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A new stability result for a flexible satellite system with viscoelastic damping

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In this paper, we study a viscoelastic flexible satellite under unknown distributed disturbances during attitude maneuvering. The system composed of a rigid central hub that represents the spacecraft with two symmetrical viscoelastic Euler–Bernoulli beams and subject to undesirable vibrations. The problem can be modeled by a set of partial differential equations (PDEs) taking into account therefore the dynamic boundary condition. The well-posedness of the closed-loop system is discussed. By applying a control force at the center body of the spacecraft, we shall suppress these vibrations; namely, we establish stability results of the system under appropriate assumptions imposed on the relaxation function. These new results generalize and improve many results in the literature. Our results are obtained by using the multiplier technique. Numerical simulation results show the effectiveness of the proposed control scheme.

KEYWORDS

boundary control, distributed disturbances, Euler–Bernoulli beams, flexible spacecraft system, memory term, relaxation function, vibration suppress, viscoelastic damping

MSC CLASSIFICATION

35L20, 93D15, 93D20

1 | INTRODUCTION

In this article, we study the asymptotic stability of a flexible satellite system with a rigid hub and long flexible solar panels subject to unknown distributed disturbances and some undesirable vibrations. The left and the right panels are modeled as two viscoelastic Euler–Bernoulli beams (see Figure 1). As the viscoelastic damping is extremely weak, it is more difficult to find a suitable control applied at the center body of the satellite to suppression and/or reduction of the vibration of the flexible spacecraft. Our contribution is that we design a suitable control force capable of suppressing the vibrations that occur during attitude maneuvering of the flexible spacecraft system.

Over the past decades, the control problems of flexible spacecraft system with different kinds of boundary control have been obtained by several investigators and many results have been obtained in this regard. We mention, among others, the work of Hu and Ma,[1] Hu et al.,[2–4] Du and Li,[5] Liu and Cao,[6] Aoues et al,[7] and Liu et al[8] and the references therein.

Particularly, in reference,[9] He and Ge used the Hamilton's principle to derive the equations of motion for a satellite with two identical flexible solar panels, while the left and the right panels are modeled as two Euler–Bernoulli beams, and the

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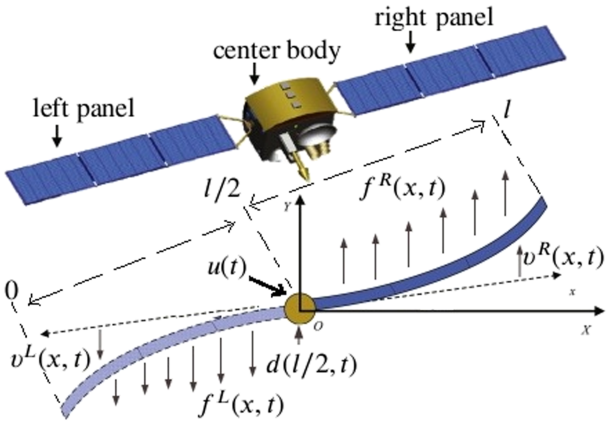


FIGURE 1 The flexible satellite system [Colour figure can be viewed at wileyonlinelibrary.com]

main body of the satellite is modeled as a lumped mass in the center of two panels. They obtained the following:

$$\begin{cases} \rho A v_{tt}^L(x,t) + EIv_{xxxx}^L(x,t) + \gamma_1 v_t^L(x,t) = f^L(x,t),\ (x,t) \in \left[0, l/2\right] \times [0,\infty) \\ \rho A v_{tt}^R(x,t) + EIv_{xxxx}^R(x,t) + \gamma_2 v_t^R(x,t) = f^R(x,t),\ (x,t) \in \left[l/2, l\right] \times [0,\infty) \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} v_x^L(l/2,t) = v_x^R(l/2,t) = 0,\ v_{xx}^L(0,t) = v_{xx}^R(l,t) = 0,\ v_{xxx}^L(0,t) = v_{xxx}^R(l,t) = 0,\ t \geq 0 \\ v^L(l/2,t) = v^R(l/2,t) = v(l/2,t),\ t \geq 0 \\ mv_{tt}(l/2,t) = u(t) + EIv_{xxx}^L(l/2,t) - EIv_{xxx}^R(l/2,t),\ t \geq 0. \end{cases} \tag{2}$$

Here, the functions $v^L(x,t)$, $v^R(x,t) f^L(x,t)$, $f^R(x,t)$, and $d(l/2,t)$ represent the transverse displacements of the left and right panels at the position x for the time t, the distributed disturbances along the flexible satellite system, and the disturbance imported on the center body, respectively, and $v(l/2,t)$ is the transverse displacements of a lumped mass. The positive constants ρ, A, EI, m, and γ_1, γ_2 represent the density of the beam material, the cross-sectional area, bending stiffness, the mass of the center body, and the coefficients of viscous damping, and $u(t)$ is a single-point control force applied on the center body of the satellite.

Under the following control force applied on the center body of the satellite

$$u(t) = -k\frac{\alpha}{\beta}v(l/2,t) - kv_t(l/2,t) - \frac{\alpha}{\beta}mv_t(l/2,t) - k_p v(l/2,t),\ t \geq 0,$$

where α and β are positive weighting constants, and k and k_p are the control gains, they established an exponential decay result for the system (1)–(2) if $f^L \equiv f^R \equiv d \equiv 0$.

