts to

$$\text{(49)} \qquad \text{minimize} \quad \sum_{t=0}^{\infty} (1+r)^{-t} \left[w_t L_t + p_t^K K_t \right]$$

such that

$$\text{(50)} \qquad \delta X(L_t, K_t, R_t) = \overline{\delta X}_t$$

$$\text{(51)} \qquad R_t = R_0 - \sum_{\alpha=0}^{t-1} \delta X(L_\alpha, K_\alpha, R_\alpha)$$

$$\text{(52)} \qquad R_0 \geq \sum_{t=0}^{\infty} \delta X(L_t, K_t, R_t)$$

(50) specifies the output sequence with respect to which cost is to be minimized. (51) is the depletion constraint, and (52) is the exhaustion constraint.

Necessary conditions for constrained cost minimization are obtained by minimizing the following Lagrangian with respect to L_t, K_t, and R_t:

$$L = \sum_{t=0}^{\infty}(1+r)^{-t}\{w_tL_t + p_t^kK_t + \mu_{1t}[\overline{\delta X}_t - \delta X(L_t,K_t,R_t)] + \mu_{2t}[R_t + \sum_{\alpha=0}^{t-1}\delta X(L_\alpha,K_\alpha,R_\alpha) - R_0]\} + \mu_3[\sum_{t=0}^{\infty}\delta X(L_t,K_t,R_t) - R_0] \tag{53}$$

$$L = \sum_{t=0}^{\infty}(1+r)^{-t}\{w_tL_t + p_t^kK_t + \mu_{2t}R_t + \mu_{1t}\overline{\delta X}_t - [\mu_{1t} - \sum_{\alpha=t+1}^{\infty}(1+r)^{t-\alpha}\mu_{2\alpha} - (1+r)^t\mu_3]\delta X(L_t,K_t,R_t)\} - [\mu_3 + \sum_{t=0}^{\infty}(1+r)^{-t}\mu_{2t}]R_0 \tag{54}$$

μ_{1t} is the Lagrangian multiplier associated with the output constraint. $(1+r)^{-t}\mu_{1t}$ is the discounted shadow value of a unit of output in period t. From (54)

$$\frac{\partial L}{\partial(\overline{\delta X}_t)} = (1+r)^{-t}\mu_{1t} \tag{55}$$

$(1+r)^{-t}\mu_{1t}$ is the amount by which discounted minimum cost

would rise if the required output in period t were increased one unit.

$(1 + r)^{-t}\mu_{2t}$ is the discounted shadow price of a unit of reserves in period t, and μ_3 is the rent associated with the exhaustion constraint. The first order conditions for cost minimization are, from (54)

$$(56) \quad (1 + r)^{-t}w_t = [(1 + r)^{-t}\mu_{1t} - \sum_{\alpha=t+1}^{\infty}(1 + r)^{-\alpha}\mu_{2\alpha} - \mu_3]\delta X_1(L_t,K_t,R_t)$$

$$(57) \quad (1 + r)^{-t}p_t^k = [(1 + r)^{-t}\mu_{1t} - \sum_{\alpha=t+1}^{\infty}(1 + r)^{-\alpha}\mu_{2\alpha} - \mu_3]\delta X_2(L_t,K_t,R_t)$$

$$(58) \quad (1 + r)^{-t}\mu_{2t} = [(1 + r)^{-t}\mu_{1t} - \sum_{\alpha=t+1}^{\infty}(1 + r)^{-\alpha}\mu_{2\alpha} - \mu_3]\delta X_3(L_t,K_t,R_t)$$

In addition, we have the Kuhn-Tucker conditions:

$$(59) \quad \mu_{1t}[\overline{\delta X}_t - \delta X(L_t,K_t,R_t)] = 0$$

$$(60) \quad \mu_{2t}[R_t + \sum_{\alpha=0}^{t-1}\delta X(L_\alpha,K_\alpha,R_\alpha) - R_0] = 0$$

$$(61) \quad \mu_3[\sum_{t=0}^{\infty}\delta X(L_t,K_t,R_t) - R_0] = 0$$

Comparison with the necessary conditions for profit maximization (43) - (47) is instructive. (56) states that employment of labor will minimize cost when the wage paid for the last unit of labor equals the net shadow value of the marginal product of that labor. The net shadow value of the marginal product of labor equals the gross shadow value of the marginal product of labor $[(1 + r)^{-t}\mu_{1t}\delta X_{1t}]$, less the shadow value of the depletion and exhaustion engendered by production

of the marginal product of labor. (57) shows that employment of capital will be optimal when the cost of the last unit of capital hired just equals the net shadow value of the marginal product of capital. (58) defines the discounted shadow value of a unit of reserves as the net shadow value of the marginal product of a unit of reserves. The necessary conditions of the cost minimization problem are just those of the profit maximization problem, with the market price sequence $\{(1 + r)^{-t} p_t\}$ replaced by the shadow price sequence $\{(1 + r)^{-t} \mu_{1t}\}$.

Dividing (56) by (57)

$$\frac{w_t}{p_t^k} = \frac{\delta X_1(L_t, K_t, R_t)}{\delta X_2(L_t, K_t, R_t)} \tag{62}$$

we see that the output required in period t will be produced with a combination of labor and capital which will minimize the cost of extraction within that period. Since the cost minimization problem begins by requiring that a certain output be produced in each period, it cannot be said that extraction is restrained because of the exhaustible and depletable nature of the nonrenewable resource. Instead, when depletion and (potential) exhaustion arise, the shadow value of a unit of output is greater than when depletion and (potential) exhaustion do not arise, all else equal. If depletion did not occur, but production involved a fixed amount R_t of reserves, we could write the optimality condition for employment of labor as

$$(1 + r)^{-t} w_t = (1 + r)^{-t} \mu_{1t}^* \, \delta X_1(L_t, K_t, R_t) \tag{63}$$

(the asterisk indicates that the shadow price refers to a situation in which depletion does not occur). Comparing (56) and (63) we have

$$(64)\qquad (1+r)^{-t}\mu_{1t} = \frac{(1+r)^{-t}w_t}{\delta X_1(L_t,K_t,R_t)} + \sum_{\alpha=t+1}^{\infty}(1+r)^{-\alpha}\mu_{2\alpha} + \mu_3 > \frac{(1+r)^{-t}w_t}{\delta X_1(L_t,K_t,R_t)} = (1+r)^{-t}\mu_{1t}^*$$

Depletion and (potential) exhaustibility manifest themselves in both the primal and the dual problem, but in different ways. In the primal problem, the effect of exhaustibility and depletability is to restrict output. In the dual problem, the effect is to increase the shadow value placed on each unit of output.

It will be assumed below that the necessary conditions (56) - (61) can be solved for the cost minimizing input sequences $\{L_t\}$ and $\{K_t\}$. These may be substituted into

$$(65)\qquad C = \sum_{t=0}^{\infty}(1+r)^{-t}\{w_tL_t + p_t^kK_t\}$$

to obtain the discounted minimum cost function

$$(66)\qquad C = C[\{\delta X_t\}, \{(1+r)^{-t}w_t\}, \{(1+r)^{-t}p_t^k\}, \{(1+r)^{-t}\mu_{2t}\}]$$

which gives the discounted minimum cost of producing a specified output sequence as a function of that sequence and of (discounted) factor prices in each period.[10]

The cost function (66) may be related to the earlier, Hotelling-type model. Given the cost function (66), we could investigate the

behavior of a profit maximizing producer by supposing that he acts to maximize

$$(67) \qquad \sum_{t=0}^{\infty} (1 + r)^{-t} p_t \delta X_t - C[.....]$$

(see (66) for the arguments of the cost function). The decision variables in (67) are the outputs in successive periods; factor inputs are buried in the cost function. If we assume the following special form for the cost function (66)

$$(68) \qquad C[.....] = \sum_{t=0}^{\infty} c[X_{t-1}, \delta X_t, \{(1+r)^{-t} w_t\}, \{(1+r)^{-t} p_t^k\}, \{(1+r)^{-t} \mu_{2t}\}]$$

and substitute in (67) we obtain

$$(69) \qquad \sum_{t=0}^{\infty} (1 + r)^{-t} \{p_t \delta X_t - c[X_{t-1}, \delta X_t, ...]\}$$

which is the discrete version of the discounted profit used in the Hotelling model which studies depletion and exhaustion (see (19)). The difference is that the cost function (66) incorporates the depletion and exhaustion constraints.

The cost function (66) will be used below to derive an empirical test of the influence of depletability and exhaustibility on the extraction of mineral resources.

IV. Profit maximization and cost minimization for renewable resources

The role of labor and capital may be investigated for a renewable resource. Suppose that the harvest in period t is

$$\delta X_t = \delta X(L_t, K_t, R_t) \tag{70}$$

where R_t is the resource stock in period t, given by

$$R_t = R_0 + \sum_{\alpha=0}^{t-1} B(R_\alpha) - \sum_{\alpha=0}^{t-1} \delta X_\alpha \tag{71}$$

$B(R_t)$ is the renewal function, which gives the gross increase in population as a function of the level of population. If the population is either very small or very large relative to the environment, the renewal rate may well be negative. If a species crowds the environment, population may decrease as some members of the species are unable to subsist. If the species is sparsely distributed, reproduction may not compensate for natural mortality. But if the population is in proportion to the environment, the renewal rate will be positive. In this case, harvesting will serve a positive role, preventing the natural increase from crowding the environment.

The industry faces an exhaustion constraint

$$R_0 + \sum_{t=0}^{\infty} B(R_t) \geq \sum_{t=0}^{\infty} \delta X_t \tag{72}$$

Cumulative harvesting cannot exceed the initial population plus cumulative additions to population.

Complete knowledge of output and factor price sequences is assumed. Suppose the industry acts to maximize

$$(73) \qquad \sum_{t=0}^{\infty} (1 + r)^{-t}\{p_t \delta X(L_t, K_t, R_t) - w_t L_t - p_t^k K_t\}$$

subject to the depletion constraint (71) and the exhaustion constraint (72). Necessary conditions for constrained maximization are obtained by maximizing the following Lagrangian with respect to L_t, K_t, and R_t:

$$\begin{aligned} L = \sum_{t=0}^{\infty} (1 + r)^{-t}\{&p_t \delta X(L_t, K_t, R_t) - w_t L_t - p_t^k K_t \\ &+ \mu_{2t}\Big[R_0 + \sum_{\alpha=0}^{t-1} B(R_\alpha) - \sum_{\alpha=0}^{t-1} \delta X(L_\alpha, K_\alpha, R_\alpha) - R_t\Big]\} \\ &+ \mu_3\Big[R_0 + \sum_{t=0}^{\infty} B(R_t) - \sum_{t=0}^{\infty} \delta X(L_t, K_t, R_t)\Big] \end{aligned} \tag{74}$$

$$\begin{aligned} L = \sum_{t=0}^{\infty}(1 + r)^{-t}\{&\Big[p_t - \sum_{\alpha=t+1}^{\infty}(1 + r)^{t-\alpha}\mu_{2\alpha} - (1 + r)^t \mu_3\Big]\delta X(L_t, K_t, R_t) \\ &- w_t L_t - p_t^k K_t - \mu_{2t} R_t + \Big[\sum_{\alpha=t+1}^{\infty}(1 + r)^{t-\alpha}\mu_{2\alpha} + (1 + r)^t \mu_3\Big]B(R_t)\} \\ &+ \Big[\sum_{t=0}^{\infty}(1 + r)^{t-\alpha}\mu_{2t} + \mu_3\Big]R_0 \end{aligned} \tag{75}$$

Differentiating (75), the necessary conditions are

$$(76) \quad (1 + r)^{-t} w_t = \Big[(1 + r)^{-t} p_t - \sum_{\alpha=t+1}^{\infty}(1 + r)^{-\alpha}\mu_{2\alpha} - \mu_3\Big]\delta X_1(L_t, K_t, R_t)$$

$$(77) \quad (1 + r)^{-t} p_t^k = \Big[(1 + r)^{-t} p_t - \sum_{\alpha=t+1}^{\infty}(1 + r)^{-\alpha}\mu_{2\alpha} - \mu_3\Big]\delta X_2(L_t, K_t, R_t)$$

$$\begin{aligned} (1 + r)^{-t}\mu_{2t} = &\Big[(1 + r)^{-t} p_t - \sum_{\alpha=t+1}^{\infty}(1 + r)^{-\alpha}\mu_{2\alpha} - \mu_3\Big]\delta X_3(L_t, K_t, R_t) \\ &+ \Big[\sum_{\alpha=t+1}^{\infty}(1 + r)^{-\alpha}\mu_{2\alpha} + \mu_3\Big]B'(R_t) \end{aligned} \tag{78}$$

plus the Kuhn-Tucker conditions

$$(79) \qquad \mu_{2t}\left[R_0 + \sum_{\alpha=0}^{t-1} B(R_\alpha) - \sum_{\alpha=0}^{t-1} \delta X(L_\alpha, K_\alpha, R_\alpha) - R_t\right] = 0$$

$$(80) \qquad \mu_3\left[R_0 + \sum_{t=0}^{\infty} B(R_t) - \sum_{t=0}^{\infty} \delta X(L_t, K_t, R_t)\right] = 0$$

μ_{2t} is the shadow price of a unit of resource stock in period t. μ_3 is the rent associated with the exhaustion constraint. (80) implies that μ_3 will equal zero unless the population is completely harvested.[11] This is unlikely for a renewable resource[12], so we expect $\mu_3 = 0$.

(76) states that employment of labor will be optimal when the cost of the last unit of labor equals the net value of the marginal product of that labor. The net price of output is the gross price of output, less the discounted shadow price of reserves in each future period, less the exhaustion rent. Through the depletion constraint (71), harvesting the marginal product of labor in period t reduces the population in each future period; the value of this reduction must be set against the gross price received in period t for the marginal product. Similarly, if exhaustion occurs, the exhaustion rent must be set against the gross price of output.

(77) has a similar interpretation: employment of capital is optimal when the cost of the marginal unit of capital equals the net value of the marginal product of capital.

