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STUDENTS' COMPETENCIES IN WORKING WITH FUNCTIONS IN SECONDARY MATHEMATICS EDUCATION—EMPIRICAL EXAMINATION OF A COMPETENCE STRUCTURE MODEL

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ABSTRACT. In the subject matter of functional relationships, a student's ability to translate from one form of representation to another is seen as a central competence. In the course of the HEUREKO project (heuristic work with representations of functional relationships and the diagnosis of mathematical competencies of students), a theoretical competence structure model comprising five competence dimensions was developed. These are based on four types of representation (graph, numerical table, algebraic equation and situational description) and correspond to the skill to translate from one type of representation to another. The following study was aimed to examine the facets of the model. The 5-dimensional model was empirically tested with a sample of $N = 645$ students of grades 9 and 10. This was accomplished by comparing competing item response models with regard to model fit using information criteria measures. In comparison with other possible model structures, our postulated 5-dimensional model showed the best model fit, suggesting that all translations are relevant for competence assessment and development. Furthermore, in order to allow for the identification of structural components of cognitive actions, the employed tasks used in the empirical testing were divided into categories with regard to the different demands of cognitive action. Our results suggest that cognitive actions may have a specific dimensional structure. These findings can contribute to a better diagnostic approach concerning specific strengths and weaknesses and can therefore foster students' competencies.

KEY WORDS: change of representation, competence structure model, elements of cognitive action, item response theory, representations of functional relationships

Introduction

Basic mathematical forms of representations and translating between them play central roles in subject-related didactics. Important contributions regarding mathematical forms of representations can be found in Kaput (1985), Goldin (1998) and Sierpinska (1992). In the area of functional relationships, typical forms of representations play a key role. These forms of representations can consist of graphs, numerical tables, algebraic equations and verbal descriptions (e.g. Duval, 1993; Schwarz & Dreyfus, 1993).

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Despite there being different emphases within the studies themselves, a general agreement exists that, using different basic mathematical forms of representations and translating between these forms, are considered key skills in mathematics (e.g. Swan, 1985; Ainsworth, Bibby & Wood, 2002).

Many studies focus on one specific translation process, especially the translation between the algebraic and graphical forms of representation (e.g. Schoenfeld, Smith & Arcavi, 1993; Zaslavsky, 1987; Kerslake, 1981; Markovits, Eylon & Bruckheimer, 1986). The results indicate that, using both forms of representations, can improve content-related understanding. Ongoing learning may, however, also be obstructed if the selected tasks are not adequate in relation to the selected form of representation (Schnotz & Bannert, 2003). Altogether, the main research interests in this area have been on particular difficulties in different forms of representations and translating between them (e.g. Kerslake, 1981; Leinhardt, Zaslavsky & Stein, 1990).

Further research findings concentrated on possible preferences that students may have when working with different forms of representations (Zaslavsky, 1987; Dreyfus & Eisenberg, 1981). These findings suggest that such a preferred form of representation exists and that students' preferences differ between contextualised and purely mathematical descriptions. For tasks containing purely mathematical descriptions, most students preferred using equations, whereas they tended to use graphs or tables in contextualised settings (Keller & Hirsch, 1998).

Translating between forms of representations was also considered in respect of aspects of cognitive psychology. Thomas et al. (2010) examined students' brain activities when working with different forms of representations. They focused on the difference between strategies used by novices and by experts. Their findings show that experts focused more on the essential characteristics of a function, which helped them execute the translation. In contrast, novices tried to capture the representation as a whole without identifying the key properties relevant to the translation. Furthermore, the results highlight the importance of representational versatility (in connection with the ability to work seamlessly within and between representations).

Another research approach focused on the effectiveness of actual teaching and learning models. Several studies confirmed that knowledge and the application of different translations between forms of representations support individual learning processes (e.g. Gagatsis & Elia, 2004; Ainsworth, Bibby & Wood, 2002). Studies in the field of media didactics examined the effects of different multimedia representations in combination with different instructional models. Most findings show that the skill

of being able to translate between different forms of representations leads to a deeper understanding, but this ability also requires a higher level of cognitive activity (e.g. Ainsworth, Bibby & Wood, 2002). Furthermore, students with little previous knowledge, in particular, find it difficult to coordinate and integrate different forms of representations (Ainsworth, 1999; Seufert, 2003).

A relatively new research perspective is the analysis of a learning progression, which focuses on the development of different competence levels. This is considered highly relevant to the organisation of the learning process. Most studies in this field can be found in Science Education (Wilson, 2008) whereas there are only few attempts in mathematics to develop and quantitatively assess the learning progression within a specific content. Wilmot et al. (2011) tried to generate an assessment tool using the item response theory to analyse the learning progression, by establishing connections across multiple representations of (linear) functions. They refer to the SOLO taxonomy, which is a generalised assessment framework predicting five levels of understanding with regard to students' abilities to connect different domains. They predicted one more level, thus yielding six levels of progression representing increased sophistication of students' understanding. The empirical results showed that such a cognitive framework is useful when measuring the learning progression, but a differentiation between all six levels could not be ascertained. Our assumption is that a closer look at the cognitive demands of the task and a more differentiated item pool is required to develop such a model.