Later, in 2020, Ji and Liu[10] improved these new results by taking into account the unknown distributed disturbance for the system; they studied the stability of the following model:

$$\begin{cases} \rho A v_{tt}^L(x,t) + EIv_{xxxx}^L(x,t) + \gamma_1 v_t^L(x,t) = f^L(x,t) + U^L(x,t),\ (x,t) \in \left[0, l/2\right] \times [0,\infty) \\ \rho A v_{tt}^R(x,t) + EIv_{xxxx}^R(x,t) + \gamma_2 v_t^R(x,t) = f^R(x,t) + U^R(x,t),\ (x,t) \in \left[l/2, l\right] \times [0,\infty) \end{cases} \tag{3}$$

with the boundary conditions

$$\begin{cases} v_x^L(l/2,t) = v_x^R(l/2,t) = 0,\ v_{xx}^L(0,t) = v_{xx}^R(l,t) = 0,\ v_{xxx}^L(0,t) = v_{xxx}^R(l,t) = 0,\ t \geq 0 \\ v^L(l/2,t) = v^R(l/2,t) = v(l/2,t),\ t \geq 0 \\ mv_{tt}(l/2,t) = u(t) + EIv_{xxx}^L(l/2,t) - EIv_{xxx}^R(l/2,t) + d(l/2,t),\ t \geq 0, \end{cases} \tag{4}$$

where f^L, f^R, and d are the distributed disturbance in the left and the right panels and the disturbance imported on the center body, such that U^L and U^R are the distributed control inputs of the system. Under the adaptive control laws, the authors established the stability of the system in the presence of partial actuator fault.

It is worth mentioning that there is no result concerning a flexible spacecraft system with viscoelastic damping, namely, when the left and the right panels are modeled as two Euler–Bernoulli beams with a finite memory. Viscoelastic materials are proved to be a good candidate and have the ability to suppress vibrations and to force the systems to reach their equilibrium in a reasonable amount of time. Many results have been established for the asymptotic behavior of wave equations and beams with finite memory; we refer the reader to see previous studies[11–19] and the references therein. The study of the Kirchhoff plates subject to a viscoelastic damping has attracted the attention of many authors, and several decay results have been established; see, for instance, previous works[20,21] and the references therein. The well-posedness and asymptotic stability of a plate equation with memory and time-varying delay has been extensively studied in literature; see, for instance, references.[22,23] In literature,[24] Feng and Soufyane studied a von Karman plate equation with memory-type boundary conditions and they established an optimal explicit and general energy decay result. It is important to note that, in reference,[25] Zhiji et al. studied a vibration and disturbance rejection problem of a wind turbine tower under exogenous perturbations. Using Galerkin approximation scheme, they established the existence of the solution of the problem, and they obtained the stability of the system.

Motivated by these results, we intend to study the asymptotic stability of the problem (3)–(4) with viscoelastic damping instead of the viscous damping (i.e., $\gamma_1 \equiv \gamma_2 \equiv 0$) subject to a distributed disturbance and without any distributed control inputs implemented at the left and the right panels (i.e., $U^L \equiv U^R \equiv 0$). Namely, according to Boltzmann principle (see Christensen[26] and Fabrizio and Morro[27] for more details) and using the constitutive relationship between the stress and the strain

$$\sigma(x,t) = EI\varepsilon(x,t) - EI\int_0^t \zeta(t-s)\varepsilon(x,s)ds,$$

where $\sigma(x,t)$ is the stress, $\varepsilon(x,t)$ is the strain, and the function $\zeta(t)$ is called the relaxation function, then, we study the following system:

$$\begin{cases} \rho A v_{tt}^L(x,t) + EIv_{xxxx}^L(x,t) - EI\int_0^t \zeta(t-s)\, v_{xxxx}^L(x,s)\, ds = f^L(x,t),\ (x,t) \in \left[0, l/2\right] \times [0,\infty) \\ \rho A v_{tt}^R(x,t) + EIv_{xxxx}^R(x,t) - EI\int_0^t \zeta(t-s)\, v_{xxxx}^R(x,s)\, ds = f^R(x,t),\ (x,t) \in \left[l/2, l\right] \times [0,\infty) \end{cases} \tag{5}$$

with the boundary conditions

$$\begin{cases} v_x^L(l/2,t) = v_x^R(l/2,t) = 0,\ v_{xx}^L(0,t) = v_{xx}^R(l,t) = 0,\ v_{xxx}^L(0,t) = v_{xxx}^R(l,t) = 0,\ t \geq 0 \\ v^L(l/2,t) = v^R(l/2,t) = v(l/2,t),\ t \geq 0 \\ mv_{tt}(l/2,t) = u(t) + EIv_{xxx}^L(l/2,t) - EI\int_0^t \zeta(t-s)\, v_{xxx}^L(l/2,s)\, ds \\ \qquad -EIv_{xxx}^R(l/2,t) + EI\int_0^t \zeta(t-s)\, v_{xxx}^R(l/2,s)\, ds + d(l/2,t),\ t \geq 0 \end{cases} \tag{6}$$

and the initial data

$$\begin{cases} v^L(x,0) = v_0^L(x),\ v_t^L(x,0) = v_1^L(x),\ x \in \left[0, l/2\right] \\ v^R(x,0) = v_0^R(x),\ v_t^R(x,0) = v_1^R(x),\ x \in \left[l/2, l\right]. \end{cases} \tag{7}$$

In (5), the integral term represents the viscoelastic damping term.