(78) defines the shadow price of a unit of reserves as the sum of two terms. The first term is the net value of the marginal product of a unit of reserves. But the level of reserves has another effect when

the resource is renewable: maintaining an extra unit of reserves in period t alters the renewal rate in period t. A slightly larger increase in population in period t increases the level of reserves in each future period; these increases are valued at the appropriate shadow price of reserves, and added to the net value of the marginal product of a unit of reserves as part of the total shadow value of a unit of reserves in period t. Similarly, if exhaustion occurs, increasing the renewal rate in period t postpones that exhaustion, and the exhaustion rent contributes to the value of a unit of reserves in period t.

Note, however, that the second right hand side term in (78) is of ambiguous sign. If the population is large relative to the environment, maintaining an extra unit of reserves will reduce the increase in population ($B'(R_t) < 0$), and the renewal term will reduce the current shadow value of a unit of reserves. If this term predominates, the current shadow value of a unit of reserves will be negative; the renewal term will induce an increase in current harvesting. The harvest will exceed the level which maximizes profit within period t, because thinning out the population in period t increases profitability in future periods. The discounted value of a unit of reserves in period t is the discounted net value of the marginal product of a unit of reserves in period t, plus the discounted shadow value of the population changes induced by maintaining an extra unit of reserves.

From (76) and (77), in profit maximizing equilibrium we have

$$(81) \qquad \frac{w_t}{p_t^k} = \frac{\delta X_1(L_t,K_t,R_t)}{\delta X_2(L_t,K_t,R_t)}$$

The scarcity value placed on the resource stock (μ_{2t}) will influence the optimal harvest in any period, but once the most profitable harvest level is determined, that harvest will be produced with a combination of labor and capital which minimizes the current cost of production.

We may consider the dual problem for a renewable resource. Suppose a competitive industry acts to minimize

$$(82) \qquad \sum_{t=0}^{\infty} (1 + r)^{-t}[w_t L_t + p_t^k K_t]$$

such that

$$(83) \qquad \delta X(L_t,K_t,R_t) = \overline{\delta X}_t$$

and subject to the depletion constraint (71) and the exhaustion constraint (72). The Lagrangian for constrained cost minimization is

$$\begin{aligned} L = \sum_{t=0}^{\infty} (1 + r)^{-t}\{ & w_t L_t + p_t^k K_t + \mu_{1t}[\overline{\delta X}_t - \delta X(L_t,K_t,R_t)] \\ & + \mu_{2t}[R_t - R_0 - \sum_{\alpha=0}^{t-1} B(R_\alpha) + \sum_{\alpha=0}^{t-1} \delta X(L_\alpha,K_\alpha,R_\alpha)]\} \\ & + \mu_3\{\sum_{t=0}^{\infty} \delta X(L_t,K_t,R_t) - R_0 - \sum_{t=0}^{\infty} B(R_t)\} \end{aligned} \qquad (84)$$

$$
\begin{aligned}
L = \sum_{t=0}^{\infty} (1+r)^{-t}\{ & w_t L_t + p_t^k K_t + \mu_{2t} R_t + \mu_{1t}\overline{\delta X}_t \\
& - \Big[\mu_{1t} - \sum_{\alpha=t+1}^{\infty}(1+r)^{t-\alpha}\mu_{2\alpha} - (1+r)^t \mu_3\Big]\delta X(L_t,K_t,R_t) \\
& - \Big[\sum_{\alpha=t+1}^{\infty}(1+r)^{t-\alpha}\mu_{2\alpha} + (1+r)^t\mu_3\Big]B(R_t)\} \\
& - \Big[\sum_{t=0}^{\infty}(1+r)^{-t}\mu_{2t} + \mu_3\Big]R_0
\end{aligned} \tag{85}
$$

From (85), necessary conditions for cost minimization are

$$(1+r)^{-t} w_t = \Big[(1+r)^{-t}\mu_{1t} - \sum_{\alpha=t+1}^{\infty}(1+r)^{-\alpha}\mu_{2\alpha} - \mu_3\Big]\delta X_1(L_t,K_t,R_t) \tag{86}$$

$$(1+r)^{-t} p_t^k = \Big[(1+r)^{-t}\mu_{1t} - \sum_{\alpha=t+1}^{\infty}(1+r)^{-\alpha}\mu_{2\alpha} - \mu_3\Big]\delta X_2(L_t,K_t,R_t) \tag{87}$$

$$
\begin{aligned}
(1+r)^{-t}\mu_{2t} = & \Big[(1+r)^{-t}\mu_{1t} - \sum_{\alpha=t+1}^{\infty}(1+r)^{-\alpha}\mu_{2\alpha} - \mu_3\Big]\delta X_3(L_t,K_t,R_t) \\
& - \Big[\sum_{\alpha=t+1}^{\infty}(1+r)^{-\alpha}\mu_{2\alpha} + \mu_3\Big]B'(R_t)
\end{aligned} \tag{88}
$$

In addition, an optimal solution will obey the Kuhn-Tucker conditions

$$\mu_{1t}\Big[\overline{\delta X}_t - \delta X(L_t,K_t,R_t)\Big] = 0 \tag{89}$$

$$\mu_{2t}\Big[R_t - R_0 - \sum_{\alpha=0}^{t-1} B(R_\alpha) + \sum_{\alpha=0}^{t-1}\delta X(L_\alpha,K_\alpha,R_\alpha)\Big] = 0 \tag{90}$$

$$\mu_3\Big[\sum_{t=0}^{\infty}\delta X(L_t,K_t,R_t) - R_0 - \sum_{t=0}^{\infty} B(R_t)\Big] = 0 \tag{91}$$

Employment of labor will minimize cost when the discounted cost of the last unit of labor hired equals the discounted net shadow value of the marginal product of labor. The discounted net shadow value of output equals the discounted gross shadow price of output less the discounted shadow value of a unit of resource stock in each future period, less the exhaustion rent. Employment of capital will minimize cost when the discounted cost of the last unit of capital hired equals the discounted net shadow value of the marginal product of capital.

(88) defines the shadow value of a unit of resource stock in period t as the net shadow value of the marginal product of a unit of resource stock in period t, plus the shadow value of the population changes induced by maintaining an additional unit of resource stock.

When the primal problem is considered (profit maximization), the effect of positive shadow prices for reserves is to restrict harvesting. The dual problem begins by specifying the output sequence; in this case, the effect of positive shadow prices for reserves is to increase the shadow price of output in any period. However, if the population looms large in the environment, the shadow price of the resource stock may be negative. This would tend to reduce the shadow price of output, compared with an otherwise equal situation in which exhaustion and depletion did not arise.[13]

I assume that the necessary conditions (86) - (91) can be solved for the cost minimizing input sequences $\{L_t\}$ and $\{K_t\}$. These optimal input sequences lead to a discounted minimum cost function

$$(92) \qquad C = C[\{\delta X_t\}, \{(1+r)^{-t} w_t\}, \{(1+r)^{-t} p_t^K\}, \{(1+r)^{-t} \mu_{2t}\}]$$

which gives the discounted minimum cost of producing a sequence of harvests as a function of the quantities to be harvested and of factor prices in all periods.

V. Optimal behavior of a nonrenewable resource price leader

Suppose that an industry which extracts a nonrenewable resource is dominated by one producer, with reserves of such magnitude and quality that he can at time zero set the sequence of prices which will rule in the market in successive periods. The competitive fringe will take this price sequence as given, and produce a sequence of outputs which will maximize the present discounted value of fringe profit. The price leader satisifies residual demand. The question is, what price sequence should the price leader choose to maximize the present discounted value of his own profit?[14]

For simplicity, consolidate the competitive fringe into one producer. Suppose the fringe acts to maximize

$$\sum_{t=0}^{T} (1 + r)^{-t}\{p_t \delta X_t^c - C^c[X_{t-1}^c, \delta X_t^c]\} \tag{93}$$

such that

$$R_0^c > \sum_{t=0}^{T} \delta X_t \tag{94}$$

where

p_t = market price in period t

δX_t^c = fringe output in period t

$X_t^c = \sum_{\alpha=0}^{t} \delta X_\alpha^c$ = cumulative fringe output at the end of period t

$C^c[X_{t-1}^c, \delta X_t^c]$ = cost of producing output δX_t^c when cumulative output is X_{t-1}^c

$C_1^c(t) > 0$ cumulative extraction increases current extraction cost

$C_2^c(t) > 0$ increasing cost of current extraction

Necessary conditions for constrained profit maximization by the fringe are derived from the following Lagrangian:

$$(95) \quad L = \sum_{t=0}^{T} (1 + r)^{-t}\{p_t \delta X_t^c - C^c[X_{t-1}^c, \delta X_t^c]\} + \mu^c[R_0^c - \sum_{t=0}^{T} \delta X_t^c]$$

$$(96) \quad L = \sum_{t=0}^{T} (1 + r)^{-t}\{[p_t - (1 + r)^t \mu^c]\delta X_t^c - C^c[X_{t-1}^c, \delta X_t^c]\} + \mu^c R_0^c$$

Using the discrete version of the Euler equation of the calculus of variations, it is seen that the optimal output sequence of the competitive fringe will satisfy

$$(97) \quad p_t = C_2^c(t) + \sum_{\alpha=t+1}^{T} (1 + r)^{t-\alpha} C_1^c(\alpha) + (1 + r)^t \mu^c$$

as well as the Kuhn-Tucker condition

$$(98) \quad \mu^c[R_0^c - \sum_{t=0}^{T} \delta X_t^c] = 0$$

(97) is the discrete version of the optimality condition for a competitive producer of a nonrenewable resource, derived in I. Fringe output will be optimal when the price received for the last unit currently produced equals the marginal cost of extracting that unit, plus the discounted value of the increases in cost imposed on future production by production of that last unit, plus the exhaustion rent. Substituting t = T in (97) gives

$$(99) \quad \mu^c = (1 + r)^{-T}[p_T - C_2(T)]$$

and (98) implies that μ^c will differ from zero only if exhaustion actually occurs.

To solve the price leader's problem, it will be necessary to have an expression for the reaction of the fringe to small changes in the price set by the price leader. To analyze this reaction, in (97) and (98) replace p_t by $p_t + \gamma\nu_t$, where ν_t is a change in the period t price set by the price leader and γ is a scalar.

Differentiate the resulting equations with respect to γ at $\gamma = 0$. The result will be a system of T + 2 equations in the T + 2 unknowns $\frac{d}{d\gamma}(X_t^c)$, t = 0, 1, ..., T, and $\frac{d\mu}{d\gamma}^c$ (all evaluated at $\gamma = 0$), which I assume can be solved for

$$\frac{d}{d\gamma}(X_t^c) = \sum_{\alpha=0}^{T} \frac{\partial}{\partial p_t}(X_t^c)\nu_t \tag{100}$$

The partial derivatives which appear in the right hand side of (100) measure the response of cumulative fringe output in any period to a small change in price in any other period. Nothing general can be said about these partial derivatives. Each will depend on the second derivatives of the cost function in all periods.

This specifies the behavior of the competive fringe. With this behavior in mind, the price leader will seek to maximize

$$\sum_{t=0}^{T}(1 + r)^{-t}\{p_t[q(p_t) - \delta X_t^c] - C^m[\sum_{\alpha=0}^{t-1} q(p_\alpha) - X_{t-1}^c, q(p_t) - \delta X_t^c]\} \tag{101}$$

subject to the exhaustion constraint

$$R_0^m \geq \sum_{t=0}^{T} q(p_t) - \sum_{t=0}^{T} \delta X_t^c \tag{102}$$

where

$q(p_t)$ = total market demand as a function of price

$q(p_t) - \delta X_t^c$ = residual demand facing price leader

$C^m[.....]$ = price leader's cost function

The first argument of the cost function is cumulative extraction. The second is current extraction. It is assumed that the partial derivative of the cost function with respect to each argument is positive.

The Lagrangian for the constrained maximization problem of the price leader is

$$L = \sum_{t=0}^{T}(1+r)^{-t}\{p_t[q(p_t) - \delta X_t^c] - C^m[\sum_{\alpha=0}^{t-1} q(p_\alpha) - X_{t-1}^c, q(p_t) - \delta X_t^c]\} + \mu^m\{R_0^m - \sum_{t=0}^{T} q(p_t) + \sum_{t=0}^{T} \delta X_t^c\} \tag{103}$$

Using the method of the calculus of variations, and using (100), the following necessary condition for the price leader is obtained:

$$p_t + \frac{q(p_t) - \delta X_t^c}{q'(p_t)} = C_2^m(t) + \sum_{\alpha=t+1}^{T}(1+r)^{t-\alpha}C_1^m(\alpha) + (1+r)^t\mu^m + \frac{1}{q'(p_t)}\sum_{\beta=0}^{T}(1+r)^{t-\beta}[p_\beta - (1+r)^\beta\mu^m - C_2^m(\beta) - \sum_{\alpha=\beta+1}^{T}(1+r)^{\beta-\alpha}C_1^m(\alpha)]\frac{\partial}{\partial p_t}(\delta X_\beta^c) \tag{104}$$

This may be compared with (30), which gives a necessary condition for the optimal price path of a nonrenewable resource monopolist.

The left hand side of (104) is the marginal revenue earned by the last unit sold by the price setter in period t. The first right hand

term in (104) is the marginal cost of producing that unit. The second right hand side term is the user cost of producing the marginal unit, the discounted sum of the increases in cost imposed on future output by production of the marginal unit in period t. The third right hand side term is the exhaustion rent. The fourth right hand side term shows the effect of the competitive fringe. It is the discounted sum of the excess of price over cost (marginal production cost plus user cost plus the exhaustion rent) times the change in fringe output in response to a change in market price. When the price leader increases the price in period t, fringe output in each period changes; the cost of this change to the price leader is the excess of price over the three cost terms, times the change in fringe output, which is to say, times the change in total demand given up to the fringe. The fourth right hand side term measures the discounted value of the share of the market lost to the fringe because of the marginal increase in price. The price leader will choose the price sequence which equates marginal revenue in each period with marginal production cost, plus user cost, plus the exhaustion rent, plus the value of the marginal market loss to the fringe.