Our study therefore includes a deeper analysis when dealing with mathematical forms of representations altogether. Are there any central forms of representations that should be focused on in learning activities or would it be necessary to involve all forms of representations and the respective translations to develop basic mathematical competencies in the field of functional relationships? In this article, a general analysis of the competence structure in the field of functional relationships is presented.

Additionally, we analyse different mathematical actions. Although we do not use SOLO taxonomy, there is a connection with this approach. Biggs & Tang (2007) use certain verbs to describe and differentiate the different levels of understanding. These verbs refer to different (mathematical) actions and could be seen as indicators for these levels. Some of the actions considered in our approach resemble these verbs or form specific combinations of them.

Considering the connection between mathematical actions and different levels of understanding, an overemphasis on specific actions cannot

be supported. For example, most studies in the field of functional relationships mainly consider interpretation tasks and only a few of them focus on construction tasks, although these are quite different actions (Leinhardt, Zaslavsky & Stein, 1990; Hattikudur et al., 2012). Bossé, Adu-Gyamfi & Cheetham (2011) found that it makes a major difference whether the task demands an action of interpretation or an action of construction and that student errors depend on the required action. Thus, the question arises if understanding functional relationships is connected with the consideration of different cognitive actions of students.

Hence, in our study, we consider two main aspects. Firstly, we would like to focus on the general competence structure and investigate the ability to translate between forms of representations systematically. Additionally, we implemented certain cognitive action demands to obtain a deeper understanding of the translation process. This is done in the context of psychometric differentiated modelling in order to meet an important prerequisite for the development of effective teaching and learning concepts.

Theoretical background

In the field of functional relationships, four main forms of representations can be considered: graph (G), numerical table (N), algebraic equation (A), and situational description (S). Each form of representation illustrates relevant aspects of a given function to varying extents. Graphical forms of representations for example, afford working with covariation because students can see directly how things change. In contrast, algebraic equations accentuate the aspect of correspondence within functions—for every x value, there is a corresponding y value (Thompson, 1994; Confrey & Smith, 1995).

This study is a study of translations among representations, something we know from the literature is important (e.g. Swan 1985; Ainsworth, Bibby & Wood, 2002).

The translation process can be referred to the psychological or cognitive processes involved by moving from one form of representation (source) to another one (target) (Janvier, 1987). It should be borne in mind "*that it is not the representations that are translated but rather the ideas or constructs expressed in them*" (Adu-Gyamfi, Stiff & Bossé, 2012, p.159). According to Janvier (1987), such translations possess directionality, for example, moving from an equation to a graph does not require the same psychological processes than moving from a graph to a

table. Janvier differentiated between direct and indirect processes, depending on whether students translate from a given representation to the target representation directly or whether a third (transitional) form of representation is used. Examining these processes in detail is particularly interesting on examining possible difficulties concerning certain translations between various forms of representation (Adu-Gyamfi, Stiff & Bossé, 2012). To investigate the underlying processes from a cognitive-psychological point of view, the analyses have to focus on an individual micro-level. Borromeo-Ferri (2006) used this approach, among others, and analysed the modelling processes of students whilst following the modelling circuit. Taking into account that our study does not consider a particular part but the translation process as a whole and also involves a written test, it appears to be quite difficult to detect underlying thinking processes with regard to directionality. Even though a given task provides an initial representation, it could be assumed that, on constructing the target representation, the thinking process is not a linear one but, instead, that multiple translations between the two forms of representations are made. In written test items, such translations cannot be traced. Thus, our study does not differentiate between the different directions, which the translation process takes when analysing the overall competence structure.

A theoretical competence model was developed, which was to be verified by empirical data. The term "competence model" is not used in literature consistently and can refer to different aspects. Basically, two general groups of theoretical models are distinguished to describe and explain competencies: models of competence levels and models of competence structures (Klieme, Hartig & Rauch, 2008). Models of competence levels help to understand the individual development of competencies as a process and describe the stage of development reached at certain points. Models of competence structure identify the general competence structure in a certain domain or, more specifically, in a certain content area within a domain. These models can be defined as a normative setting based on educational decisions or as a result of empirical analysis of the underlying competence structure. Competence structure models form the basis for the analysis of competence levels, as the general structure needs to be known before different levels can be identified. The aim of our study is therefore a domain-specific and empirically founded specification of the competence structure within the field of functional relationships. The basic idea of such structural modelling consists in understanding competencies as a multidimensional construct and in identifying the underlying dimensions. First of all, these potential dimensions have to be defined by means of theoretical

hypotheses of the composition of the content area focused on. Subsequently, this theoretical postulated dimensional structure has to be verified by empirical analysis.

In our theoretical competence structure model, the following forms of representations were integrated into the developed competence structure model: graph (G), numerical table (N), algebraic equation (A) and situational description (S). The translations between the forms of representations constitute the various dimensions within the anticipated competence structure model (whereas directionality was not considered):

1. Translation between graph and algebraic equation (GA)
2. Translation between graph and numerical table (GN)
3. Translation between graph and situational description (GS)
4. Translation between numerical table and algebraic equation (NA)
5. Translation between situational description and algebraic equation (SA).

A translation from situational description to numerical table, although possible theoretically, was not included in the model. With regard to the examined age group, this type of translation did not play a role on its own in mathematics classes and thus did not contribute to the examination of the competence structure. The five dimensions are presented in Fig. 1.