Our goal in this article is to improve earlier results by establishing a uniform stability result for the problem (5) to (7) under an unknown distributed disturbance and for a suitable control u applied at the center body of the spacecraft and a

large class of kernels ζ satisfying the following condition (see reference[28]):

$$\zeta'(t) \leq 0 \text{ and } \gamma(t)\zeta(t) \in L^1(0,\infty),$$

where γ is a non-negative function.

Remark 1. This condition is satisfied by a large class of functions. It is important to consider such class as it allows the use of several types of materials that could be more convenient. For more examples and for more other types of kernels, one can consult.[28–33]

Remark 2. An important question on vibration suppress of the flexible satellite has been raised by Ji and Liu.[10] It has been proved there that solutions of the closed-loop system decay exponentially to the equilibrium state provided that we have a viscous damping (i.e., $\omega_t^R(x,t)$ and $\omega_t^L(x,t)$) under the boundary control $u(t)$ applied on the center body of the satellite and the distributed control inputs.

This damping is known as a *frictional damping*. In this article, we improve these results by establishing a simple control force $u(t)$ yielding the stability of the problem under *weaker damping* with unknown distributed disturbances and without the need for a distributed control input, namely, a viscoelastic damping induced by the material itself.

The rest of work is organized as follows: In Section 2, we present some preliminaries results and assumptions on the relaxation function needed to prove our main results. The well-posedness of the closed-loop system is proved in Section 3. The uniform stability of the problem (5)–(7) under unknown distributed disturbance is stated and proved in Section 4. Then, numerical simulations are given to show the effectiveness of the designed control scheme, which is offered at the end of the paper.

2 | PRELIMINARIES RESULTS

In this section, we introduce some lemmas and notation which will be utilize throughout this paper. We use the same notations in references,[34,35] for every measurable set $\mathcal{Z} \subset \mathbb{R}^+$; we define the probability measure $\widehat{\zeta}$ by

$$\widehat{\zeta}(\mathcal{Z}) = \frac{1}{\kappa}\int_{\mathcal{Z}} \zeta(s)\, ds. \tag{8}$$

The flatness set and the flatness rate of ζ are defined by

$$\chi_\zeta = \left\{ s \in \mathbb{R}^+ \, \zeta(s) > 0 \text{ and } \zeta'(s) = 0 \right\} \tag{9}$$

and

$$\mathfrak{R}_\zeta = \widehat{\zeta}(\chi_\zeta), \tag{10}$$

respectively; we also have

$$\widetilde{\chi}_\zeta = \left\{ s \in \mathbb{R}^+ \, : \, 0 \leqslant s \leqslant t, \zeta(t-s) > 0 \text{ and } \zeta'(t-s) = 0 \right\}. \tag{11}$$

Let $t_\star > 0$ be a real number such that $\int_0^{t_\star} \zeta(s)\, ds =: \zeta_\star > 0$.
We suppose that the kernel $\zeta(t)$ satisfies (see reference[33]):

(H_1) For all $t \geqslant 0, \zeta(t) \geqslant 0$, ζ is continuously differentiable satisfying $0 < \kappa := \int_0^{+\infty} \zeta(s)\, ds < 1$.
(H_2) For all $t \geqslant 0, \zeta'(t) \leqslant 0$.
(H_3) There exists a nondecreasing function $\gamma(t) > 0$ such that $\gamma'(t)/\gamma(t) = \delta(t)$ is a decreasing function and

$$\gamma(t)\,\zeta(t) \in L^1(0,\infty).$$

The unknown distributed disturbances are assumed to fulfill the conditions:

(H_4) The functions f^L, f^R, and d are continuous in t such that $f^L \in L^2(0,l/2)$, $f^R \in L^2(l/2,t)$, and there exist a positive constant $\overline{d}$ such that $d(l/2,t) \leq \overline{d}$ for all $t \geq 0$.

To simplify, we denote

$$(\zeta\square\upsilon)(t) := \int_0^t \zeta(t-s)(\upsilon(x,t)-\upsilon(x,s))^2 ds \text{ and } \left(\zeta\diamondsuit\upsilon\right)(t) := \int_0^t \zeta(t-s)(\upsilon(l/2,t)-\upsilon(l/2,s))^2 ds,\ t\geqslant 0.$$

The following lemmas will be applied repeatedly throughout this article.

Lemma 1. *We have*

$$xy \leqslant \eta x^2 + \frac{y^2}{4\eta}, x, y \in \mathbb{R}, \eta > 0.$$

Lemma 2 *(The Poincaré inequalities) (See reference[9]). For any $\psi(x,t)$ continuously differentiable on $[L_1, L_2]$, we have*

$$\int_{L_1}^{L_2} [\psi(x,t)]^2 dx \leqslant 2(L_2-L_1)\psi^2(L_2,t) + 4(L_2-L_1)^2 \int_{L_1}^{L_2} [\psi_x(x,t)]^2 dx \tag{12}$$

and

$$\int_{L_1}^{L_2} [\psi(x,t)]^2 dx \leqslant 2(L_2-L_1)\psi^2(L_1,t) + 4(L_2-L_1)^2 \int_{L_1}^{L_2} [\psi_x(x,t)]^2 dx \tag{13}$$

for all $t \geq 0$.