VI. An empirical test for nonrenewable resources

In III we derived a discounted minimum cost function for the extraction of a sequence of outputs of a nonrenewable resource:

$$(105) \qquad C = C[\{\delta X_t\},\{(1+r)^{-t}w_t\},\{(1+r)^{-t}p_t^k\},\{(1+r)^{-t}\mu_{2t}\}]$$

According to Shephard's Lemma [1970, p. 171; see also Samuelson, 1947, pp. 67-68] the cost minimizing demand for a factor of production is given by the derivative of the minimum cost function with respect to the price of that factor. For the cost function (105) this implies in particular[15]

$$(106) \qquad L_0 = \frac{\partial C}{\partial w_0}$$

from which

$$(107) \qquad \frac{w_0 L_0}{C} = \frac{w_0}{C}\frac{\partial C}{\partial w_0} = \frac{\partial \ln C}{\partial \ln w_0}$$

which in turn gives

$$(108) \qquad \ln w_0 L_0 = \ln \frac{\partial \ln C}{\partial \ln w_0} + \ln C$$

Equation (108) explains current expenditures on labor. It is derived under the assumption that current input decisions are made with complete and certain knowledge of the extent of the resource stock, of current and future factor prices, and of the output which will be required in each period. Technology is assumed to be unchanging. In fact, although producers may know current output and

factor prices, they can at best have expectations of the extent of the resource stock, of future outputs, of future factor prices, and in particular of the appropriate shadow price of reserves. Further, producers do not make input decisions for all periods at the start of some initial period and then passively carry them out. Expectations will be updated at the start of each period, and used to form current input decisions.[16] Because of uncertainty, fixed capital, labor contracts, and other impediments to cost minimization, we can at best expect to observe a tendency toward the labor expenditure indicated by (108):

$$\ln w_0 L_0 = \ln \frac{\partial \ln C}{\partial \ln w_0} + \ln C + \epsilon \tag{109}$$

where ϵ is a disturbance from the cost minimizing expenditure on labor, the properties of which will be specified below. Arguments on the right hand side of (109) which deal with the future will be represented by expected values, which will also have to be specified.

Because of technical progress, we expect the cost of extracting a given output sequence at given wages and capital costs from a given resource stock to decline over time. To account for this, alter (109) to include a simple time trend:

$$\ln w_0 L_0 = \ln \frac{\partial \ln C}{\partial \ln w_0} + \ln C + \lambda t + \epsilon \tag{110}$$

We will expect $\lambda < 0$, to reflect the influence of technical progress.

I will make strong assumptions concerning expectations and the functional form of the cost function, to obtain versions of (110) which can be easily estimated.

Suppose that the cost function takes a Cobb-Douglas form[17], which can be written in logarithmic form as:

$$\ln C = k + \sum_{t=0}^{\infty} [A_t \ln \delta X_t + B_t \ln (1+r)^{-t} w_t + D_t \ln (1+r)^{-t} p_t^k + E_t \ln (1+r)^{-t} \mu_{2t}] \tag{111}$$

Separating out the discount factor in (111) gives

$$\ln C = k + \sum_{t=0}^{\infty} [A_t \ln \delta X_t + B_t \ln w_t + D_t \ln p_t^k + E_t \ln \mu_{2t}] - \ln (1+r) \sum_{t=0}^{\infty} t(B_t + D_t + E_t) \tag{112}$$

The assumption that the cost function displays first degree homogeneity in factor prices implies that

$$\sum_{t=0}^{\infty} (B_t + D_t + E_t) = 1 \tag{113}$$

It is an implication of the assumed Cobb-Douglas functional form that

$$B_t > 0 \qquad D_t > 0 \qquad E_t > 0 \tag{114}$$

for all t.

I will assume that the sequence $\{\mu_{2t}\}$ can be collapsed into a single value, μ, the expected shadow value of a unit of reserves. This is an empirical simplification, made to allow a test of the importance of the scarcity of reserves. If information were available describing an active market in reserve bearing land, the scarcity of reserves could be directly measured, by comparing the market price of

reserve bearing land with the market price of land not so endowed. In the absence of such information, we must look for a proxy variable which will measure the expected scarcity value of reserves. This will be the crucial variable in the empirical tests presented below: some physical or market measure of the scarcity value which producers use to guage the extent to which reserves should be economized.

I assume that the wage sequence can be collapsed into two values, current wage w_0 and expected normal wage, w. The capital cost sequence $\{p_t^k\}$ is represented by the current price p_0^k and the expected normal price p^k. Similarly, the output sequence is condensed into current output δX_0 and expected normal output δX. This specification keeps the number of variables to a minimum, while providing some mechanism to study arbitrage between the present and the future. On this basis, the cost function (112) becomes

$$\begin{aligned} \ln C = k + A_0 \ln \delta X_0 + \left(\sum_{t=1}^{\infty} A_t\right) \ln \delta X + B_0 \ln w_0 + \left(\sum_{t=1}^{\infty} B_t\right) \ln w \\ + D_0 \ln p_0^k + \left(\sum_{t=1}^{\infty} D_t\right) \ln p^k + \left(\sum_{t=0}^{\infty} E_t\right) \ln \mu \\ - \ln (1 + r) \sum_{t=0}^{\infty} t(B_t + D_t + E_t) \end{aligned} \tag{115}$$

which I rewrite

$$\begin{aligned} \ln C = k + A_0 \ln \delta X_0 + A \ln \delta X + B_0 \ln w_0 + B \ln w \\ + D_0 \ln p_0^k + D \ln p^k + E \ln \mu + F \ln (1 + r) \end{aligned} \tag{116}$$

Comparison of coefficients in (115) and (116) defines the coefficients in (116). The homogeneity constraint (113) becomes

$$B_0 + B + D_0 + D + E = 1 \tag{117}$$

with

$$B_0 > 0 \quad B > 0 \quad D_0 > 0 \quad D > 0 \quad E > 0 \tag{118}$$

in addition, the homogeneity constraint implies

$$F < 0 \tag{119}$$

Substitution of the cost function (116) into (110) gives the following equation explaining current expenditures on labor:

$$\begin{aligned} \ln w_0 L_0 = {} & K + A_0 \ln \delta X_0 + A \ln \delta X + B_0 \ln w_0 + B \ln w \\ & + D_0 \ln p_0^k + D \ln p^k + E \ln \mu + F \ln (1 + r) \\ & + \lambda t + \epsilon \end{aligned} \tag{120}$$

where $K = \ln B_0 + k$.

(120) is a linear equation explaining expenditures on labor. It provides a framework within which both the tradeoff between current and future prices, and the influence of the scarcity of reserves, may be studied. However, (120) contains three pairs of variables, one of which is currently observed (as, w_0) and one of which is an expected normal value (as, w). We may expect some collinearity between the current and the expected normal value, no matter what assumption is made to represent expectations. With this problem in mind, I derive

a simpler model from (120) by assuming that the expected normal values of output, wages, and capital costs are given by the respective current values. This gives

$$\ln w_0L_0 = K + (A_0 + A)\ln \delta X_0 + (B_0 + B)\ln w_0 + (D_0 + D)\ln p_0^k + E \ln \mu + F\ln (1 + r) + \lambda t + \epsilon \tag{121}$$

or, rewriting,

$$\ln w_0L_0 = K + A^*\ln \delta X_0 + B^*\ln w_0 + D^*\ln p_0^k + E \ln \mu + F \ln (1 + r) + \lambda t + \epsilon \tag{122}$$

The homogeneity constraint for (122) is

$$B^* + D^* + E = 1 \tag{123}$$

Using the homogeneity constraint to eliminate E in (120) and (122) gives the following estimating equations:

$$\ln \frac{w_0L_0}{\mu} = K + A^*\ln \delta X_0 + B^*\ln \frac{w_0}{\mu} + D^*\ln \frac{p_0^k}{\mu} + F \ln (1 + r) + \lambda t + \epsilon \tag{124}$$

$$\ln \frac{w_0L_0}{\mu} = K + A_0\ln \delta X_0 + A \ln \delta X + B_0\ln \frac{w_0}{\mu} + B \ln \frac{w}{\mu} + D_0\ln \frac{p_0^k}{\mu} + D \ln\frac{p^k}{\mu} + F \ln (1 + r) + \lambda t + \epsilon \tag{125}$$

These equations may be estimated, using appropriate assumptions concerning expectations and the shadow price μ. The coefficient E is recovered via the homogeneity constraint.[18]

Equations (124) and (125) have been estimated for four mineral

industries: U. S. coal and copper, and Canadian gold and iron ore. The primary purpose of this test is to locate proxies for the shadow price of reserves which are significant explanatory variables for expenditures on labor, not to study expectations. Consequently, I have assumed perfect foresight. Expected normal output is represented by the actual output for 24 months ahead. Expected normal wage is represented by the actual wage for 12 months ahead. Expected normal capital cost is represented by the actual capital cost for 36 months ahead. The consequences of alternative assumptions concerning expectations are reported in Appendix II.

There is, unfortunately, no way to directly observe the industry estimate of the scarcity value of a unit of reserves. I will assume that the expected shadow price μ is proportional to some observable variable θ

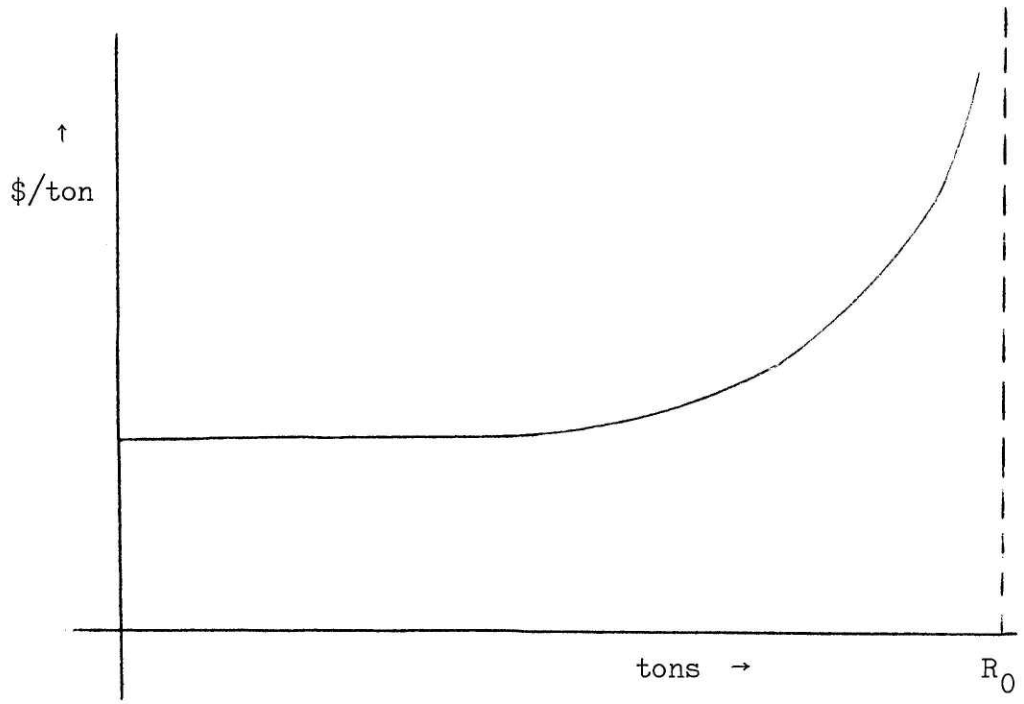
(126) $$\mu = \alpha\theta$$

On this assumption, (124) and (125) may be estimated, substituting θ for μ and altering the value of the constant term. Estimates will be presented using (126) and using an instrumental variables approximation to (126).

Three proxies for the shadow price of reserves will be tested for the U. S. coal industry. The first proxy is a physical index. The 1973 Minerals Yearbook reports the average BTU yield per pound of coal extracted in the U. S., on an annual basis, for the period 1955 to 1973 inclusive. The reported values decline over time, which is what one would expect if the best ores were mined first and depletion forced

attention to less well endowed deposits. A straight line regression was made on the annual data[19], and used to interpolate monthly values. The inverse of these fitted values, a series which rises over time, is used as the first proxy for the shadow price of coal.

The second proxy used for the shadow price of coal is cumulative extraction of coal. The Bureau of the Census' Historical Statistics of the United States gives annual production for the period 1800 to 1956. This was combined with the monthly output series to construct a monthly cumulative output series. For scaling purposes, cumulative output is divided by cumulative output at the end of December, 1950. Use of cumulative output as a proxy for scarcity value may be defended as follows. Suppose that before extraction of coal begins, the following curve depicts the number of tons of coal which can be extracted for less than a certain unit cost[20]:



Presumably, there is some minimum cost below which no ore, no matter how rich, can be extracted. Further, the unit cost of extracting the very last, poorest reserves will be very great, rising to infinity. If extraction proceeds from those deposits which are least expensive to extract to those which are most expensive to extract, and if the frequency curve is approximately linear for the output range involved, then cumulative extraction will provide an approximate measure of the value placed on a unit of reserves.

By their nature, both these shadow price proxies rise over time. It is not unreasonable to suppose that the value placed on reserves will rise over time, if demand is steady and if extraction progressively increases the scarcity of reserves. But demand may not be steady, and discovery may reveal new reserves which causes revision of the scarcity value downward. New and unexpected techniques may open previously unprofitable reserves to profitable exploitation. The market estimate of the scarcity of reserves, in a world of uncertainty and technical advance, may rise or fall over time. For this reason, the third proxy tested for the coal industry is the current market price of coal. The assumption is that the scarcity value placed on a unit of reserves is proportional to the price for which that unit could be sold if placed on the market. Since it is to be expected that the current price of output will be related to expenditures on labor for reasons which have nothing to do with the scarcity of reserves (in particular, both expenditure on labor and price will be determined partly by the current wage), an instrumental variables approximation to the market price

will have to be used to avoid problems of simultaneity.

Three proxies for the shadow price of reserves are tested for the U. S. copper industry. The first proxy is based on the average yield of copper ore, in per cent, as reported by the Minerals Yearbook for the period 1924 - 1972. The average yield declines over time, and a hyperbola yields a very close fit to the reported annual values.[21] The fitted hyperbola is used to interpolate monthly values, which are inverted to obtain a proxy for the shadow price. The hypothesis is that as extraction forces attention to ores which yield less refined copper, the value placed on a unit of reserves will rise.

The second shadow price proxy for the U. S. copper industry is cumulative output, and the third is based on market price.

Two shadow price proxies are used for each of the Canadian industries. The first is based on cumulative output, and the second is based on current market price.

VII. Description of the data

Output

Monthly mine production of recoverable copper in the United States is reported in annual editions of the U. S. Bureau of Mines' Minerals Yearbook for the period April 1941 - December 1973. The same source provides monthly production of bituminous coal and lignite for the United States for the period January 1932 to December 1973.