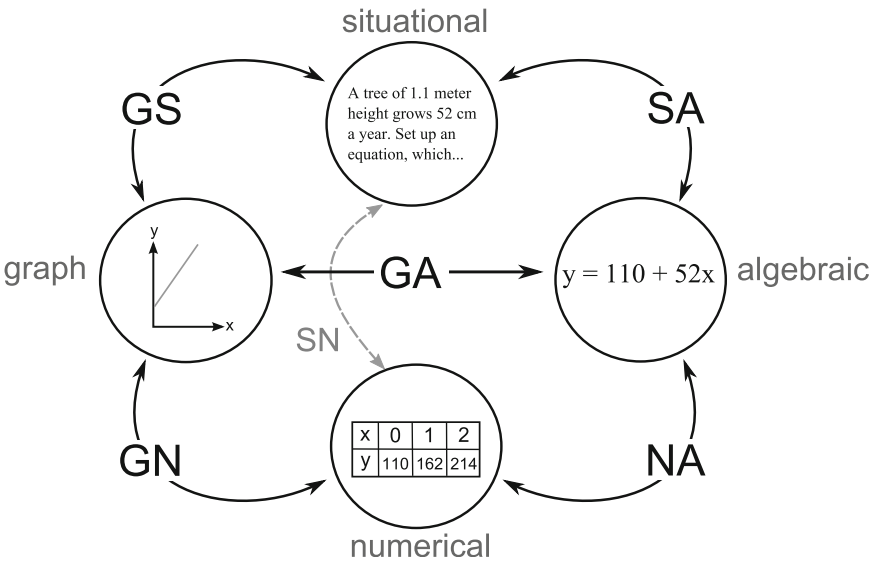


Figure 1. Five-dimensional competence structure model: The dimensions are depicted using arrows, showing the different translations between forms of representations (GS, SA, NA, GN, GA)

This study is limited to linear and quadratic functions, as these are the two types the examined age group is familiar with in the German curriculum. With regard to the validation of such competence structure model, other possible models with different dimensions have to be taken into account also. For this reason, the following alternative models will be analysed and compared to the postulated model:

- One-dimensional competence structure (1D)

The student's translation skills are assumed to be one single construct of skills and competencies. Hence, a single dimension was anticipated, containing all translations (GA, GN, GS, NA and SA).

In addition, four two-dimensional models were created subsequently. For these models, the assumption was that students' competencies could be separated into two dimensions differing in one dominating form of representation. The consideration was that the mathematics curriculum often focuses on specific forms of representations, whilst underrepresenting others (Leinhardt, Zaslavsky & Stein, 1990).

- Algebraic vs. non-algebraic dimension (2D A)

The algebraic form of representation is assumed to be the main influence of students' competencies (2D A). One dimension is made up of all translations including the algebraic equation (GA, NA, SA); the other dimension contains all translations without algebraic functions (GN, GS)

- Graphic vs. non-graphic dimension (2D G)

The graph is considered the predominant component (2D G). One dimension is made up of all translations containing a graph (GA, GN, GS); the other dimension contains all translations without a graph (NA, SA).

- Numerical vs. non-numerical dimension (2D N)

The numerical table is considered the main factor (2D N). The first dimension contains all translations with a numerical table (GN, NA). The second dimension contains all translations without a numerical table (GA, GS, SA).

- Situational vs. non-situational dimension (2D S)

The situational description is assumed to be the main influencing factor (2 D S). The first dimension contains translations with situational descriptions (GS, SA); the second dimension contains all other translations (GA, GN, NA).

In addition to an analysis of different translations, we also integrated different action demands into the tasks. Previous research findings suggest the assumption that the difficulty of translation tasks depends on the demanded translation action (Janvier, 1987; Bossé, Adu-Gyamfi & Cheetham, 2011). In order to demonstrate that mathematical action consists of different parts or elements, whilst taking into account that this refers to a cognitive process, we use the term "elements of cognitive action" to describe mathematical actions. These elements of cognitive action represent the students' different activities when working on mathematical tasks and highlight different educational requirements.

There are only few studies referring to students' mathematical actions during the translation process. Janvier (1987) considered a translation action to be a type of action associated with a specific translation. For example, moving from table to graph involves "plotting", whereas moving from graph to table contains "reading off". As we would like to investigate the elements of cognitive actions irrespective of any translation, we refer to a more general classification. Such a general categorisation of translation actions can be found in Leinhardt, Zaslavsky & Stein (1990). Here, tasks from previous studies were analysed and assigned to two main categories: interpretation and construction. Interpretation refers to all actions by which a student makes sense of or acquires a meaning from a specific form of representation. Construction requires an action of generating new parts that are not provided. With such categorisation in mind, different types of tasks are considered, differentiated by the aspect of "translation": "(*a*) *the act of recognizing the same function in different forms of representations*; (*b*) *identifying a specific transformation of a function in one representation and its corresponding transformation in another representation* (…); *or* (*c*) *constructing one representation of a function given another one*" (Leinhardt, Zaslavsky & Stein, 1990, p. 16).