Remark 3 (See reference[9]). From Poincaré inequalities, we obtain for $x \in [L_1, L_2]$

$$\int_{L_1}^{L_2} [\psi(x,t)]^2 dx \leqslant 2(L_2-L_1)[\psi(L_2,t)]^2 + 8(L_2-L_1)^3[\psi_x(L_2,t)]^2 + 16(L_2-L_1)^4 \int_{L_1}^{L_2} [\psi_{xx}(x,t)]^2 dx, \tag{14}$$

and

$$\int_{L_1}^{L_2} [\psi(x,t)]^2 dx \leqslant 2(L_2-L_1)[\psi(L_1,t)]^2 + 8(L_2-L_1)^3[\psi_x(L_1,t)]^2 + 16(L_2-L_1)^4 \int_{L_1}^{L_2} [\psi_{xx}(x,t)]^2 dx, \tag{15}$$

where $t \geq 0$.

Lemma 3 (See reference[33]). *We have, for $\zeta \in C(0,\infty)$ and $\upsilon \in C((0,\infty);[0,L])$*

$$\int_0^L \upsilon \int_0^t \zeta(t-s)\upsilon(s)\, ds dx = \frac{1}{2}\left(\int_0^t \zeta(s)\, ds\right)\|\upsilon\|_2^2 + \frac{1}{2}\int_0^t \zeta(t-s)\|\upsilon(s)\|_2^2\, ds - \frac{1}{2}\int_0^L (\zeta\square\upsilon)\, dx,\ t\geqslant 0. \tag{16}$$

Lemma 4 (See reference[36]). *Let $Q(t), \sigma(t), \beta(t) \in C[0,\infty)$. If there exists a positive function $\Psi(t) \in C^1[0,\infty)$ such that*

$$0 \leq \sigma(t) \leq \frac{\Psi(t)}{2}\left(\chi(t) - \frac{\Psi'(t)}{\Psi(t)}\right),\ \beta(t) \leq \frac{1}{2\Psi(t)}\left(\chi(t) - \frac{\Psi'(t)}{\Psi(t)}\right),\ t \geq 0,$$

then a non-negative solution ϑ of the following inequality

$$\frac{d}{dt}\vartheta(t) \leq -Q(t)\vartheta(t) + \sigma(t)\vartheta^2(t) + \beta(t),\ t \geq 0$$

such that $\Psi(0)\,\vartheta(0) < 1$, *satisfies the estimate*

$$\vartheta(t) < \frac{1}{\Psi(t)},\ t \geq 0.$$

Lemma 5. *The energy functional associated to the problem (5)–(7) is defined by*

$$\begin{aligned}\mathcal{E}(t) = {} & \frac{\rho A}{2}\int_0^{l/2}\left[v_t^L(x,t)\right]^2 dx + \frac{m}{2}\left[v_t(l/2,t)\right]^2 + \frac{\rho A}{2}\int_{l/2}^{l}\left[v_t^R(x,t)\right]^2 dx + \frac{EI}{2}\int_0^{l/2}\left(\zeta\square v_{xx}^L\right)(t)\,dx \\ & + \frac{EI}{2}\int_{l/2}^{l}\left(\zeta\square v_{xx}^R\right)(t)\,dx + \frac{EI}{2}\left[1-\left(\int_0^t \zeta(s)\,ds\right)\right]\left(\int_0^{l/2}\left[v_{xx}^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[v_{xx}^R(x,t)\right]^2 dx\right)\end{aligned} \tag{17}$$

satisfies

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) = {} & \frac{EI}{2}\left(\int_0^{l/2}(\zeta'\square v_{xx}^L)(t)dx + \int_{l/2}^{l}(\zeta'\square v_{xx}^R)(t)dx\right) - \frac{EI}{2}\zeta(t)\left(\int_0^{l/2}\left[v_{xx}^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[v_{xx}^R(x,t)\right]^2 dx\right) \\ & + \int_0^{l/2} v_t^L(x,t)\,f^L(x,t)dx + \int_{l/2}^{l} v_t^R(x,t)\,f^R(x,t)dx + (u(t)+d(l/2,t))\,v_t(l/2,t),\ t \geqslant 0.\end{aligned} \tag{18}$$

Proof. By multiplying the first equation of system (5) by $v_t^L(x,t)$ and integrating over $\left[0, l/2\right]$, and similarly, multiplying the second equation of (6) by $v_t^R(x,t)$ and integrating over $\left[l/2, l\right]$, then multiplying the last equation of (6) by $v_t(l/2,t)$, using the conditions of the borders and summarizing the findings, we obtain

$$\begin{aligned}& \frac{d}{dt}\left(\frac{\rho A}{2}\int_0^{l/2}\left[v_t^L(x,t)\right]^2 dx + \frac{\rho A}{2}\int_{l/2}^{l}\left[v_t^R(x,t)\right]^2 dx + \frac{m}{2}\left[v_t(l/2,t)\right]^2 + \frac{EI}{2}\int_0^{l/2}\left[v_{xx}^L(x,t)\right]^2 dx + \frac{EI}{2}\int_{l/2}^{l}\left[v_{xx}^R(x,t)\right]^2 dx\right) \\ & = \int_0^{l/2} v_{txx}^L(x,t)\int_0^t \zeta(t-s)\,v_{xx}^L(x,s)\,dsdx + \int_{l/2}^{l} v_{txx}^R(x,t)\int_0^t \zeta(t-s)\,v_{xx}^R(x,s)\,dsdx + \int_0^{l/2} v_t^L(x,t)\,f^L(x,t)dx \\ & \quad + \int_{l/2}^{l} v_t^R(x,t)\,f^R(x,t)dx + (u(t)+d(l/2,t))\,v_t(l/2,t),\ t \geq 0.\end{aligned} \tag{19}$$