Statistics Canada publications (General Review of the Mineral Industries, 1961 and 1962; Monthly Statistical Review, 1963 - 1975) report monthly output for Canadian gold and iron ore industries for the period January 1948 to August 1975.

Wages

The U. S. Bureau of Labor Statistics' Employment and Earnings gives average hourly earnings of production workers in the coal industry, on a monthly basis, for January 1934 to December 1973. It reports average hourly earnings of production workers in the copper industry for January 1939 to September 1975.

Various Statistics Canada publications (Man hours and hourly earnings with average weekly wages; Statistical yearbook) give average hourly earnings, on a monthly basis, for the gold industry for the period January 1961 to June 1975, and for the iron ore industry for the same period.

Labor input

The Bureau of Labor Statistics' Employment and Earnings gives monthly figures for thousands of employees and thousands of production

workers for the coal industry for January 1934 to December 1973. It reports production worker average weekly hours in coal for the same period. Labor input in thousands of hours per month is computed as production worker average weekly hours, times thousands of production workers, plus forty hours, times thousands of nonproduction workers, all times the number of weeks per month. From the same sources, a similar time series of labor input in thousands of hours per month was constructed for the copper industry for the period January 1947 to September 1973.

From various Statistics Canada sources[22] figures for labor input are constructed as average weekly hours, times average monthly employment, times the number of weeks per month. The period is January 1961 to June 1975 for both gold and iron ore.

Capital cost and discount rate

The preferred capital rental rate series has the form

$$p_t^k = r_t p_{t-1} - (p_t - p_{t-1}) \tag{127}$$

where r_t is an appropriate interest rate and p_t is a price series or index for the particular species of capital used by the industry in question, deflated by a general price index.

For both U. S. industries, such an index was constructed using as an interest rate the average yield on newly issued A corporate bonds, reported in Moody's Industrial Manual for January 1951 to May 1975. The price index used was the wholesale price index for mining machinery as reported by the Bureau of Labor Statistics, deflated by the consumer

price index.[23]

For the Canadian industries, a capital rental rate series was constructed using the yield on 3 month Canadian treasury bills[24] and a price index for thirty industrial materials, reported in Statistics Canada's Canadian Statistical Review, deflated by the Canadian consumer price index, from the same source, for the period June 1949 to March 1976.

The results presented below were first estimated using these capital rental rate series to represent the cost of capital. The results were totally unsatisfactory - coefficients extremely close to zero, and statistically insignificant. It was concluded that the purchase price series used to construct (127), although the best that could be located, did not really capture fluctuations in the purchase price of capital for these industries. For this reason, the neutral assumption was made that the cost of capital for these industries has fluctuated with the general price level; on this basis the cost of capital series becomes

(128) r_t

which was represented by the Moody's index for the U. S. industries and by the treasury bill index for the Canadian industries. The variables were measured in per cent, which amounts to measuring capital in units of one hundred dollars worth, and cost of capital in dollars per one hundred dollars.

The same series, in per cent divided by one hundred, were used for the discount rate variable, for which

(129) $$\ln (1 + r_t) \approx r_t = p_t^k/100$$

where r_t is small. There is thus a functional relation between these variables, although it is not a linear relation, since the cost of capital appears in the form

(130) $$\ln p_t^k$$

Price indices

All price values for the U. S. industries were deflated by the consumer price index, base 1957-59 = 100. Canadian money values were deflated by the Canadian consumer price index, reported in the Canadian Statistical Review, base 1949 = 100. Although more broadly based deflators might have been preferred, the need for a monthly index compelled these choices.

Shadow price proxies

The annual average BTU yield of coal extracted in the U. S. is reported in the 1973 Minerals Yearbook. Cumulative coal output is derived from the Minerals Yearbook (annual) and Historical Statistics of the United States. The Bureau of Labor Statistics' Prices and Price Indexes gives a monthly wholesale price index for bituminous coal for December 1951 through December 1973.

Annual issues of Minerals Yearbook report the annual average yield of copper ore, in per cent, for 1924 - 1972. Cumulative output is constructed from information reported in Minerals Yearbook and Historical Statistics of the United States. Metals Week reports the

export f.o.b. refinery price of electrolytic copper.

Cumulative output for the Canadian mineral industries is derived from the monthly output figures and from figures reported in Historical Statistics of Canada. Output for both industries is reported from 1886. Statistics Canada, Prices and Price Indexes, reports a price index for iron products for January 1950 to June 1975. The Bank of England Quarterly Bulletin reports monthly gold prices, in U. S. dollars per fine ounce, for January 1960 to February 1976. The U. S. dollar price was converted to a Canadian dollar price by the appropriate exchange rate.

The sources described above provide complete data for the following sample periods:

		Observations
U. S. Coal	January 1951 - December 1973 (Price index from December 1951)	276 (265)
U. S. Copper	January 1951 - December 1973	276
Canadian Gold	January 1961 - June 1975	174
Canadian Iron Ore	January 1961 - June 1975	174

In each case, months in which output was seriously affected by strikes were eliminated from the sample.

VIII. U. S. Coal

I will refer to the BTU shadow price proxy as MU1, the cumulative output proxy as MU2, and the current price proxy as MU3.

MU1 rises from 0.949953 in January 1951 to 1.04591 in December 1973. Over the same period, MU2 rises from 1.00205 to 1.44682. From December 1951 to December 1973, MU3 fluctuates between 0.851557 (April 1966) and 2.14585 (October 1973). Table 1 describes the three proxies.

Table 1: Shadow price proxies for U. S. Coal

	Mean	Standard Deviation	Correlation with MU1	MU2	MU3
MU1	0.998251	0.026777	1.000000	0.999183	0.532444
MU2	1.22108	0.120915		1.000000	0.560171
MU3	1.01058	0.190113			1.000000

The two physical measures move in very much the same fashion. Their collinearity no doubt derives much from the fact that both rise over time, by construction. The price proxy moves in a different fashion.

Table 2 gives estimates of (124) and (125) for the U. S. coal industry. With monthly data, serial correlation was expected. The equations were estimated using the Cochrane-Orcutt iterative technique for first order serial autocorrelation of the error term. The Durbin-Watson statistics for the equations estimated by the Cochrane-Orcutt technique indicate that serial correlation is not a problem for the corrected estimates.

Table 2 - U. S. Coal Industry

	Constant	Current Output	Expected Output	Current Wage	Expected Wage	Current Capital Cost	Expected Capital Cost	Shadow Price Proxy	Discount Factor	Time	Rho	R^2	DOF
2.1	7.3324	0.2798		1.0176		0.2413		-0.2589	-3.1801	-0.002224	0.8858	0.8873	254
MU1	(15.9458)	(8.9399)		(3.8206)		(0.7301)		(0.6312)	(0.4988)	(3.4681)	(30.7867)		
2.2	7.5972	0.2857		0.8666		0.0841		0.0493	-1.0506	-0.002273	0.9038	0.9369	265
MU2	(17.3969)	(9.4491)		(3.3594)		(0.2789)		(0.1284)	(0.1728)	(3.1234)	(34.7647)		
2.3	7.6080	0.2824		0.9219		-0.0225		0.1006	2.1065	-0.002495	0.8545	0.8873	254
MU3	(21.9796)	(8.9295)		(4.1953)		(0.1058)		(1.4680)	(0.4942)	(4.5290)	(26.5195)		
2.4	7.1904	0.3796	-0.0496	0.7386	0.0634	-0.2566	0.1594	0.2953	11.0038	-0.004986	0.5620	0.9053	215
MU1	(14.1442)	(8.2506)	(1.0700)	(2.8245)	(0.2718)	(1.0681)	(2.3658)	(0.7578)	(2.6102)	(12.3558)	(10.1689)		
2.5	7.3961	0.3316	-0.0003	0.5841	0.0724	-0.3040	0.1401	0.5074	11.1313	-0.005334	0.6819	0.9474	226
MU2	(14.5296)	(7.4418)	(0.0057)	(2.0998)	(0.3023)	(1.1832)	(1.7570)	(1.4180)	(2.3227)	(12.1981)	(14.2902)		
2.6	7.4134	0.3623	-0.0264	0.3524	0.0423	-0.0590	0.1849	0.4794	5.3545	-0.004168	0.5656	0.8733	215
MU3	(17.1220)	(8.1865)	(0.5949)	(1.2427)	(0.2068)	(0.2061)	(2.8697)	(3.2976)	(1.4237)	(9.1183)	(10.2651)		

MU1: Average BTU yield
MU2: Cumulative extraction
MU3: Current price

T-statistics in parentheses

Normal distribution critical values: 90% - 1.282; 95% - 1.645; 99% - 2.326 (one-tail test)

In Table 2, the coefficient of current output is consistently positive and highly significant. The coefficient of expected output is negative and insignificant, no matter which shadow price proxy is tested.

The coefficient of current wage is positive and significant in each case. It exceeds one for the BTU proxy. The coefficient declines in magnitude when expected wage is present. Although the estimated coefficient of expected wage is positive and less than one for all shadow price proxies, as expected, it is never significantly different from zero.

The estimated coefficient of current capital cost is consistently insignificant, and frequently negative. On the other hand, the coefficient of expected capital cost is positive, less than one, and highly significant. It may be that the coefficient of current capital cost is contaminated by collinearity with the discount factor.

The coefficient of MU1 is negative in the simple model, positive in the more complex model, and insignificant in each case. The coefficient of MU2 is positive and less than one in both cases, and significantly different from zero in the complex model. The coefficient of MU3 is positive, less than one, and statistically significant in both models. For each proxy, the coefficient rises in magnitude and significance in passing from the simple to the expanded model.

The coefficient of the discount factor is insignificant when it is negative, and frequently positive, in contradiction to the implication of the assumed Cobb-Douglas cost function. This coefficient

may be confused by collinearity with the capital cost variable.

The estimated time trend is consistently negative and highly significant. This may reflect neutral technical advance, the basis upon which it was introduced into the model, or it may reflect a reduction in labor input over time.

The high estimated serial correlation coefficient for the simple model indicates that important explanatory variables are omitted. The estimated serial correlation coefficient falls in magnitude in the expanded model.

Since we are trying to study the influence of an unobserved and unobservable variable, the estimated shadow price of mineral reserves, the proxies ought to be approximated by instrumental variables to ensure consistent estimates. In addition, for the current price proxy, one might expect some simultaneity to be involved (market price is likely to change with expenditures on labor, and vice versa). This also argues for the use of instrumental variables. Table 3 presents estimates for the coal industry when the three proxies were first approximated by instrumental variables. The list of instrumental variables included all explanatory variables other than the price proxy, nine month lagged output, wages, and capital cost, and the average yield on Moody's A bonds (as opposed to the yield on new A bonds, which is used as the capital cost variable). Use of instrumental variables has the side effect of reducing the sample period for the simple model by 36 observations, since one of the instrumental variables is expected capital cost, which is the 36 month ahead observed capital cost.

Table 3 - U. S. Coal Industry (Instrumental variables)

	Constant	Current Output	Expected Output	Current Wage	Expected Wage	Current Capital Cost	Expected Capital Cost	Shadow Price	Discount Factor	Time	Rho	R^2	DOF
3.1	7.3448	0.3459		0.7015		-0.2915		0.5900	10.9299	-0.004871	0.6477	0.9366	220
MU1	(16.6190)	(9.7728)		(2.6014)		(1.1500)		(1.9137)	(2.3513)	(12.9710)	(12.7821)		
3.2	7.3470	0.3462		0.7277		-0.2857		0.5580	10.9238	-0.004365	0.6455	0.9046	220
MU2	(16.3896)	(9.7704)		(2.7666)		(1.1282)		(1.8738)	(2.3435)	(11.7257)	(12.7068)		
3.3	7.4312	0.3350		0.5207		0.0784		0.4008	3.1396	-0.003713	0.6308	0.8590	218
MU3	(15.4196)	(9.3949)		(1.5413)		(0.3550)		(1.9141)	(0.7538)	(6.5945)	(12.1672)		
3.4	7.2171	0.3534	-0.0257	0.6366	0.1802	-0.2840	0.1562	0.3100	11.2201	-0.005273	0.6124	0.9381	217
MU1	(14.0147)	(7.5457)	(0.5479)	(2.3536)	(0.7326)	(1.1552)	(2.1121)	(0.8869)	(2.5355)	(13.6193)	(11.6463)		
3.5	7.2237	0.3523	-0.0252	0.6455	0.1881	-0.2835	0.1568	0.2931	11.2436	-0.005006	0.6125	0.9069	217
MU2	(13.9927)	(7.5348)	(0.5361)	(2.4069)	(0.7752)	(1.1512)	(2.1203)	(1.3082)	(2.5220)	(11.2372)	(11.6492)		
3.6	7.5244	0.3105	0.0150	0.2668	0.0690	-0.0097	0.1917	0.4822	4.8181	-0.004174	0.5936	0.8643	215
MU3	(15.3958)	(6.2391)	(0.3028)	(0.7349)	(0.3232)	(0.0433)	(2.8043)	(2.1024)	(1.1286)	(7.0709)	(11.0384)		

MU1: Average BTU yield
MU2: Cumulative extraction
MU3: Current price

T-statistics in parentheses

Normal distribution critical values: 90% - 1.282; 95% - 1.645; 99% - 2.326 (one-tail test)

The magnitude and statistical significance of the output coefficients are little changed by the use of instrumental variables. The current wage coefficient is reduced in magnitude (which should be considered an improvement), remains positive, and is significantly different from zero. The coefficient of expected wage is slightly greater in magnitude when instrumental variables are used.

The capital cost coefficients are about the same as those in Table 2: the current capital cost coefficient is negative and statistically insignificant, the expected capital cost coefficient is positive, less than one, and significantly different from zero.

The shadow price coefficients in the simple model increase in both magnitude and significance. In the expanded model, the coefficient of MU1 increases in magnitude and significance compared with Table 2, the coefficient of MU2 decreases in magnitude and significance, while the coefficient of MU3 increases slightly in magnitude but decreases in statistical significance.

The coefficient of the discount factor is consistently positive, and significant when MU1 and MU2 are tested. The time trend coefficient is not much changed by the use of instrumental variables. The estimated serial correlation coefficient is generally reduced by the use of instrumental variables.