This basic differentiation of actions is also used to categorise typical errors when working with different translations. Bossé, Adu-Gyamfi & Cheetham (2011), for example, differentiate between interpretation errors and implementation errors. The two main categories of interpretation and construction resemble the categories found within the framework of a conceptual system with a view to characterising students' actions, put forward by Bruder & Brückner (1989), which serves as the basis of our categorisation. In this conceptual system, the students' predominant action when working on tasks is considered a combination of cognitive operations and their respective links. Different terminological levels are introduced to describe the complexity of the activities performed: basic operations, complex operations of thinking, basic actions of acquirement,

basic actions and complex actions. Furthermore, a more precise classification has been undertaken at individual levels.

Concerning the field of functional relationships, two more actions were considered essential: *description* and *explanation*. Altogether, we examined four elements of cognitive action: When working with mathematical functions and their forms of representations and translations, students have to identify the most important aspects of the depicted form(s) of representation(s) as a basic step. The information provided has to be compared with the facts known already, so that relevant decisions in respect of this task can be taken. Compared to the categorisation of Leinhardt, Zaslavsky & Stein, the term of *interpretation* is changed to identification (I). This is done because interpretation can include validation of the results of a translation task, which involves more than just identifying the key properties in both representations. Here, a combination of different actions is therefore on hand (proving correctness, explaining why the solution is wrong and so forth). Identification refers to a more elementary action. Similarly, the verb "Identify" is used in SOLO taxonomy on the lowest level of understanding (Biggs & Tang, 2007). A second cognitive action is described as construction. In this case, the student is asked to translate a given form of representation of the function into another one he/she will create, with no new parts given in the task. To gain insight into the student's thinking process, the student can be asked to describe his/her approach verbally. Either the student has solved the construction task and is then asked to describe his/her solution process, or the student describes generally what he/she would do if he/she had to solve such task. The fourth element of cognitive action refers to tasks where the student is asked to explain his/her solution verbally or to demonstrate why a specific (given) solution is correct or incorrect. With regard to description, the student is thus asked what he/she does (or would do), whereas with regard to explanation, the student is asked to justify why the solution is (in)correct. These four elements of cognitive action were related to all types of translations.

Combining the types of translations and the elements of cognitive action within one competence structure model would result in far too many dimensions. Such a complex model would be very difficult to validate. Hence, we decided not to include both aspects in one model but to separate our analyses into two different parts. Hence, the elements of cognitive actions were not included in the five-dimensional competence structure model mentioned above, but they form a separate model, instead. In this model, we no longer differentiate between forms of

representations but, instead, postulate a structure encompassing the elements of cognitive action. We anticipated the following three-dimensional competence structure encompassing these actions: identification (I) and construction (C) form two separate dimensions. In the third dimension, the elements of cognitive action, description (D) and explanation (E), are summarised as a combined action (DE). In this model, we assume that the dimensions of Identification and Construction differ in their cognitive demands so that they cannot be represented in a single dimension. Explanation and description are combined, as both elements of cognitive action demand verbalisation, which is not necessarily a requirement for identification and construction. Hence, these elements have a different type of requirement profile. An additional specific feature of our model is the option of multiple item loading, i.e. one item can be related to more than one element of cognitive action. Identification of important values or characteristics in the initial representation is essential for task solving, so that all items are related to this element of cognitive action. The process of construction is included in all items, which requires the student to construct the target representation of the function and/or to describe such process. The third dimension combining description and explanation includes all items, which demand a description of the solving process or an explanation of the identified form of representation (see Fig. 2).

With regard to the dimensional structure, the following competing models are possible and should be borne in mind:

- One-dimensional competence structure (1D)

The one-dimensional competence structure model (1D) is based on the assumption that there are no distinguishable aspects of competence when it comes to the different elements of cognitive actions.

- Two-dimensional competence structure (2D)

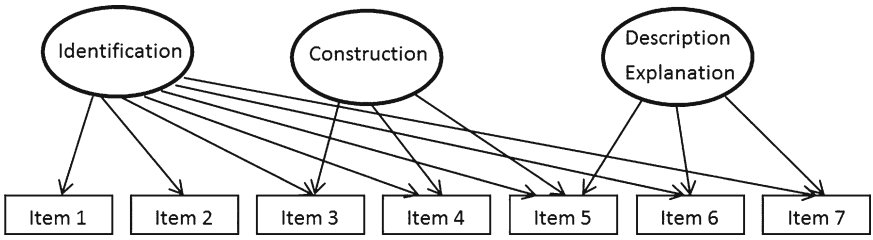


Figure 2. Three-dimensional model encompassing cognitive actions

Alternatively, the two-dimensional model considers identification and construction jointly as a first dimension, as these are the most basic actions. Hence, all items are related to this dimension. The second dimension includes all items, which consider description or explanation, as here the verbal aspect is the most important one.

- Four-dimensional competence structure (4D)

A four-dimensional model assumes that all four elements of cognitive action are dependent on different aspects of competence and can thus be divided into different dimensions. The main difference between this model and the other models is the assumption that the two actions description and explanation can be separated.

The aim of the following empirical testing can be summarised as follows: The ability to translate from one form of representation to another requires a specific collection of competencies. The aim of this study is a psychometric validation of the proposed underlying multidimensional structure of such competencies. Furthermore, different essential elements of cognitive action, required by the nature of the tasks, are examined from an empirical point of view.

Method

The following analysis is embedded in the HEUREKO project, which concerns heuristic discoveries of representations of functional relationships and the diagnosis of mathematical competencies of students. This project is, in turn, anchored in the core area programme of "Competence models for the recognition of individual learning results and the evaluation of educational processes" of the Deutsche Forschungsgemeinschaft (DFG; German Research Foundation) (SPP 1293).