Now, for $t \geq 0$, we have

$$\begin{aligned}2\int_0^{l/2} v_{txx}^L(x,t)\int_0^t \zeta(t-s)\,v_{xx}^L(x,s)\,dsdx = {} & \int_0^{l/2}\left(\zeta'\square v_{xx}^L\right)(t)\,dx - \zeta(t)\int_0^{l/2}\left[v_{xx}^L(x,t)\right]^2 dx \\ & - \frac{d}{dt}\left[\int_0^{l/2}\left(\zeta\square v_{xx}^L\right)(t)\,dx - \left(\int_0^t \zeta(s)\,ds\right)\int_0^{l/2}\left[v_{xx}^L(x,t)\right]^2 dx\right]\end{aligned} \tag{20}$$

and

$$2\int_{l/2}^{l} v_{txx}^{R}(x,t)\int_{0}^{t}\zeta(t-s)\,v_{xx}^{R}(x,s)\,dsdx=\int_{l/2}^{l}\left(\zeta'\square v_{xx}^{R}\right)(t)\,dx-\zeta(t)\int_{l/2}^{l}\left[v_{xx}^{R}(x,t)\right]^{2}dx - \frac{d}{dt}\left[\int_{l/2}^{l}\left(\zeta\square v_{xx}^{R}\right)(t)\,dx-\left(\int_{0}^{t}\zeta(s)\,ds\right)\int_{l/2}^{l}\left[v_{xx}^{R}(x,t)\right]^{2}dx\right]. \tag{21}$$

□

Therefore, using the relations (20) and (21) in (19), we get (18).

Remark 4. From the equality (18), we can observe that the derivative of the modified energy functional is of an undefined sign. Now, we construct a Lyapunov functional K which plays an important role in the proof of our stability results.

Let

$$K(t)=\mathcal{E}(t)+\sum_{i=1}^{5}\lambda_i\Xi_i(t),\ t\geqslant 0, \tag{22}$$

where $\lambda_i, i=1,\ldots,5$ are positive constants which will be selected later such that $\lambda_3=\lambda_4=1$, and

$$\Xi_1(t):=\rho A\int_0^{l/2} v^L(x,t)\,v_t^L(x,t)\,dx+\rho A\int_{l/2}^{l} v^R(x,t)\,v_t^R(x,t)\,dx+mv_t(l/2,t)\,v(l/2,t),\ t\geqslant 0$$

$$\Xi_2(t):=-\rho A\int_0^{l/2} v_t^L(x,t)\int_0^t\zeta(t-s)\left(v^L(x,t)-v^L(x,s)\right)dsdx-mv_t(l/2,t)\int_0^t\zeta(t-s)\left(v(l/2,t)-v(l/2,s)\right)ds - \rho A\int_{l/2}^{l} v_t^R(x,t)\int_0^t\zeta(t-s)\left(v^R(x,t)-v^R(x,s)\right)dsdx,\ t\geqslant 0$$

$$\Xi_3(t):=\frac{1}{2}(\zeta\diamondsuit v)(t)+\frac{1}{2}\left[v(l/2,t)\right]^2,\ t\geq 0,$$

$$\Xi_4(t):=\int_0^t\varphi_\alpha(t-s)\left(v(l/2,t)-v(l/2,s)\right)^2ds,\ t\geq 0,$$

$$\Xi_5(t):=EI\int_0^t\varphi_\gamma(t-s)\int_0^{l/2}\left[v_{xx}^L(x,s)\right]^2dxds+EI\int_0^t\varphi_\gamma(t-s)\int_0^{l/2}\left[v_{xx}^R(x,s)\right]^2dxds,\ t\geq 0,$$

where

$$\varphi_\alpha(t):=e^{-\alpha t}\int_t^\infty\zeta(s)\,e^{\alpha s}ds,\ \varphi_\gamma(t):=\gamma(t)^{-1}\int_t^\infty\zeta(s)\,\gamma(s)\,ds,\ t\geq 0, t\geq 0,$$

such that α is a positive constant to be selected later.

Remark 5. The functionals $\Xi_i, i=1,\ldots,5$ should be chosen in such a way that their derivatives will provide us with similar terms to the ones in the energy $\mathcal{E}(t)$, but with negative coefficients. In this, we will control these terms and have the energy with a negative sign in the right-hand side of the estimate of the rate of change of the energy.

Finally, in order to stabilize our problem, we propose the following control force $u(t)$:

$$u(t) = -k_p v_t(l/2, t) - k_r v(l/2, t), \; t \geqslant 0, \tag{23}$$

where k_p and k_r are positive "control gains."

3 | WELL-POSEDNESS

Now, we state the existence result of the problem (5)–(7), which will be proven by using the Galerkin approximation method. For this end, we denote

$$\mathbb{V} = \left\{ (y^L, y^R) \in H^1(0, l/2) \times H^1(l/2, l), \; y_x^L(l/2) = y_x^R(l/2) = 0, \; y^L(l/2) = y^R(l/2) = y(l/2) \right\},$$

and

$$\mathbb{W} = \left\{ (y^L, y^R) \in H^2(0, l/2) \times H^2(l/2, l) \cap \mathbb{V}, \; y_{xx}^L(0) = y_{xx}^R(l) = 0, \; y_{xxx}^L(0) = y_{xxx}^R(l) = 0 \right\},$$

where $H^1(0, l/2)$, $H^1(l/2, l)$, $H^2(0, l/2)$, and $H^2(l/2, l)$ are the usual Sobolev spaces.