In general, the use of instrumental variables brings the coefficients of the simple model closer to those of the expanded model. This indicates that the main effect of the instrumental variables may be through the reduction in sample size for the simple model.

The model developed to explain expenditures on labor works fairly well. Both physical and market measures of the scarcity of reserves are significant explanatory variables for expenditures on labor for the U. S. coal industry, which confirms the importance of depletion in the extraction of coal.

IX. U. S. Copper

MU1, the first shadow price proxy for the U. S. copper industry, is based on the average yield of copper ore. MU2 is based on cumulative extraction of copper, and MU3 is the real price of copper in dollars per ton.

MU1 rises from 1.17180 in January 1951 to 1.60449 in December 1973. Over the same period, MU2 rises from 1.00220 to 1.76148. MU3 fluctuates between 402.385 (February 1958) and 1262.40 (November 1973) over the same period. Table 4 describes the three proxies.

Table 4: Shadow price proxies for U. S. Copper

	Mean	Standard	Correlation with MU1	MU2	MU3
MU1	1.41012	0.124590	1.000000	0.986148	0.401160
MU2	1.33908	0.216136		1.000000	0.455277
MU3	699.515	181.295			1.000000

As with the coal industry proxies, the two physical measures are highly correlated, while the price index moves in a different fashion.

Table 5 gives the results of estimating (124) and (125) for the U. S. copper industry.

The coefficient of current output is positive and quite significant in all cases. The coefficient of expected output is positive, of slight magnitude, and not significantly different from zero.

The coefficient of current wage is always highly significant, but also greater than one. This indicates that the assumed Cobb-Douglas

Table 5 - U. S. Copper Industry

	Constant	Current Output	Expected Output	Current Wage	Expected Wage	Current Capital Cost	Expected Capital Cost	Shadow Price Proxy	Discount Factor	Time	Rho	R^2	DOF
5.1	10.4313	0.4529		1.2788		-0.6187		0.3396	14.4947	-0.001432	0.7570	0.9108	234
MU1	(20.6065)	(10.4837)		(4.2581)		(2.4590)		(1.0086)	(3.0076)	(3.0624)	(17.9455)		
5.2	10.5938	0.4477		1.2739		-0.6919		0.4181	16.0019	-0.001868	0.7350	0.8434	247
MU2	(21.2961)	(10.7403)		(5.4428)		(3.3164)		(1.6155)	(3.8919)	(4.0465)	(17.2460)		
5.3	9.9369	0.4385		1.3425		-0.4269		0.0844	11.3372	-0.001325	0.6945	0.9482	234
MU3	(19.5925)	(10.0857)		(6.2255)		(2.0687)		(2.3639)	(2.6812)	(2.9547)	(14.9522)		
5.4	9.9315	0.4468	0.0225	1.3857	0.0053	-0.5480	0.1930	-0.0360	15.1241	-0.002680	0.5820	0.8740	199
MU1	(15.6946)	(9.4661)	(0.5016)	(4.6770)	(0.0239)	(2.6133)	(3.3984)	(0.0986)	(3.7857)	(6.1225)	(10.2651)		
5.5	9.8733	0.4385	0.0296	1.5136	-0.0158	-0.5831	0.1939	-0.1086	15.9667	-0.002684	0.5778	0.8208	212
MU2	(16.0083)	(9.6559)	(0.6774)	(6.6028)	(0.0760)	(3.2056)	(3.5096)	(0.4186)	(4.5180)	(6.9972)	(10.5214)		
5.6	9.5325	0.4165	0.0286	1.0724	-0.0023	-0.3119	0.1019	0.1398	9.4311	-0.001751	0.5218	0.9396	199
MU3	(16.0815)	(8.9697)	(0.6670)	(4.2116)	(0.0131)	(1.6841)	(1.8511)	(3.6108)	(2.4591)	(3.9494)	(8.8212)		

MU1: Average yield of refined copper per ton of ore extracted
MU2: Cumulative extraction
MU3: Current price

T-statistics in parentheses

Normal distribution critical values: 90% - 1.282; 95% - 1.645; 99% - 2.326 (one-tail test)

form of the cost function, which may be viewed as a first order log-linear approximation to a general, unspecified cost function, is unduly restrictive. Since the homogeneity constraint forces the estimated coefficients of the price terms to sum to one, the large wage coefficient will have effect elsewhere.

The coefficient of expected wage is of slight magnitude, uncertain sign, and statistically insignificant.

Although statistically significant, the capital cost coefficient is negative. In constrast, the coefficient of expected capital cost is positive, less than one, and statistically different from zero.

In the simple model, the estimated coefficient of each shadow price proxy is positive and less than one. The estimated coefficient of MU2 is statistically significant, as is the coefficient of MU3. The coefficients of MU1 and MU2 change sign in the expanded model, and only the coefficient of MU3 remains positive and significant in the expanded model.

The coefficient of the discount factor is consistently positive and highly significant. The time trend is negative, as expected, and significant. The serial correlation coefficient is quite large in the simple model, and declines in magnitude for the expanded model.

Table 6 gives estimates of the same equations when instrumental variables are used to approximate the shadow price variables. The list of instrumental variables is the same as that used for the U. S. coal industry.

Coefficients other than those of the shadow price proxies are

Table 6 - U. S. Copper Industry (Instrumental variables)

	Constant	Current Output	Expected Output	Current Wage	Expected Wage	Current Capital Cost	Expected Capital Cost	Shadow Price Proxy	Discount Factor	Time	Rho	R^2	DOF
6.1	10.3371	0.4652		1.3081		-0.6210		0.3129	15.7362	-0.002335	0.6539	0.8082	206
MU1	(19.5028)	(10.3775)		(4.6940)		(2.9979)		(1.1086)	(3.8537)	(5.4028)	(12.5842)		
6.2	10.2447	0.4675		1.3183		-0.6221		0.3038	15.9136	-0.002113	0.6560	0.8750	206
MU2	(20.0018)	(10.3644)		(4.3364)		(2.8863)		(0.9081)	(3.7798)	(4.8938)	(12.6562)		
6.3	9.8594	0.4512		1.3783		-0.4542		0.0759	12.4051	-0.001877	0.6242	0.9214	206
MU3	(18.7693)	(9.9279)		(6.4474)		(2.3478)		(1.4905)	(2.9441)	(3.9543)	(11.6328)		
6.4	9.9186	0.4469	0.0258	1.3891	-0.0088	-0.5802	0.1926	0.0007	15.6074	-0.002690	0.5849	0.8176	203
MU1	(15.6691)	(9.5568)	(0.5819)	(5.1667)	(0.0415)	(3.3180)	(3.0753)	(0.0245)	(4.2381)	(6.8350)	(10.4985)		
6.5	9.9085	0.4468	0.0257	1.3957	-0.0257	-0.5764	0.1934	-0.0009	15.5695	-0.002693	0.5841	0.8813	203
MU2	(15.9063)	(9.5221)	(0.5808)	(4.9080)	(0.0165)	(2.9162)	(3.4226)	(0.0240)	(4.1142)	(6.3168)	(10.4777)		
6.6	9.9797	0.4494	0.0281	1.4440	-0.0205	-0.6077	0.2091	-0.0250	16.4759	-0.002852	0.5832	0.9250	203
MU3	(16.3214)	(9.5558)	(0.6296)	(5.3529)	(0.1097)	(3.2067)	(3.1712)	(0.4294)	(3.9800)	(5.3194)	(10.4528)		

MU1: Average yield of refined copper per ton of ore extracted
MU2: Cumulative extraction
MU3: Current price

T-Statistics in parentheses

Normal distribution critical values: 90% - 1.282; 95% - 1.645; 99% - 2.326 (one-tail test)

little changed by the use of instrumental variables. In particular, the wage coefficient remains greater than one, and the current capital cost coefficient remains negative. The shadow price coefficients remain positive in the simple model, although of reduced statistical significance. On the other hand, the coefficient of MU3, which was positive and significantly different from zero in the simple model, becomes slightly negative and statistically insignificant in the expanded model.

The estimated coefficient of the discount factor, and the estimated time trend, are largely unchanged by the use of instrumental variables.

The Cobb-Douglas cost function is not a suitable approximation for the U. S. copper industry. Further, the expectational assumptions made do not seem to perform very well for this industry. However, all three shadow price proxies perform as expected in the simple model, which provides modest confirmation for the role of depletion in the extraction of copper.

X. Canadian Gold

The first shadow price proxy for Canadian gold, cumulative depletion, increases from 1.03345 in January 1961 to 1.31336 in June 1975. Over the same period, the real price of gold fluctuated between 21.4369 1949 Canadian dollars per fine ounce (in August 1970) and 81.3998 1949 Canadian dollars per fine ounce (December 1974).

Table 7: Shadow price proxies for Canadian gold

	Mean	Standard Deviation	Correlation with MU1	MU2
MU1	1.19823	0.081167	1.000000	0.513260
MU2	33.5430	15.4301		1.000000

The correlation between the physical index MU1 and the market index MU2 is positive but not great.

The first four equations of Table 8 present estimates of (124) and (125) for the Canadian gold industry for MU1 and MU2.

Both current and expected output have positive coefficients which are statistically significant. In the simple model, the coefficient of current wage, although large, is positive and statistically significant. In the expanded model, the wage coefficient becomes small, negative, and statistically insignificant, for MU1. The wage coefficient declines in magnitude, which is an improvement, but remains positive in the expanded model, for MU2. The coefficient of the expected wage variable is negative for both proxies. The capital cost coefficient is negative for the cumulative extraction proxy, and positive for the price proxy;

Table 8 - Canadian Gold Industry

	Constant	Current Output	Expected Output	Current Wage	Expected Wage	Current Capital Cost	Expected Capital Cost	Shadow Price Proxy	Discount Factor	Time	Rho	R^2	DOF
8.1	14.1673	0.3092		0.7995		-0.1855		0.3860	4.0201	-0.006381	0.8337	0.9801	167
MU1	(29.7469)	(4.7739)		(6.2234)		(0.9424)		(1.5855)	(0.9364)	(8.5472)	(19.8580)		
8.2	13.6169	0.3168		0.8211		0.0359		0.1430	0.0391	-0.006749	0.7150	0.9912	167
MU2	(27.6671)	(4.7218)		(7.7284)		(0.3769)		(3.2283)	(0.0186)	(11.1469)	(13.4532)		
8.3	11.0803	0.5698	0.1874	-0.0457	-0.2805	-0.0077	0.0894	1.2445	1.6748	-0.003052	0.2080	0.9838	128
MU1	(15.6649)	(6.5212)	(2.2329)	(0.3523)	(2.1304)	(0.0572)	(3.5568)	(5.6021)	(0.5481)	(4.0821)	(2.4895)		
8.4	10.6866	0.5026	0.1935	0.4963	-0.0229	0.2720	0.0800	0.1746	-4.0932	-0.003758	0.5060	0.9648	128
MU2	(13.1428)	(5.8457)	(2.3147)	(4.0262)	(0.1754)	(1.8713)	(2.0501)	(1.7930)	(1.1487)	(4.2982)	(6.8667)		
Instrumental variables:													
8.5	12.9214	0.4967		0.4092		0.1514		0.4394	-2.9137	-0.005557	0.4612	0.9820	122
MU1	(21.7833)	(5.9340)		(2.9510)		(0.7514)		(1.7654)	(0.6849)	(8.2170)	(5.8813)		
8.6	11.9708	0.5214		0.5188		0.2972		0.1840	-5.7910	-0.004844	0.4849	0.9627	122
MU2	(15.6462)	(6.0778)		(4.4120)		(2.1615)		(1.6826)	(1.9087)	(6.3768)	(6.2734)		
8.7	11.7942	0.5030	0.1520	0.1739	-0.1368	0.0540	0.0735	0.8356	0.2355	-0.004306	0.2796	0.9832	119
MU1	(15.4785)	(5.7297)	(1.7810)	(1.2777)	(0.9886)	(0.3037)	(2.6780)	(3.2260)	(0.0601)	(4.9813)	(3.2949)		
8.8	10.1070	0.5523	0.0995	0.3858	-0.0908	0.1653	0.1217	0.4180	-0.9068	-0.003551	0.2654	0.9661	119
MU2	(11.6170)	(6.2216)	(1.1649)	(3.2333)	(0.7264)	(1.1406)	(3.8609)	(3.8329)	(0.2649)	(4.0129)	(3.1146)		

MU1: Cumulative extraction
MU2: Current price

T-statistics in parentheses

it is significant for MU2 in the expanded model ((8.4)).

The coefficient of the shadow price proxy is positive and significant in both the simple and expanded equations. It is greater than one for MU1 in the expanded equation; this is related to the fact that the wage coefficient is negative in that equation.

The coefficient of the discount factor is of ambiguous sign and in any event statistically insignificant. The time trend coefficient is negative, as expected, and highly significant. The serial correlation coefficient declines in magnitude and significance in passing from the simple to the expanded model.

The second half of Table 8 shows estimates of the same equations when instrumental variables are used to approximate the shadow price proxies. The list of instrumental variables includes all explanatory variables except shadow price, nine month lagged values of output, wage, and capital cost, and the natural logarithm of the average yield on Moody's "A" bonds.

The wage coefficient in the expanded model becomes positive, while the coefficient of MU1 in the same equation becomes less than one (equation (8.7)). In other equations, the current wage coefficient declines in magnitude, although remaining statistically significant. The coefficient of current capital cost is positive in all four equations. The shadow price coefficients increase in magnitude, and remain significantly different from zero.

The coefficient of the discount factor remains of ambiguous sign and significance. The time trend is little changed by the use of

instrumental variables.

Both physical and market proxies for the scarcity value of reserves are significant explanatory variables for expenditures on labor; futher, the estimated coefficients are of the sign and magnitude predicted by the underlying model. The importance of depletion for the Canadian gold industry is confirmed by these estimates.

XI. Canadian Iron Ore

The cumulative extraction shadow price proxy, MU1, for Canadian iron ore, rises from 1.37593 in January 1961 to 6.18074 in June 1975. Over the same period, the real price index of iron products varied between 1.73963 (August 1969) and 2.20362 (April 1975).

Table 9: Shadow price proxies for Canadian iron ore

	Mean	Standard Deviation	Correlation with MU1	Correlation with MU2
MU1	3.50206	1.44104	1.000000	-0.0123423
MU2	1.88549	0.0987976		1.000000

The correlation between the cumulative extraction index, MU1, and the price index, MU2, is negative and slight.