Item Construction

Altogether, 278 mathematical tasks involving functions were constructed for the empirical testing. Each of these tasks dealt with the translation of a function in combination with a certain element of cognitive action. In order to achieve high curricular validity, current schoolbooks were used as the basis for the development of these tasks. A task may be composed of several items that are relevant to evaluation, i.e. a calculation task can include a request for a description of how the solution was found.

For example, four different task types were developed considering the various cognitive actions when translating from graph to equation (GA): GA-I, GA-C, GA-D and GA-E. Figure 3 shows an example item referring to a graphic–algebraic translation with construction action, thus categorised as GA-C. Prior to constructing the target representation, the characteristics of the represented mathematical function in this very representation have to be identified. Hence, this item refers to the dimension of identification additionally.

The following task is separated into two subtasks where the first part refers to the construction of the target representation, which is an equation (see Fig. 4). Prior to the construction of the adequate equation, the student has to identify the relevant values and characteristics of the mathematical function represented in the graph. This item therefore refers to the elements of cognitive action of identification and construction. The second item requests a description of the construction process. This description can only be given if the student is able to identify the relevant values and to construct the target representation (apart from possible errors related to a lack of concentration, which we will not take into account). Hence, the elements of cognitive actions of identification, construction, and explanation were related to this item.

The task in Fig. 5 is also divided into two items. Item 1 involves identification of the appropriate equation, which could be found by way of comparison between the relevant values of both forms of representations. It thus considers the element of cognitive action of identification only. Item 2 involves justification of the chosen equation. For an appropriate explanation, previous identification of the correct equation is essential, and, consequently, this item requires the elements of cognitive action of identification and explanation.

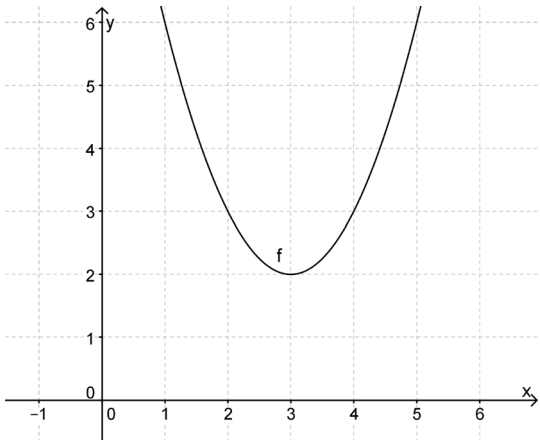


Figure 3. Task example 1

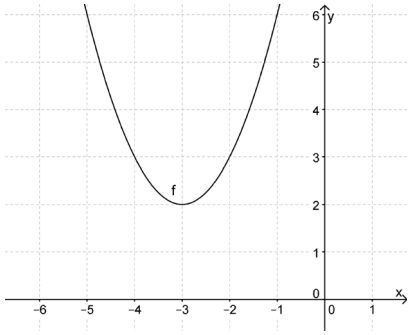


Figure 4. Task example 2

On combining the translations between different forms of representations with the different elements of cognitive action, it turned out that the following variations were not suitable for assessing pragmatic and learning theory-based reasoning with regard to their item construction: GN-E, GN-C, GS-C, and GS-D. The main reason for this is that the tests were designed for students of the ninth and tenth grades. These students are already at an advanced level when translating between forms of representations in the area of functional relationships. We can therefore presuppose a certain knowledge and skill set that students can access already. For example, drawing a graph based on a given numerical table (GN-C) does not provide any additional information about the competencies that we would like students to develop, as this translation is only of a structural nature that students should very well know by that time.

In a pilot project at two high schools (Gymnasium) ($N = 87$, grade 9), 168 of the designed tasks were tested. In order to identify students' potential difficulties, they were asked about the tasks and the solution-finding processes when solving the tasks. In the interviews, think-aloud techniques (Ericsson & Simon, 1980) and the stimulated recall method (Gass & Mackey, 2000) were used. With the help of students' answers, inadvertent difficulties and problems with the tasks (e.g. unknown terms) were identified, and the tasks concerned were further revised for the main study.

Consider the following numerical table and the two equations:

x	0	1	2	3
y	2	3	6	11

(1) $y = 2x + 2$ (2) $y = x^2 + 2$

Decide which of the equations refer to the numerical table. Explain your choice.

Response: The numerical table can be related to equation ☐.

Explanation:

Figure 5. Task example 3

Main Study

Instruments. In the main study, 120 designed tasks were used, based on the results of the pilot tasks. They were divided into four different assessment booklets in the form of a multimatrix design (Gonzalez & Rutkowski, 2010). On average, there were six items per translation between two forms of representations for each assessment booklet. Tasks of different types were arranged in alternating order, so that students would work on tasks of every type. Each assessment booklet contained 30 tasks with a test time of 40 min provided for working on the tasks. Students were not expected to finish the test booklet and were told to work on the tasks in the right order. On the last page of each assessment booklet, there were ten additional questions concerning students' working methods (motivation, perceived level of difficulty and their ability to concentrate when working on the test). In order to reduce students' time and effort for the assessment, another short test was used at another time to equate all four assessment booklets.