Theorem 1. *Let $\left(v_0^L, v_0^R\right) \in \mathbb{V}$, $\left(v_1^L, v_1^R\right) \in L^2(0, l/2) \times L^2(l/2, l)$ be given. Suppose that (H_1) to (H_4) holds. Then, under the external force $u(t)$ defined in (23), there exists a unique weak solution $\left(v^L, v^R\right)$ of the problem (5)–(7) which satisfies*

$$\left(v^L, v^R\right) \in C([0, T); \mathbb{V}), \; \left(v_t^L, v_t^R\right) \in C\left([0, T); L^2(0, l/2) \times L^2(l/2, l)\right),$$

where $T > 0$.

Proof. Let us solve the variational problem associated with (5)–(7), which is given by: find $\left(v^L, v^R\right) \in \mathbb{W} \times \mathbb{W}$ such that

$$\begin{aligned}
&\rho A \int_0^{l/2} v_{tt}^L(x, t) \, y^L(x) \, dx + \rho A \int_{l/2}^{l} v_{tt}^R(x, t) \, y^R(x) \, dx + EI \int_0^{l/2} v_{xx}^L(x, t) \, y_{xx}^L(x) \, dx + EI \int_{l/2}^{l} v_{xx}^R(x, t) \, y_{xx}^R(x) \, dx \\
&- EI \int_0^{l/2} y_{xx}^L(x) \int_0^t \zeta(t - s) \, v_{xx}^L(x, s) \, ds dx - EI \int_{l/2}^{l} y_{xx}^R(x) \int_0^t \zeta(t - s) \, v_{xx}^R(x, s) \, ds dx \\
&+ y(l/2) \left[m v_{tt}(l/2, t) + k_p v_t(l/2, t) + k_r v(l/2, t) - d(l/2, t) \right] = \int_0^{l/2} f^L(x, t) \, y^L(x) \, dx + \int_{l/2}^{l} f^R(x, t) \, y^R(x) \, dx,
\end{aligned}$$

for any $\left(y^L, y^R\right) \in \mathbb{V} \times \mathbb{V}$.

Let $\left\{ \left(y^{L0}, y^{R0}\right), \left(y^{L1}, y^{R1}\right), \left(y^{L2}, y^{R2}\right), \ldots \right\}$ be a complete orthogonal system of $\mathbb{W}$ for which

$$\left(v_0^L, v_0^R\right), \left(v_1^L, v_1^R\right) \in \operatorname{span}\left\{ \left(y^{L0}, y^{R0}\right), \left(y^{L1}, y^{R1}\right) \right\}.$$

For each $m \in \mathbb{N}$, let us put

$$\mathbb{W}_m = \operatorname{span}\left\{ \left(y^{L0}, y^{R0}\right), \left(y^{L1}, y^{R1}\right), \left(y^{L2}, y^{R2}\right), \ldots \right\}.$$

We seek solutions of the form

$$\begin{cases}
v^{Lm}(x, t) = \sum_{j=1}^{m} a_{j,m}(t) y^{Lj}(x), \; x \in \left[0, l/2\right], \; t \geq 0 \\
v^{Rm}(x, t) = \sum_{j=1}^{m} b_{j,m}(t) y^{Rj}(x), \; x \in \left[l/2, l\right], \; t \geq 0
\end{cases}$$

to the following approximate problems in $\mathbb{W}_m$

$$\begin{cases} \rho A\int_0^{l/2} v_{tt}^{Lm}(x,t)\,y^L(x)\,dx+\rho A\int_{l/2}^{l} v_{tt}^{Rm}(x,t)\,y^R(x)\,dx+EI\int_0^{l/2} v_{xx}^{Lm}(x,t)\,y_{xx}^L(x)\,dx+EI\int_{l/2}^{l} v_{xx}^{Rm}(x,t)\,y_{xx}^R(x)\,dx \\ -EI\int_0^{l/2} y_{xx}^L(x)\int_0^t \zeta(t-s)\,v_{xx}^{Lm}(x,s)\,dsdx-EI\int_{l/2}^{l} y_{xx}^R(x)\int_0^t \zeta(t-s)\,v_{xx}^{Rm}(x,s)\,dsdx \\ +y(l/2)\left[mv_{tt}^m(l/2,t)+k_pv_t^m(l/2,t)+k_rv^m(l/2,t)-d(l/2,t)\right]=\int_0^{l/2} f^L(x,t)\,y^L(x)\,dx+\int_{l/2}^{l} f^R(x,t)\,y^R(x)\,dx, \\ \left(v^{Lm}(x,0)=v_0^{Lm}(x),\,v^{Rm}(x,0)=v_0^{Rm}(x)\right)\to\left(v_0^L,\,v_0^R\right)\ \text{in}\ \mathbb{W} \\ \left(v_t^{Lm}(x,0)=v_1^{Lm}(x),\,v_t^{Rm}(x,0)=v_1^{Rm}(x)\right)\to\left(v_1^L,\,v_1^R\right)\ \text{in}\ L^2(0,l/2)\times L^2(l/2,l). \end{cases} \tag{24}$$

Note that Equation (24) leads to a system of ordinary differential equations (ODEs) for unknown m functions $a_{j,m}$ and $b_{j,m}$. Standard ODE theory guarantees the existence of unique $\left(v^L,v^R\right)$ on the maximal interval $[0,t_m)$, $t_m\in(0,T]$ for each $m\geq 1$. It remains to extend t_m to T.