The first half of Table 10 gives estimates of (124) and (125) for the Canadian iron ore industry. The estimated coefficient of current output, although of slight magnitude, is always quite significant. The coefficient of expected output is negative and insignificant.

The coefficient of current wage changes sharply between the simple and expanded models. The wage coefficient is positive, near one, and highly significant in the simple model. In the expanded model, the coefficient is negative, near zero, and not significantly different from zero. In the expanded model, the coefficient of expected wage is negative, fairly large, and statistically significant.

The coefficient of capital cost is insignificant and of ambiguous sign in the simple model. Both current and expected capital cost have

Table 10 - Canadian Iron Ore Industry

	Constant	Current Output	Expected Output	Current Wage	Expected Wage	Current Capital Cost	Expected Capital Cost	Shadow Price Proxy	Discount Factor	Time	Rho	R^2	DOF
10.1	13.9304	0.0367		0.9780		-0.0186		0.0406	1.4701	0.001264	0.7401	0.9425	157
MU1	(38.2937)	(2.9490)		(4.7244)		(0.0757)		(0.1503)	(0.2773)	(0.5745)	(14.0513)		
10.2	13.8325	0.0385		0.8785		0.0033		0.1182	1.1114	0.001811	0.7277	0.9316	157
MU2	(92.8468)	(3.1093)		(3.2069)		(0.0178)		(0.4608)	(0.2783)	(2.4575)	(13.5454)		
10.3	15.1819	0.0780	-0.0109	-0.0102	-0.5085	0.2423	0.1516	1.1248	-4.3365	-0.007334	0.1083	0.9573	122
MU1	(57.0509)	(4.0222)	(0.5508)	(0.0545)	(3.0617)	(1.4054)	(5.0324)	(6.1806)	(1.1091)	(4.6739)	(1.2468)		
10.4	13.1990	0.0923	-0.0245	-0.2074	-0.7012	0.7841	0.1513	0.9731	-14.0907	0.004594	0.2085	0.9142	122
MU2	(69.6592)	(4.6576)	(1.0898)	(0.7161)	(2.9093)	(4.7206)	(4.1390)	(2.6034)	(3.4336)	(4.3145)	(2.4402)		
Instrumental variables:													
10.5	14.5969	-0.0579		-0.2845		0.6775		0.6070	-14.7942	-0.002443	0.4141	0.9375	116
MU1	(31.8224)	(4.6030)		(1.2127)		(2.0067)		(1.9496)	(2.0723)	(0.8560)	(5.0246)		
10.6	13.7623	0.0548		-0.0230		0.9717		0.0513	-20.3196	0.002778	0.5115	0.8698	116
MU2	(53.9261)	(3.8870)		(0.0488)		(3.0784)		(0.0887)	(2.8070)	(2.2109)	(6.5740)		
10.7	14.6225	0.0663	-0.0112	0.1184	-0.6427	0.6343	0.1571	0.7329	-11.8541	-0.003711	0.0961	0.9505	113
MU1	(45.8450)	(3.3841)	(0.5300)	(0.5459)	(3.5720)	(2.6872)	(5.0208)	(3.3576)	(2.3225)	(1.8618)	(1.0661)		
10.8	13.0247	0.0727	-0.0079	-0.4033	-1.0441	0.9492	0.1520	1.3461	-16.9836	0.005817	0.1227	0.8974	113
MU2	(59.9219)	(3.6374)	(0.3616)	(1.2375)	(4.5809)	(5.0011)	(4.6844)	(2.8716)	(3.7793)	(5.1712)	(1.3661)		

MU1: Cumulative extraction
MU2: Current price

T-statistics in parentheses

coefficients which are positive, less than one, and significant.

In the simple equation, the shadow price coefficients are positive and less than one, but not statistically significant. In the expanded equation, the coefficients are much larger - near one - and quite significant. The coefficient of cumulative extraction exceeds one.

The coefficient of the discount factor is of ambiguous sign and magnitude. The time trend is positive in three of the four equations. The serial correlation coefficient falls sharply in going from the simple to the expanded equation.

The second half of Table 10 shows the same estimates when instrumental variables are used to approximate the shadow prices. The same instrumental variables were used for the Canadian gold and the Canadian iron ore industries.

The current wage coefficients are negative in the simple equations, and in the expanded model for MU2. The coefficient of expected wage is negative and significant. The coefficient of current capital cost is positive for both proxies, and fairly large: near 0.65 for MU1, near 0.95 for MU2. The coefficient of expected capital cost is very much like that in the first half of Table 10.

The coefficient of MU1 in the simple model is larger and more significant when instrumental variables are used. In the expanded model, the coefficient is less than one and still significant. The coefficients for MU2 are little changed by the use of instrumental variables.

The coefficient of the discount factor is negative and quite

significant in all four equations. The time trend coefficient is negative when MU1 is tested, but positive for MU2. The serial correlation coefficient is smaller when instrumental variables are used.

The physical depletion index, MU1, performs best, in the context of this model, for the Canadian iron ore industry. It may be that the price index for iron products, used to construct MU2, is fairly far removed from a price index for iron ore. This is consistent with the fact that the two proxies are negatively correlated.

XII. Summary and conclusion.

This essay begins by clarifying the role of depletion, as opposed to exhaustion, in the extraction of natural resources. In the presence of depletion, "a progressive change in the quality of a resource or in the current cost of producing the resource product" [Herfindahl, 1967, p. 63], output will be restricted, whether exhaustion occurs or not. This will occur under competition or monopoly. Substantially the same effect occurs when the resource is renewable: a nonrenewable resource is, in the limit, just a special case of a renewable resource.

When the role of factors of production is considered, the same output restriction effect manifests itself. Once the optimal, restricted, output level is selected, it will be produced with that combination of labor and capital which minimizes the current cost of extraction. This is true for both renewable and nonrenewable resources.

From consideration of the cost minimization problem for a nonrenewable resource, an empirical framework is derived to test the influence of depletability and exhaustibility on the extraction of natural resources. The test is applied to four mineral industries, and several proxies for the scarcity value of reserves are tested for each industry. The proxies are constructed to reflect both physical and market measures of scarcity.

In each industry, at least one proxy for the scarcity of reserves is a significant explanatory variable in equations determining cost minimizing expenditures on labor. In three of the four industries, both physical and market measures of depletion work well within the

context of the model.

The empirical tests proceed under the assumption of a Cobb-Douglas form for the cost function. Although the results indicate that this is not a completely suitable choice, no weaker assumption will yield tractable, linear estimating equations.

The statistically significant role of proxies for the scarcity value of reserves in explaining expenditures on labor in diverse mineral industries confirms the importance of depletion as a force in the management of mineral industries.

Appendix I: Derivation of (24)

For the maximand

$$I[X(t),q(t),t] \tag{A1.1}$$

where $q(t) = \dot{X}(t)$, the Euler equation of the calculus of variations is

$$\frac{\partial I}{\partial X} = \frac{d}{dt}\left(\frac{\partial I}{\partial q}\right) \tag{A1.2}$$

Integrating both sides of (A1.2) from time t to time T gives

$$\int_{\alpha=t}^{T} I_1(\alpha)d\alpha = I_2(T) - I_2(t) \tag{A1.3}$$

or

$$\int_{\alpha=t}^{T} I_1(\alpha)d\alpha + I_2(t) = I_2(T) \tag{A1.4}$$

Thus (24) amounts to the statement that the constant of integration, $I_2(T)$, in the integral form of the Euler equation, is zero for this problem. This may be seen by deriving the integral form of the Euler equation directly. Suppose the object is to maximize the following integral:

$$\int_{t=0}^{T} I\{\int_{\alpha=0}^{t} q(\alpha)d\alpha, q(t), t\}dt \tag{A1.5}$$

where α is a dummy variable which represents time under the integral sign in the first argument of I. Supposing an optimal path for $q(t)$ to have been found, we may impose an arbitrary variation $\nu(t)$ about that solution:

$$(A1.6) \qquad J(\gamma) = \int_{t=0}^{T} I\{ \int_{\alpha=0}^{t} [q(\alpha) + \gamma v(\alpha)]d\alpha, q(t) + \gamma v(t), t\}dt$$

where γ is a scalar. Take the first derivative of $J(\gamma)$ with respect to γ at $\gamma = 0$ and set the resulting expression equal to zero; the result will be a necessary condition for the optimal path $q(t)$.

$$(A1.7) \qquad \frac{dJ(0)}{d\gamma} = \int_{t=0}^{T} \{I_1(t) \int_{\alpha=0}^{t} v(\alpha)d\alpha + I_2(t)v(t)\}dt$$

$$(A1.8) \qquad = \int_{t=0}^{T} v(t)\{ \int_{\alpha=t}^{T} I_1(\alpha)d\alpha + I_2(t)\}dt$$

The transition from (A1.7) to (A1.8) involves reversal of the order of integration of the first right hand side term of (A1.7). Since the end point of this problem is not fixed, we do not assume

$$(A1.9) \qquad \int_{t=0}^{T} v(t)dt = 0$$

Thus the necessary condition is obtained by setting the bracketed term in (A1.8) equal to zero for all t, and this is just (24).

Less directly, (25) may be obtained by integrating (23) from t to T:

$$(A1.10) \qquad p(t) = C_2(t) + \int_{\alpha=t}^{T} e^{-r(\alpha-t)}C_1(\alpha)d\alpha + e^{-r(T-t)}[p(T) - C_2(T)]$$

and substituting in (19) to obtain

$$(A1.11) \qquad L = \int_{t=0}^{T} e^{-rt}\{[C_2(t) + \int_{\alpha=t}^{T} e^{-r(\alpha-t)}C_1(\alpha)d\alpha]q(t) - C(t)\}dt + e^{-rT}[p(T) - C_2(T)] \int_{t=0}^{T} q(t)dt$$

If exhaustion occurs, $\int_{t=0}^{T} q(t)dt = R$; substituting in (A1.11) implies

$$(A1.12) \qquad \lambda = \frac{\partial L}{\partial R} = e^{-rT}[p(T) - C_2(T)]$$

If exhaustion does not occur,

$$(A1.13) \qquad \lambda = \frac{\partial L}{\partial R} = 0$$

since in that case R does not appear in the Lagrangian L.

Appendix II: Alternative expectational assumptions

The empirical results presented in the text assume perfect foresight. I will discuss the results of testing two alternative assumptions against the data for the U. S. coal industry. An adaptive expectations format cannot be used: imposing an adaptive expectations scheme on a model in which serial correlation is present produces a model which yields inconsistent estimates under least squares estimation. A second approach is based on an Almon polynomial approximation. Suppose that

$$\ln \delta X = \sum_{i=0}^{11} \beta_i \ln \delta X_0(-i) \tag{A2.1}$$

Expected normal output is a function of the observed output for the previous year. The coefficients β_i are unknown; I assume that they can be approximated by a third degree polynomial

$$\beta_i = a_0 + a_1 i + a_2 i^2 + a_3 i^3 \tag{A2.2}$$

Substituting (A2.2) into (A2.1) and rearranging terms gives

$$\ln \delta X = \sum_{j=0}^{3} a_j \sum_{i=0}^{11} i^j \ln \delta X_0(-i) \tag{A2.3}$$

$$= \sum_{j=0}^{3} a_j Z^{xj} \tag{A2.4}$$

Comparison of (A2.4) and (A2.3) defines the variables Z^{xj}, $j = 0, 1, 2, 3$. Similarly we may define

$$\ln w = \sum_{j=0}^{3} b_j Z^{wj} \tag{A2.5}$$

$$\text{(A2.6)} \qquad \ln p^k = \sum_{j=0}^{3} c_j Z^{kj}$$

Substituting from (A2.4) - (A2.6) into (120) gives

$$\begin{aligned}\ln w_0 L_0 = {} & K + A_0 \ln \delta X_0 + Aa_0 Z^{x0} + Aa_1 Z^{x1} + Aa_2 Z^{x2} + Aa_3 Z^{x3} \\ & + B_0 \ln w_0 + Bb_0 Z^{w0} + Bb_1 Z^{w1} + Bb_2 Z^{w2} + Bb_3 Z^{w3} \\ & + D_0 \ln p_0^k + Dc_0 Z^{k0} + Dc_1 Z^{k1} + Dc_2 Z^{k2} + Dc_3 Z^{k3} \\ & + E \ln \mu + F \ln (1 + r) + \lambda t + \epsilon\end{aligned} \qquad \text{(A2.7)}$$

Thus the Almon polynomial procedure requires the following steps. Construct the 12 Z variables, defined by (A2.4) - (A2.6). Regress $\ln w_0 L_0$ on a constant term and the 17 variables on the right hand side of (A2.7). This has the advantage of allowing the data to provide estimates of the weights a, b, and c which determine the coefficients of the sum (A2.2) (similarly for ln w and $\ln p^k$). Unfortunately, the procedure provides estimates only of products, as Aa_0, Aa_1, and so on. It is not possible to recover estimates of a_0, a_1, ..., c_2, c_3. The procedure also masks the estimates of A, B, and D. The coefficient E must be directly estimated; it cannot be derived from the homogeneity constraint. Table 11 gives the results of estimating (A2.7) for MU1 (BTU yield), MU2 (cumulative extraction) and MU3 (current price) for the U. S. coal industry.