In addition, further assessment tools were developed to keep a record of possible confounding variables concerning the outcome of the assessments and the relationships between the test results. These included a test for basic skills and knowledge as well as student and teacher questionnaires. The basic skills and knowledge test was used for gathering general information on the student's basic competencies regarding functions. To test this, 16 tasks were developed; these were similar in content to the main test. The level of difficulty of these tasks was low compared to the main test, as the main goal of this part was meant to disclose the student's general ability to understand functional relationships. Furthermore, student and teacher questionnaires were developed and used to gather information about the learning habits of students, about mathematics classes, etc. The teacher questionnaire concerned the teacher's assessment of the class and also an assessment of the tasks employed with regard to their difficulty. The results of the student and the teacher questionnaires are reported on separately.

Participants. Eight high schools (gymnasium) from the South of Hesse (Germany) took part in our study with 27 forms of ninth (19) and tenth (8) graders. $N = 650$ students took part in the main test. Five of these had to be excluded from evaluation due to a lack of serious commitment to the test. This was based on the extremely low number of tasks they had worked on and /or inappropriate comments on the test papers. The resulting sample consisted of $n = 645$ students. Of these, 330 (51 %) were female and 315 (49 %) were male.

Statistical Analysis Methods. In order to test the anticipated competence structure model, the results of the main test are considered. The tasks were scored based on criteria established beforehand. Half or sub-points were not allotted at this stage. Items with a wrong answer were coded 0; items with a correct answer were coded 1. Tasks that students were unable to work on or that they skipped due to lack of time were marked as missing. Prior to fitting any item response models, tasks with an extremely low solution rate (< 25 %) were identified. The following problems were observed regarding item construction. Some tasks contained too many different forms of representation (e.g. a numerical table with a situational description and the request to create an algebraic equation) and could therefore not be allocated to one dimension only. Hence, eight tasks of this type were excluded from the evaluation. One of the tasks was not solved by any participant, resulting in nine tasks being excluded from the study.

The modelling of the competence structure was carried out using probabilistic test models (item response models). Due to the fact that the number of items was too small to also integrate translation and elements of cognitive action in one model, we divided the analysis into two parts. In both cases, the data set was analysed in the following way: We optimised our data set based on a one-parameter logistic model (1PL model) according to Rasch (1960/1980) (regarding wmnsq and T value) and compared it with a two-parameter logistic model (2PL model) (Birnbaum, 1968; Moosbrugger & Kelava, 2007), using a chi-square difference test (Bollen, 1989). Based on the better model fit, we optimised the data set a second time (with regard to item-total correlations and the p value). To verify the various models of dimensional structure (1D, 2D, …), we applied the Bayesian information criterion (BIC) and the adjusted BIC. The model with the lowest BIC value, with a minimal distance of 10 points from the other models, fits the data best (Raftery, 1995). Additionally, we analysed the correlations between the dimensions of the best fitting model and the item thresholds (for further information see Appendix). Data analysis was conducted using CONQUEST software (Wu et al., 2007) and Mplus software (Muthén & Muthén, 1998–2010).

Results

Testing the Anticipated Competence Structure Model

On average, students worked on 19.31 (SD = 5.22) of the 30 tasks provided from the booklet. Of these, they solved an average of 7.62 (SD = 3.90) tasks correctly.

The performance of the ninth graders was not significantly different from that of the tenth graders (p = .247). The difference between the solution rates with regard to the tasks and the requested translation was relatively small. As a result, all students were classified into the same random sample (see Table 1).

As mentioned above, some of the tasks consist of multiple parts. These were divided into different items resulting in a total of 129 items, taken as the basis for the following analysis.

Analysis of Translations Between Different Forms of Representations. With a view to verifying the five-dimensional competence structures of the translations, we first optimised the data set and compared the 1PL and 2PL models, as described in "Statistical analysis methods" under "Main study". The results show that the 2PL model can be assumed based on 108 items.

Regarding dimensional structure, six different models were tested. A comparison between the adjusted BIC values ascertained our postulated five-dimensional competence structure model [with the five dimensions of graph/algebraic equation (GA), graph/numerical table (GN), graph/situational description (GS), situational description/algebraic equation (SA) and numerical table/algebraic equation (NA)], due to its lowest value (see Table 2).

Analysis of Elements of Cognitive Action. After clarifying the subject of dimensions with regard to translations, a special interest arose in finding a possible structure of the four elements of cognitive action that were required by the items. Hence, a procedure similar to the one used for analysis of the competence structure was chosen. In this case, the dimensions consisted of elements of cognitive action. A comparison between AIC values of the 1PL model and 2PL model yielded that the 1PL model had a notably lower value than the 2PL model than could be presupposed. The resulting data set includes 131 items. There were different models tested for an evaluation of

TABLE 1

Descriptive statistics of item performance in the main study

		Minimum	*Maximum*	*Mean*	*SD*
Tasks worked on	Total	4	30	19.31	5.22
	Grade 9	4	30	19.55	5.20
	Grade 10	7	30	18.77	5.24
Tasks solved	Total	0	23	7.62	3.90
	Grade 9	0	30	7.74	3.90
	Grade 10	0	23	7.36	3.91

TABLE 2

BIC values of models to be compared (data set of 108 items)

Model	*1D*	*2D A*	*2D G*	*2D N*	*2D S*	*5D*
Sample size adjusted BIC	17,396.716	17,329.716	17,370.135	17,356.986	17,398.417	17,268.081
Difference from the 5D model	128.635	61.635	102.054	88.905	130.336	–

the dimensional structure (see section 2). In Table 3, the values are listed for the adjusted BICs of the different models.