A priori estimate. Let $y^L=v_t^{Lm}$, $y^R=v_t^{Rm}$ in (24), we get for any $t\geq 0$

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\rho A}{2}\int_0^{l/2}\left[v_t^{Lm}(x,t)\right]^2dx+\frac{\rho A}{2}\int_{l/2}^{l}\left[v_t^{Rm}(x,t)\right]^2dx+\frac{EI}{2}\int_0^{l/2}\left[v_{xx}^{Lm}(x,t)\right]^2dx+\frac{EI}{2}\int_{l/2}^{l}\left[v_{xx}^{Rm}(x,t)\right]^2dx\right. \\ &\left.+\frac{m}{2}\left[v_t^m(l/2,t)\right]^2\right)=EI\int_0^{l/2} v_{txx}^{Lm}(x,t)\int_0^t\zeta(t-s)\,v_{xx}^{Lm}(x,s)\,dsdx+EI\int_{l/2}^{l} v_{txx}^{Rm}(x,t)\int_0^t\zeta(t-s)\,v_{xx}^{Rm}(x,s)\,dsdx \\ &-k_p\left[v_t^m(l/2,t)\right]^2-k_rv_t^m(l/2,t)v^m(l/2,t)+\int_0^{l/2} f^L(x,t)\,v_t^{Lm}(x,t)\,dx+\int_{l/2}^{l} f^R(x,t)\,v_t^{Rm}(x,t)\,dx+v_t^m(l/2,t)d(l/2,t). \end{aligned} \tag{25}$$

Let

$$X^m(t)=\mathcal{E}^m(t)+\frac{k_r}{2}\left[v^m(l/2,t)\right]^2, \tag{26}$$

where

$$\begin{aligned} \mathcal{E}^m(t)=&\frac{\rho A}{2}\int_0^{l/2}\left[v_t^{Lm}(x,t)\right]^2dx+\frac{m}{2}\left[v_t^m(l/2,t)\right]^2+\frac{\rho A}{2}\int_{l/2}^{l}\left[v_t^{Rm}(x,t)\right]^2dx+\frac{EI}{2}\int_0^{l/2}\left(\zeta\square v_{xx}^{Lm}\right)(t)\,dx \\ &+\frac{EI}{2}\int_{l/2}^{l}\left(\zeta\square v_{xx}^{Rm}\right)(t)\,dx+\frac{EI}{2}\left[1-\left(\int_0^t\zeta(s)\,ds\right)\right]\left(\int_0^{l/2}\left[v_{xx}^{Lm}(x,t)\right]^2dx+\int_{l/2}^{l}\left[v_{xx}^{Rm}(x,t)\right]^2dx\right) \end{aligned} \tag{27}$$

A differentiation of $X^m(t)$ gives

$$\frac{d}{dt}X^m(t)=\frac{d}{dt}\mathcal{E}^m(t)+k_rv_t^m(l/2,t)\,v^m(l/2,t). \tag{28}$$

Equation (25) and the relations (20) and (21) permit to obtain

$$\frac{d}{dt}\mathcal{E}^m(t) = \frac{EI}{2}\left(\int_0^{l/2}(\zeta'\square v_{xx}^{Lm})(t)dx + \int_{l/2}^{l}(\zeta'\square v_{xx}^{Rm})(t)dx\right) - \frac{EI}{2}\zeta(t)\left(\int_0^{l/2}\left[v_{xx}^{Lm}(x,t)\right]^2dx + \int_{l/2}^{l}\left[v_{xx}^{Rm}(x,t)\right]^2dx\right)$$
$$+\int_0^{l/2} v_t^{Lm}(x,t)\,f^L(x,t)dx + \int_{l/2}^{l} v_t^{Rm}(x,t)\,f^R(x,t)dx - k_r v_t^m(l/2,t)v^m(l/2,t) - k_p\left[v_t^m(l/2,t)\right]^2 + v_t^m(l/2,t)d(l/2,t). \tag{29}$$

Now, by using Lemma 1, we have

$$v_t^m(x,t)d(l/2,t) \le \eta\left[v_t^m(x,t)\right]^2 + \frac{1}{4\eta}d^2(l/2,t),\ \eta = k_p. \tag{30}$$

From (28), (29), and (30), we arrive at

$$\frac{d}{dt}X^m(t) = \int_0^{l/2} v_t^{Lm}(x,t)\,f^L(x,t)dx + \int_{l/2}^{l} v_t^{Rm}(x,t)\,f^R(x,t)dx + \frac{1}{4k_p}d^2(l/2,t) \tag{31}$$

for any $t \in [0, t_n)$. Integrating (31) over $[0, t]$ using Hölder and Young's inequality and assumption (H_2), we obtain

$$X^m(t) \le X^m(0) + \frac{1}{4k_p}\int_0^t\left[d(l/2,s)\right]^2ds + \frac{1}{4\eta'}\int_0^t\int_0^{l/2}\left[f^L(x,s)\right]^2dxds + \eta'\int_0^t\int_0^{l/2}\left[v_t^{Lm}(x,s)\right]^2dxds$$
$$+\frac{1}{4\eta'}\int_0^t\int_{l/2}^{l}\left[f^R(x,s)\right]^2dxds + \eta'\int_0^t\int_{l/2}^{l}\left[v_t^{Rm}(x,s)\right]^2dxds,\ \eta' > 0. \tag{32}$$

From (26), choosing η' sufficiently small, taking into account assumption (H_4) and $\left(v_0^{Lm}, v_0^{Rm}\right)$, $\left(v_1^{Lm}, v_1^{Rm}\right)$ are bounded in $\mathbb{W}$, and employing Gronwall's lemma, we infer

$$\mathcal{E}^m(t) \le M, \tag{33}$$

where M is a positive constant independent of t and m.