The equations were estimated using the Cochrane-Orcutt iterative technique. In the event that collinearity between a set of variables is perfect, to the precision of the computer package, the last variable

Table 11: Almon polynomial scheme, U. S. coal industry

Variable	Coefficient MU1	MU2	MU3
Constant	23.5609 (12.2382)	3.1794 (1.7740)	0.3744 (0.3558)
$\ln \delta X_0$	0.2233 (4.7984)	0.2336 (4.3830)	0.2101 (3.9319)
z^{x0}	0.1325 (2.6051)	0.0769 (1.2315)	0.1529 (3.0763)
z^{x1}	-0.0290 (2.8339)	-0.0143 (0.6049)	-0.0370 (1.8603)
z^{x2}	0.0	0.0012 (0.6519)	0.0028 (1.7029)
z^{x3}	0.0002 (2.1424)	0.0	0.0
$\ln w_0$	0.7058 (1.3022)	0.6141 (1.1207)	1.0150 (2.3006)
z^{w0}	0.4924 (1.2290)	0.2040 (0.4576)	0.0041 (0.0161)
z^{w1}	-0.3672 (1.3399)	-0.2144 (0.6821)	0.0233 (0.2350)
z^{w2}	0.0765 (1.4868)	0.0367 (0.6145)	-0.0032 (0.4009)
z^{w3}	-0.0045 (1.5887)	-0.0019 (0.5662)	0.0
$\ln p_0^k$	0.5419 (2.3521)	-0.0194 (0.0620)	-0.1984 (0.8510)
z^{k0}	0.0503 (0.6073)	0.0215 (0.2064)	-0.0045 (0.4348)
z^{k1}	-0.0143 (0.4422)	0.0012 (0.0289)	0.0
z^{k2}	0.0009 (0.3345)	-0.0011 (0.3075)	0.0

Table 11: Continued

Variable	Coefficient		
	MU1	MU2	MU3
Z^{k3}	0.0	0.0	0.0
ln μ	383.746 (14.4135)	42.3801 (4.9154)	0.3920 (6.7760)
ln (1 + r)	-10.9532 (3.1416)	-0.0521 (0.0094)	5.8577 (1.3495)
Time	-0.1386 (14.9690)	-0.5497 (5.3735)	-0.004034 (10.0464)
R^2	0.9138	0.8901	0.8874
Durbin-Watson	2.0621	2.3849	2.1605
Degrees of freedom	241	241	241
Rho	0.2909 (4.9021)	0.6568 (14.0440)	0.4751 (8.7070)

T-statistics in parentheses

is suppressed from the regression and a zero reported as the coefficient of the suppressed variable.

In the equation using MU1 (BTU yield) as a shadow price, the coefficient of current output is positive and significant. The first two expected output variables are significant, and the third is suppressed due to collinearity. The fourth output term is significant, but of slight magnitude.

The coefficient of the current wage term is positive, less than one, and moderately significant. The expected wage terms vary in sign and magnitude but are always significant. The coefficient of current capital cost is positive, less than one, and quite significant. The coefficients of current wage and current capital cost sum to more than one, which violates the homogeneity constraint. The coefficients of the expected capital cost variables are of slight magnitude and significance.

The coefficient of the BTU index, although highly significant, is about 383, which is certainly greater than one.

The discount factor has a negative and significant coefficient. The time trend is negative, but is approximately 100 times larger than the time trend estimated when perfect foresight is assumed.

The pattern of the output and wage coefficients is little changed when MU2 (cumulative output) is substituted for MU1 as a shadow price proxy. The coefficient of current capital cost becomes negative and insignificant, and none of the expected capital cost coefficients are significant. The coefficient of MU2 is about 42, far too large. It is, however, statistically significant. The time trend is also quite large.

When MU3 (current price) is used as a shadow price proxy, the coefficient of current wage exceeds one in magnitude. The expected wage variables have small and insignificant coefficients. The capital cost coefficients are negative and insignificant. The coefficient of MU3, however, is positive, less than one, and highly significant. The time trend is of reasonable magnitude.

I view these results as mixed. All three shadow price proxies have coefficients which are highly significant. But the pattern of coefficients violates the homogeneity constraint. In particular, these results seem to cast little light on the role of expectations.

An alternative assumption allows imposition of the homogeneity constraint. Suppose that expected normal output is a weighted average of output 3, 6, 9 and 12 months ago:

$$\text{(A2.8)} \quad \ln \delta X = a_3 \ln \delta X_0(-3) + a_6 \ln \delta X_0(-6) + a_9 \ln \delta X_0(-9) + a_{12} \ln \delta X_0(-12)$$

where

$$\text{(A2.9)} \quad a_3 + a_6 + a_9 + a_{12} = 1$$

(A2.9) may be substituted into (A2.8) to eliminate a_{12} from the expression for ln δX. Similar expressions may be derived for expected normal wage and expected normal capital cost. The resulting expressions may be substituted into (120) to obtain an equation upon which the homogeneity constraint may be imposed. Table 12 gives the results of estimating that equation for the U. S. coal industry.

The pattern of estimated coefficients is remarkably similar for

Table 12: Weighted average expectations, U. S. coal industry

Variable	Coefficient		
	MU1	MU2	MU3
Constant	9.5638 (11.7657)	9.6065 (12.0005)	9.9287 (13.9232)
ln δX_0	0.3168 (8.0432)	0.3172 (8.0532)	0.3173 (8.0100)
ln δX	-0.2787 (3.9639)	-0.2782 (3.9567)	-0.2645 (3.7996)
ln w_0	1.0620 (4.2758)	1.0588 (4.2512)	0.9812 (4.1757)
ln w	-0.2880 (0.6403)	-0.2998 (0.6624)	-0.5696 (1.7064)
ln p_0^k	0.4243 (1.3504)	0.4178 (1.3305)	0.2861 (1.0162)
ln p^k	0.3250 (2.0121)	0.3238 (1.9970)	0.2777 (1.8185)
ln μ	-0.5234 (0.8970)	-0.5006 (0.8475)	0.0246 (0.3586)
ln (1 + r)	-8.0566 (1.3235)	-7.9308 (1.3041)	-5.4215 (0.9882)
Time	-0.002094 (2.1062)	-0.001605 (1.8285)	-0.001753 (2.0923)
R^2	0.9097	0.9372	0.9036
Durbin-Watson	2.5510	2.5507	2.5409
Degrees of freedom	241	241	241
Rho	0.9258 (39.4231)	0.9262 (39.5371)	0.9205 (37.9100)

the three shadow price proxies. The output coefficient is about 0.3, and highly significant. The expected output coefficient is about -0.27, and highly significant. The current wage coefficient is very near one, and statistically significant. The expected wage coefficient is negative, and insignificant except for MU3.

The current capital cost coefficient is positive, less than one, and slightly significant for MU1 and MU2. The expected capital cost coefficient is positive, less than one, and significant in each case.

The estimated coefficients of MU1 and MU2 are about -0.5 in magnitude, and not statistically significant. The coefficient of MU3 is barely positive, and not significantly different from zero.

The coefficient of the discount factor is in each case negative and slightly significant. The time trend is significant and of reasonable magnitude.

For each shadow price proxy, the serial correlation coefficient is about 0.9, and immensely significant. This is high even compared with the estimates presented in the body of the paper. This indicates that important forces are not captured by the weighted average expectations assumption.

Having made these preliminary investigations with little success, it was decided to concentrate on the perfect foresight assumption.

Appendix III: Empirical results for two Canadian mineral industries

The empirical framework derived in Section VI was applied to the Canadian asbestos industry and the Canadian coal industry. Although the shadow price proxies were frequently significant, the pattern of coefficients is far out of the bounds implied by the Cobb-Douglas form of the cost function. This indicates that the Cobb-Douglas approximation is not a suitable one for these two industries. For completeness, the results of estimating (124) and (125) for these industries are reported here.

The shadow price proxies for asbestos are MU1, cumulative extraction, and MU2, a monthly price index from Statistics Canada, Prices and Price Indexes. The sample period is January 1962 to September 1972, 129 observations in all. Over that period, MU1 rises from 1.00308 to 1.65788. Within that period, MU2 fluctuates between 2.40253 (January 1969) and 2.60216 (January 1962).

Table 13: Shadow price proxies for Canadian asbestos

	Mean	Standard Deviation	Correlation with MU1	Correlation with MU2
MU1	1.31159	0.191551	1.000000	-0.818934
MU2	2.48288	0.0575367		1.000000

The two proxies move in distinctly opposite ways. Table 14 gives estimates of (124) and (125) for the Canadian asbestos industry; where instrumental variables are used, they are the same as those used for the Canadian gold industry.

Table 14 - Canadian Asbestos Industry

	Constant	Current Output	Expected Output	Current Wage	Expected Wage	Current Capital Cost	Expected Capital Cost	Shadow Price Proxy	Discount Factor	Time	Rho	R^2	DOF
14.1	11.8961	0.0882		1.6692		0.2469		-0.9161	-3.9844	0.003791	0.1215	0.5805	122
MU1	(26.1622)	(3.6443)		(7.7558)		(1.1630)		(2.5426)	(0.9103)	(3.5209)	(1.3845)		
14.2	13.0050	0.0792		1.3267		-0.1209		-0.2058	3.3534	0.000839	0.1885	0.9175	122
MU2	(36.2233)	(3.1523)		(6.1568)		(0.6781)		(0.6818)	(0.8894)	(1.3171)	(2.1710)		
14.3	12.9436	0.0610	-0.0269	1.2453	-0.1142	0.5375	-0.1002	-0.5686	-10.4001	0.003206	0.0658	0.6406	83
MU1	(11.2705)	(1.6741)	(0.6693)	(2.9297)	(0.2402)	(1.9662)	(2.6122)	(0.7321)	(1.7546)	(1.0931)	(0.6328)		
14.4	13.7794	0.0567	-0.0297	1.0860	-0.3132	0.4551	-0.1123	-0.1057	-8.4421	0.001009	0.0467	0.8796	83
MU2	(34.0189)	(1.5577)	(0.7430)	(3.2416)	(0.8312)	(1.9498)	(3.3531)	(0.3424)	(1.6852)	(1.2392)	(0.4488)		
Instrumental variables:													
14.5	11.4517	0.0604		2.2279		0.3226		-1.5505	-7.4135	0.006889	0.0185	0.6195	77
MU1	(18.5171)	(2.6123)		(6.9520)		(1.0569)		(2.9404)	(1.2188)	(4.0242)	(0.1682)		
14.6	13.8976	0.0368		2.0776		-0.0177		-1.0599	-0.8438	0.000641	0.0506	0.8899	77
MU2	(40.4782)	(1.6116)		(8.0657)		(0.0841)		(3.3375)	(0.2085)	(0.9757)	(0.4618)		
14.7	11.8147	0.0613	-0.0073	1.8449	0.0839	0.4102	-0.0420	-1.2971	-9.3079	0.006162	0.0030	0.6263	74
MU1	(14.8244)	(1.6629)	(0.1802)	(3.9318)	(0.2967)	(1.2809)	(1.0913)	(2.0118)	(1.4623)	(3.2673)	(0.0270)		
14.8	14.0615	0.0499	-0.0141	1.5843	0.3601	0.2492	-0.0510	-1.1426	-6.0744	0.000194	-0.0174	0.8950	74
MU2	(36.4712)	(1.4160)	(0.3603)	(4.5288)	(1.2250)	(1.0088)	(1.5225)	(3.0901)	(1.2330)	(0.2018)	(0.1589)		

MU1: Cumulative extraction
MU2: Current price

T-statistics in parentheses

Recall that it is an implication of the assumed Cobb-Douglas form of the cost function that the coefficient of each factor price should be positive and less than one. This is clearly not the case in Table 14. The estimated time trend is positive. Further, the serial correlation coefficient is small (negative in 14.8) and not statistically significant. This is surprising for monthly data, and may reflect on the quality of the data.

The shadow price proxies for the Canadian coal industry are cumulative extraction, from Historical Statistics of Canada, and a price index for coal, from Prices and Price Indexes. Over the period January 1950 to March 1972, MU1 rises from 1.00234 to 1.39089. MU2 fluctuates between 1.18479 (March 1972) and 1.67365 (February 1951).

Table 15: Shadow price proxies for Canadian coal

	Mean	Standard Deviation	Correlation with MU1	MU2
MU1	1.20566	0.103570	1.000000	-0.813045
MU2	1.46082	0.109767		1.000000

Table 16 gives estimates of (124) and (125) for the Canadian coal industry; the same list of instrumental variables is used. Again, the shadow price proxies (especially MU2) are sometimes significant, but the pattern of coefficients runs against the implications of the Cobb-Douglas assumption.

Table 16 - Canadian Coal Industry

	Constant	Current Output	Expected Output	Current Wage	Expected Wage	Current Capital Cost	Expected Capital Cost	Shadow Price Proxy	Discount Factor	Time	Rho	R^2	DOF
16.1	13.8478	0.1840		1.1706		-0.0568		-0.1138	0.1762	-0.004109	0.3980	0.9289	260
MU1	(63.0727)	(5.7931)		(6.9685)		(1.0711)		(0.5973)	(0.0976)	(15.6263)	(7.0760)		
16.2	14.0435	0.1724		1.2841		-0.0161		-0.2680	-0.5560	-0.004793	0.3889	0.8665	260
MU2	(58.6637)	(5.6017)		(11.2125)		(0.3076)		(1.9881)	(0.3152)	(10.8712)	(6.8841)		
16.3	13.7846	0.1966	-0.0270	1.1887	0.4906	-0.1212	0.0363	-0.5944	2.4375	-0.005239	0.3860	0.9253	221
MU1	(52.6381)	(3.8001)	(0.5222)	(2.9459)	(1.4465)	(1.7400)	(0.9122)	(1.1564)	(0.8966)	(6.6066)	(6.3455)		
16.4	14.2601	0.2269	-0.0533	2.0671	-0.1249	-0.0532	0.0537	-0.9428	1.0956	-0.005917	0.3349	0.8715	221
MU2	(50.9355)	(4.4884)	(1.0614)	(7.0211)	(0.4762)	(0.8664)	(1.4500)	(1.9832)	(0.4495)	(9.8717)	(5.3900)		
Instrumental variables:													
16.5	13.8202	0.1825		1.3160		-0.1409		-0.1752	3.5830	-0.004223	0.4000	0.9173	215
MU1	(54.0554)	(5.1328)		(3.2558)		(1.9025)		(0.4114)	(1.2396)	(11.0675)	(6.4886)		
16.6	14.0493	0.1933		1.9313		-0.0246		-0.9067	-0.3325	-0.005512	0.4201	0.8556	215
MU2	(50.3259)	(5.3952)		(4.4449)		(0.2489)		(1.8184)	(0.0925)	(7.1158)	(6.8826)		
16.7	13.8566	0.2037	-0.0045	1.2714	0.5574	-0.1582	0.0551	-0.7257	3.7645	-0.004257	0.3567	0.9186	212
MU1	(53.7960)	(3.8919)	(0.8427)	(3.0032)	(1.5560)	(1.9379)	(1.3994)	(1.3432)	(1.2005)	(10.7629)	(5.6754)		
16.8	14.3907	0.2723	-0.1060	2.0736	0.5840	-0.0222	0.0850	-1.7204	-0.4974	-0.006916	0.3260	0.8606	212
MU2	(47.7634)	(4.8488)	(1.9128)	(4.1566)	(2.0045)	(0.2357)	(2.2229)	(3.3805)	(0.1449)	(7.8108)	(5.1256)		

MU1: Cumulative extraction
MU2: Current price

T-statistics in parentheses

Appendix IV: An alternative depletion function

The model developed in Sections III and IV assumes that the extraction of one unit of output reduces reserves by precisely one unit. Implicitly, this model presupposes a mineral lying in pure form at greater and greater depths; as extraction proceeds from deposits which are near the surface to those at greater depths, the cost of extraction rises because the ore must be moved over a greater distance to reach the surface.