The three-dimensional model [with the three dimensions of identification (I), construction (C) and the combined dimension of description–explanation (DE)] shows the lowest adjusted BIC value and can be assumed to describe the data best.

Discussion

Different mathematical representations of functional dependency and the translations between these forms of representation have shown to be crucial factors for success in individual learning. For students to develop a holistic understanding of the concept of mathematical functions, they have to be able to identify the connecting elements of a functional dependency and to combine these. The ability to translate between various functional representations is essential for this purpose (Ainsworth, Bibby & Wood, 2002; Seufert, 2003). The aim of this study was to better understand the underlying structure of these competencies. This would provide a possibility for improved teaching and fostering methods.

TABLE 3

Adjusted BIC values of the 1PL model (data set of 131 items)

Model	*1D*	*2D*	*3D*	*4D*
Sample size adjusted BIC	20,109.844	20,115.295	20,058.650	20,076.428
Difference from the 3D model	51.194	56.645	–	17.778

The consideration that translations have a five-dimensional competence structure model, as suggested, was validated empirically. Evaluation of the data showed that the anticipated five-dimensional competence structure model could be confirmed. It was thus ascertained that translations between the graphical, numerical, situational, and algebraic forms of representation of a function (GA, GN, GS, NA, and SA) are essential when describing students' competencies in the area of functional relationships. It can be assumed that these translations include different aspects of competence. Hence, the translation process is not a one-dimensional construct in which the respective translations would be insignificant. The ability to translate between different forms of representations depends on the translations represented in the task. For example, it makes a difference whether a task requires a translation between a graph and an equation or between a graph and a situational description (e.g. Adu-Gyamfi, Stiff & Bossé, 2012). In addition, there is no predominant factor in a given form of translation that would have a higher influence on the competence structure than other ones. The consequence for mathematics classes is that all five translations would have to be considered. The usual overemphasis on certain forms of representation (e.g. Leinhardt, Zaslavsky & Stein, 1990; Bossé, Adu-Gyamfi & Cheetham, 2011) is not supported by our results due to the non-existence of a predominant factor. Instead, all translations represent different parts of the competence structure and students should be given the opportunity to familiarise themselves with all translations in order to develop broad skills in the field of functional relationships. All types of translation should therefore be represented in textbooks, equally spread, and exercises should vary across all types of translation. Hence different forms of representations and translating between them should also be considered to design curriculum (e.g. Ford, 2008).

Furthermore, the results lead to various assumptions regarding the diagnosis of student understanding and possible learning difficulties. All translations should be embedded in a diagnostic process, as all of them impact the competencies students have. In addition, learning difficulties depend also on the translation focused on so that underdeveloped competence areas can only be identified if different types of translation are distinguished (e.g. Bossé, Adu-Gyamfi & Cheetham, 2011).

With regard to an analysis of the elements of cognitive action, we showed that a multidimensional structure can be assumed, too. Hence, the elements of cognitive actions considered cannot be viewed as a one-dimensional construct, as they rather include various requirements at the same time. An important finding is the separation of the elementary mathematical actions of identification and construction. These are particularly meaningful for the learning process. Identifying significant values or characteristics within one/

several form(s) of representation is essential for all other actions. This thus constitutes an elementary skill in mathematics and should be included in the exercises explicitly. Construction differs from identification, as no target representation is given but has to be built, instead (e.g. Hattikudur et al., 2012). Hence, this has to be considered in mathematics classes separately. Nevertheless, it always includes a component of identification, as this is an essential requirement for working with all types of tasks. For a deeper understanding of functional relationships, the cognitive actions of description and explanation are highly relevant. These are based on previous identification and/or construction requested in the task and constitute an additional requirement whose common ground is the verbal aspect. Apart from the verbal skills required, we presuppose other aspects that impact the ability to describe or explain a solution: Students need a fundamental understanding of the task provided and also have to be aware of the solving process, as otherwise this process cannot be articulated. Taking a closer look at the written solution, we further suggest that such verbalisation provides an insight into the thinking process. Considering previous studies, which focused on the diagnosis of students' understanding of translations, the overemphasis on Identification tasks by these (e.g. Leinhardt, Zaslavsky & Stein, 1990) cannot be supported. Our study shows that these differ from construction tasks significantly. Hence, in order to diagnose an elementary understanding, both elements of cognitive actions have to be considered. For example, it is not sufficient to primarily concentrate on tasks, which focus on "following a procedure", e.g. graphing a linear function with the equation provided. This is indeed an important type of task but it should not be the only one to further or to assess students' understanding (Adu-Gyamfi, 2007). Instead, other types of tasks concerning different elements of cognitive action should be integrated in the teaching practice as well. Whilst the construction of a graph with the equation provided could be solved using a simple scheme, the justification of the correctness or incorrectness of a given translation requires other and less procedural components.

Outlook

In the future, the current results of the main test will be complemented by applying the basic skills and knowledge test. On considering this test, further analysis of the correlations between the different dimensions of the competence structure model regarding translations could be implemented. In this way, a possible common influencing factor could be ascertained.

Additionally, with regard to the practical consequences of the respective results, a developed competence structure model could be the basis for the creation of a diagnostic instrument. This could serve as a benchmark for teachers to diagnose the course of development of the learning groups. In order to enable the developed scale to be used as an individual diagnostic instrument, the item pool has to be differentiated even further, according to the principle of "item growing". The objective is to gain insights into individual students' skills and the different ways in which students can be supported. In this context the aim should be the development of a competence level model. Carrying out a longitudinal analysis with regard to competencies and dimensions could possibly provide indications as to the various competence levels. To achieve this, the analysis of elements of cognitive actions could be used to differentiate different levels of understanding. For example, the two basic actions of identification and construction are helpful to identify students' understanding of the concept of function. Will they be able to identify functions and non-functions with several examples provided? Could they create their own examples of functions and non-functions? Creating examples refers to a higher form of Construction and shows a better understanding of the function concept than just identifying the examples provided. This assumption is supported by the fact that the action/the verb "create" is an indicator of the highest level of understanding in SOLO taxonomy. For the purpose of differentiating various levels of understanding, however, further analysis of the elements of cognitive action is needed, in particular with regard to the level of difficulty. This will show whether the anticipated hierarchy of the elements of cognitive action can be ascertained. Additionally, an analysis should be carried out as to whether the actions themselves have different levels. As seen in the above example, the action of construction can be viewed as "following simple procedures" (which is at the lowest level of understanding with reference to SOLO taxonomy) or as "creating something new", which is at a higher level. Our assumption is that more actions are needed to obtain a hierarchical structure of the various levels of understanding.

In this context, further research on the elements of the cognitive actions of description and explanation is needed to find out whether these can be further differentiated. The mathematical action of explaining, in particular, can be required at different levels. In our study, the explanation of the connection between two different forms of representations is at a relatively low level. We suppose that a significant difference between description and explanation can be established on considering higher levels of explanation. In

addition, there probably are other mathematical actions, which can be differentiated in mathematical tasks, depending on research focus.

In this purpose, it would be interesting to analyse the impact of graphical calculators on student's ability to execute different translation actions. Current studies suggest that students' understanding increases when using graphical calculators (e.g. Ford, 2008). Nevertheless basic mathematical actions should be learned by hand (Hattikudur et al. 2012). For a decision what these basic mathematical actions would be, further research is needed.

Furthermore, a test design combining elements of cognitive action and the content area would be useful to connect the two different competence structure models within a multidimensional construct. Such combination of elements of cognitive action and the content area seems to have a high potential in respect of the analysis and diagnosis of learning processes. Further research will show whether our assumption can be ascertained that these elements of cognitive action are relevant to all mathematical subject-matters.

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Appendix

TABLE 4

AIC, BIC, and sample size adjusted BIC of the 1 PL and 2 PL model (data set of 121 items) (concerning the five-dimensional competence structure model)

Model	*AIC*	*BIC*	*Adjusted BIC*
1PL	18,718.52	19,326.34	18,894.55
2PL	18,694.54	19,820.80	19,020.71

TABLE 5

Chi-square difference test: comparison of the 1PL and 2PL models for the analysis of the change of representation

	1PL	*2PL*
Log likelihood (L):		
H0 value	−9,223.261	−9,095.272
H0 scaling correction factor (c)	0.997	1.060
Number of free parameters (p)	136	252
Difference test scaling correction (cd): $cd = \frac{p_{1PL} \cdot c_{1PL} - p_{2PL} \cdot c_{2PL}}{p_{1PL} - p_{2PL}}$	Cd	1.133862069
Chi-square difference (TRd): $TRd = -2 \cdot \frac{L_{1PL} - L_{2PL}}{cd}$	TRd:	225.7576

TABLE 6

BIC values of the models to be compared (data set of 108 items)

Model	*Sample size adjusted BIC*	*Difference from the 5D model*
1D	17,396.716	128.635
2D A	17,329.716	61.635
2D G	17,370.135	102.054
2D N	17,356.986	88.905
2D S	17,398.417	130.336
5D	17,268.081	–

TABLE 7

Correlation of the various dimensions (data set of 108 items)

	GA	*GN*	*GS*	*NA*
GA	–	–	–	–
GN	0.603	–	–	–
GS	0.668	0.734	–	–
NA	0.712	0.640	0.673	–
SA	0.671	0.755	0.891	0.901

TABLE 8

Chi-square difference test: comparison of the 1PL and 2PL models for an analysis of the elements of cognitive action

	1PL	*2PL*
Log likelihood (L):		
H0 value	−9.837.499	−9.533.268
H0 scaling correction factor (c)	1.002	210.134
Number of free parameters (p)	132	369
Difference test scaling correction (cd): $cd = \frac{p_{1PL} \cdot c_{1PL} - p_{2PL} \cdot c_{2PL}}{p_{1PL} - p_{2PL}}$	cd	326,612.5823
Chi-square difference (TRd): $TRd = -2 \cdot \frac{L_{1PL} - L_{2PL}}{cd}$	TRd:	1.86295

TABLE 9

BIC values of the 1PL model (data set of 131 items)

Model	*1D*	*2D*	*3D*	*4D*
Sample size adjusted BIC	20,109.844	20,115.295	20,058.650	20,076.428
Difference from the 3D model	51.194	56.645	−	17.778

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