Such a model may not tell the whole story for a mineral like copper, which lies in the ground in ore-bearing dirt. A ton of ore-bearing dirt may yield 1% pure copper or 2% pure copper, depending on the application of cooperating factors (labor and capital) to the ton of dirt. We may imagine that production of an output $\delta X(L_t,K_t,R_t)$ engenders a depletion of the reserve stock

$$D[\delta X(L_t,K_t,R_t),L_t,K_t] \tag{A4.1}$$

Depletion is a function not only of output δX_t, but of the labor and capital combination chosen to produce that output. The reserve stock at time t is given by

$$R_t = R_0 - \sum_{\alpha=0}^{t-1} D[\delta X(L_\alpha,K_\alpha,R_\alpha),L_\alpha,K_\alpha] \tag{A4.2}$$

instead of (38). The exhaustion constraint alternative to (39) is

$$R_0 \geq \sum_{t=0}^{\infty} D[\delta X(L_t,K_t,R_t),L_t,K_t] \tag{A4.3}$$

The Lagrangian corresponding to the industry constrained profit maximization problem is

$$L = \sum_{t=0}^{\infty}(1 + r)^{-t}\{p_t\delta X(L_t,K_t,R_t) - w_tL_t - p_t^kK_t$$

$$+ \mu_{2t}[R_0 - \sum_{\alpha=0}^{t-1} D[\delta X(L_\alpha,K_\alpha,R_\alpha),L_\alpha,K_\alpha] - R_t]\} \tag{A4.4}$$

$$+ \mu_3\{R_0 - \sum_{t=0}^{\infty} D[\delta X(L_t,K_t,R_t),L_t,K_t]\}$$

$$L = \sum_{t=0}^{\infty}(1 + r)^{-t}\{p_t\delta X(L_t,K_t,R_t) - w_tL_t - p_t^kK_t - \mu_{2t}R_t$$

$$- [(1 + r)^t\mu_3 + \sum_{\alpha=t+1}^{\infty}(1 + r)^{\alpha-t}\mu_{2\alpha}]D[\delta X(L_t,K_t,R_t),L_t,K_t]\} \tag{A4.5}$$

$$+ R_0[\mu_3 + \sum_{t=0}^{\infty}(1 + r)^{-t}\mu_{2t}]$$

The corresponding conditions for profit maximization are

$$(1 + r)^{-t}w_t = \{(1 + r)^{-t}p_t - [\sum_{\alpha=t+1}^{\infty}(1 + r)^{-\alpha}\mu_{2\alpha} + \mu_3]D_{1t}\}\delta X_1(L_t,K_t,R_t)$$

$$- [\sum_{\alpha=t+1}^{\infty}(1 + r)^{-\alpha}\mu_{2\alpha} + \mu_3]D_{2t} \tag{A4.6}$$

$$(1 + r)^{-t}p_t^k = \{(1 + r)^{-t}p_t - [\sum_{\alpha=t+1}^{\infty}(1 + r)^{-\alpha}\mu_{2\alpha} + \mu_3]D_{1t}\}\delta X_2(L_t,K_t,R_t)$$

$$- [\sum_{\alpha=t+1}^{\infty}(1 + r)^{-\alpha}\mu_{2\alpha} + \mu_3]D_{3t} \tag{A4.7}$$

$$(1 + r)^{-t}\mu_{2t} = \{(1 + r)^{-t}p_t - [\sum_{\alpha=t+1}^{\infty}(1 + r)^{-\alpha}\mu_{2\alpha} + \mu_3]D_{1t}\}\delta X_3(L_t,K_t,R_t) \tag{A4.8}$$

plus the Kuhn-Tucker conditions

$$(A4.9) \qquad \mu_{2t}\left[R_0 - \sum_{\alpha=0}^{t-1} D\left[\delta X(L_\alpha, K_\alpha, R_\alpha), L_\alpha, K_\alpha\right] - R_t\right] = 0$$

$$(A4.10) \qquad \mu_3\left\{R_0 - \sum_{t=0}^{\infty} D\left[\delta X(L_t, K_t, R_t), L_t, K_t\right]\right\} = 0$$

Having interpreted (43) - (45), the interpretation of (A4.6) - (A4.8) is straightforward. In this alternative model, employment of labor produces a marginal unit of output, and contributes to a marginal reduction in depletion; thus labor simultaneously produces two outputs, the mineral and reserves. (A4.6) states that employment of labor will be optimal when the discounted cost of the last unit of labor hired just equals the net value of the marginal mineral product of that unit of labor, plus the shadow value of the saving of reserves which results from employment of the last unit of labor. In other words, labor will be employed until its cost equals the sum of the value of the marginal products of labor - the value of the marginal mineral output plus the value of the marginal reserve output. (A4.7) is subject to a similar interpretation. (A4.8) defines the shadow value of a unit of reserves as the net value of the marginal product of a unit of reserves.

Dividing (A4.6) by (A4.7), we see that in this alternative model, (48) does not hold. Output in period t will not necessarily be produced with that combination of labor and capital which minimizes the within period cost of extraction. Some additional labor or capital may be employed to economize reserves for the future; the future saving will justify the current expense.

This model has some attractive features, and the substitution it allows between current factor inputs and future reserves is sometimes described in the natural resource literature. However, it seems to make the concept of reserves even more ambiguous than usual. In particular, it would seem that processed ore-bearing dirt, deposited in slag heaps, could yield further refined mineral product upon application of sufficient labor and capital.

The alternative depletion function (A4.1) may be substituted in the cost minimization problem, and used to derive the discounted minimum cost function (66).

Footnotes

1. Hotelling [1931, pp. 140-141] actually derives the competitive equilibrium condition by a written argument, which is quoted below. He does not use the calculus of variations, although it is clear that he has such techniques in mind, and they are used elsewhere in his paper. Gordon [1967, p. 277] uses just the format of (5) and (6), allowing for notational differences. Note that (5) and (6) treat $q(t)$ as a state variable when it is clearly a flow variable. It would be better to write

$$(5)' \qquad \frac{\partial I}{\partial X} = \frac{d}{dt}\left(\frac{\partial I}{\partial q}\right)$$

for

$$(6)' \qquad I[X(t),q(t),t] = e^{-rt}\{[p(t) - \lambda e^{rt}]q(t) - C[q(t)]\}$$

which gives as a necessary condition (since $X(t)$ does not appear on the right hand side of (6)')

$$(7)' \qquad 0 = \frac{d}{dt}\{p(t) - \lambda e^{rt} - C'[q(t)]\}$$

Integrating (7)' from time 0 to time t gives (13) directly. The main difference seems to be that the role of the Lagrangian multiplier λ appears in a much less obvious way when $q(t)$ is regarded as a flow variable. When the alternative form (24) of the Euler equation is applied to (5)' and (6)', the results of the text are obtained.

2. See Nordhaus [1973] for a similar interpretation of the price-marginal cost gap.

3. See Scott [1953] for a general discussion of user cost, and Nordhaus [1973] for an application to natural resources.

4. In Hotelling's terminology, it remains possible to argue that net price rises at the rate of interest, provided that it is clear that net price equals gross price, less marginal cost, less user cost. In particular, price less marginal cost does not rise over time at the rate of interest.

5. See Vernon Smith [1975] for a discussion of the management of renewable resources.

6. Define a new cost function C^* by the relation

$$C[X(t),q(t)] = C^*[R_0 - X(t),q(t)] = C^*[R(t),q(t)]$$

The first argument of C^* is the reserve base at time t, rather than cumulative extraction; Hotelling's analysis carries through using C^* rather than C, with a few sign changes.

7. See Adelman [1971, p. 40] and Zimmerman [1975, pp. 196A-198] for discussions of user cost in the oil industry and the U. S. coal industry, respectively.

8. See Appendix IV for a discussion of an alternative model of depletion.

9. It should be clear that in this model both R_t and μ_{2t} are endogenous: they are determined completely in terms of labor and capital inputs. One can imagine setting the problem up without a variable R_t, by substituting repeatedly from (37) to eliminate R_t. As may be seen

$$R_1 = R_0 - \delta X(L_0,K_0,R_0)$$

$$R_2 = R_0 - \delta X(L_0,K_0,R_0) - \delta X[L_1,K_1,R_0 - \delta X(L_0,K_0,R_0)]$$

the notation involved in such a formulation rapidly becomes unwieldy. It is more workable to define the endogenous variable R_t and constrain its movement via the shadow price μ_{2t}.

10. In (66), the sequence $\{(1 + r)^{-t}\mu_{2t}\}$ should be regarded as an implicit function of the wage sequence, the capital cost sequence, and initial reserves R_0. When uncertainty is introduced, it makes more sense to work with the expected shadow price of a resource stock of uncertain magnitude, than the expected magnitude of the uncertainly known resource stock.

11. "Exhaustion" may occur in finite or infinite time; at any moment population may be positive, but population may asymptotically go to zero.

12. However, see Vernon Smith [1975] for evidence that extinction through overharvesting may have occurred.

13. See page 107 for a similar argument concerning nonrenewable resources.

14. See Salant [1976], Stiglitz [1976], and Hoel [1976] for discussions of related problems.

15. More generally

$$L_t = \frac{\partial C}{\partial(1+r)^{-t}w_t} \qquad K_t = \frac{\partial C}{\partial(1+r)^{-t}p_t^k} \qquad R_t = \frac{\partial C}{\partial(1+r)^{-t}\mu_{2t}}$$

16. It is still assumed that expected profit is maximized. Risk aversion does not appear.

17. It is useful to think of the Cobb-Douglas function as a second-order translog function with the second order terms set equal to zero. A second order approximation would be preferable to the linear Cobb-Douglas form, but the second order terms in the cost function would lead to a nonlinear equation explaining expenditures on labor. The influence of the omitted second order terms may explain some of the instances below in which estimated coefficients do not obey the implications of the Cobb-Douglas assumptions.

18. Of course, any of the other factor price coefficients (A_0, A, B_0, B) could be eliminated in place of E. Most of the equations presented below for the U. S. copper industry were reestimated with the coefficient of current wage (A_0) eliminated via the homogeneity constraint; the results were virtually identical to those presented in the text. Although the shadow price proxies are of arbitrary scale, since they appear in logarithmic form, any scale change will separate out in an additive manner, combining with the constant term. In particular, this allows the use of price indexes for the market price proxy.

19. The reported BTU values are (average BTU per pound)

1954	13100	1961	12790	1968	12530
1955	13000	1962	12790	1969	12450
1956	12990	1963	12760	1970	12290
1957	12990	1964	12750	1971	12120
1958	12990	1965	12710	1972	12025
1959	12840	1966	12650	1973	12005
1960	12830	1967	12580		

The equation which yielded the best fit was

$$BTU_t = 13330.9 - 4.44611T \qquad R^2 = 0.9101 \quad D.W. = 0.2968, \ DOF = 18$$

for T such that 1954 = 27, 1955 = 49, ..., 1973 = 265. The equation used to generate the BTU index was

$$MU1 = 12659.5/(13330.9 - 4.44611T)$$

for T = 1 to T = 265. The numerator of the above fraction is the mean of the reported BTU values; it is introduced for scaling purposes only.

20. See Zimmerman [1975, Chapter VII] for an application of this concept to the U. S. coal industry.

21. The reported average yield is (per cent)

1936	1.54	1949	0.91	1962	0.75
1937	1.29	1950	0.89	1963	0.74
1938	1.34	1951	0.90	1964	0.73
1939	1.25	1952	0.85	1965	0.70
1940	1.20	1953	0.85	1966	0.67
1941	1.15	1954	0.83	1967	0.63
1942	1.09	1955	0.83	1968	0.60
1943	1.04	1956	0.78	1969	0.60
1944	0.99	1957	0.77	1970	0.59
1945	0.93	1958	0.79	1971	0.55
1946	0.91	1959	0.74	1972	0.55
1947	0.90	1960	0.73		
1948	0.92	1961	0.75		

The equation which yielded the best fit was

$$AVYLD_t = 0.356291 + 158.575/T \qquad R^2 = 0.9601 \qquad D.W. = 0.5105$$

$$DOF = 34$$

The equation used to generate MU1 (copper) was

$$MU1 = 1/\{0.356291 + 158.575/(318 + T)\}$$

for T such that January 1951 = 1 and December 1973 = 276.

22. Dominion Bureau of Statistics, Review of Employment and Average Weekly Wages and Salaries (1957-67); Dominion Bureau of Statistics, Employment and average weekly wages and salaries (1968 - February 1971); Statistics Canada, Employment, earnings, and hours (March 1971 - end of sample period).

23. Since the factor price variables appear in the estimating equations in logarithmic form, we may use price indices as an alternative to price series; the constant factor will separate out additively, changing the constant term in the equation, but not the estimated coefficient of the factor price involved.

24. Industrial bond averages are reported by the Bank of Canada Statistical Supplement for January 1964 - March 1976. Both series performed equally well in preliminary tests, so the longer time series was used for tests reported here.

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Biographical Note

Stephen Martin was born in Mount Holly, New Jersey on November 21, 1948. He graduated from Mainland Regional High School in Linwood, New Jersey in June 1966, and attended Michigan State University as a mathematics major. In September 1969 he married Linda Louise Medica. In June 1970 he graduated from Michigan State University with a B. A., having been elected to Phi Beta Kappa. Following three years active duty with the United States Army, completed as a Specialist Fifth Class, he began graduate study at M.I.T. in September 1973, where he was a National Science Foundation Fellow. During 1974 he was a research assistant for Professor Morris Adelman, and for the 1976-77 academic year he was a teaching assistant in the Department of Economics. In August 1975 he became the proud father of Beth Ritchie Martin. Beginning in September 1977, he will be an Assistant Professor in the Michigan State University Department of Economics.