Therefore, using the fact that (H_1) the estimate (33) together with (27) give us, for all $m \in \mathbb{N}$, $t_m = T$, we deduce

$$\begin{aligned}&\left(v^{Lm}\right) \text{ and } \left(v^{Rm}\right) \text{ are bounded in } L^\infty((0,T),\mathbb{V}),\\ &\left(v_t^{Lm}\right) \text{ and } \left(v_t^{Rm}\right) \text{ are bounded in } L^\infty\left((0,T),L^2(0,l/2)\times L^2(l/2,l)\right).\end{aligned} \tag{34}$$

It follows from (34) that there exists a subsequence $\left(v^{Lv}, v^{Rv}\right)$ of $\left(v^{Lm}, v^{Rm}\right)$ such that

$$\begin{aligned}&v^{Lv} \rightharpoonup v^L,\ v^{Rv} \rightharpoonup v^R \text{ weakly star in } L^\infty((0,T),\mathbb{V})\\ &v_t^{Lv} \rightharpoonup v_t^L,\ v_t^{Rv} \rightharpoonup v_t^R \text{ weakly star in } L^\infty\left((0,T),L^2(0,l/2)\times L^2(l/2,l)\right).\end{aligned} \tag{35}$$

Then from Aubin–Lions theorem,[37] for any $T > 0$

$$\begin{aligned} &v^{Lv} \to v^L,\ v^{Rv} \to v^R \text{ strongly in } C((0,T),\mathbb{V}) \\ &v_t^{Lv} \to v_t^L,\ v_t^{Rv} \to v_t^R \text{ strongly in } C\left((0,T),\ L^2(0,l/2)\times L^2(l/2,l)\right). \end{aligned} \tag{36}$$

These results are sufficient to pass to the limit in (24) to get

$$\begin{aligned} &\rho A\int_0^{l/2} v_{tt}^L(x,t)\,y^L(x)\,dx + \rho A\int_{l/2}^{l} v_{tt}^R(x,t)\,y^R(x)\,dx + EI\int_0^{l/2} v_{xx}^L(x,t)\,y_{xx}^L(x)\,dx + EI\int_{l/2}^{l} v_{xx}^R(x,t)\,y_{xx}^R(x)\,dx \\ &- EI\int_0^{l/2} y_{xx}^L(x)\int_0^t \zeta(t-s)\,v_{xx}^L(x,s)\,dsdx - EI\int_{l/2}^{l} y_{xx}^R(x)\int_0^t \zeta(t-s)\,v_{xx}^R(x,s)\,dsdx \\ &+ y(l/2)\left[mv_{tt}(l/2,t) + k_p v_t(l/2,t) + k_r v(l/2,t) - d(l/2,t)\right] = \int_0^{l/2} f^L(x,t)\,y^L(x)\,dx + \int_{l/2}^{l} f^R(x,t)\,y^R(x)\,dx, \end{aligned}$$

for any $\left(y^L, y^R\right) \in \mathbb{V}\times\mathbb{V}$. The rest of the proof is a matter of routine. □

4 | ASYMPTOTIC BEHAVIOR

In this section, we state and prove the uniform stability of the system (5)–(7) under a suitable control force $u(t)$ applied on the center body of the spacecraft. Now, we are ready to state our main result.

Theorem 2. *Assume that (H_1) to (H_4) hold. Under the control force $u(t)$ defined in (23), if $\mathfrak{R}_\zeta$ is sufficiently small, there exists a positive function $\Psi \in C^1[0,\infty)$ such that*

$$0 \le \left(Q(t) - \frac{\Psi'(t)}{\Psi(t)}\right),\ \int_0^{l/2}\left[v_t^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[v_t^R(x,t)\right]^2 dx + d^2(l/2,t) \le \frac{B}{\Psi(t)}\left(Q(t) - \frac{\Psi'(t)}{\Psi(t)}\right),\ t \ge 0,$$

where B is given in (42). Then, for some positive constant C

$$\mathcal{E}(t) \leqslant \frac{C}{\Psi(t)},\ t \geqslant 0,$$

provided that $\Psi(0)K(0) < 1$ in the cases

(a) $\lim_{t\to+\infty}\delta(t) = 0$ *and* $Q(t) = c_1\delta(t)$ *(c_1 will be chosen in the proof), or*
(b) $\lim_{t\to+\infty}\delta(t) = \overline{\delta} \neq 0$ *and* $Q(t) = c_2$ *(c_2 is as in 42).*

Proof. Differentiating $K(t)$ with respect to t, gathering the estimates in Lemmas 7–10 (i.e., for more detailed derivations, see Appendix A), making use of (23) in the expression (18), we get, for any $t \geqslant t_\star > 0$,

$$
\begin{aligned}
\frac{d}{dt}K(t) \leqslant\ & B_1\left[\upsilon_t\left(l/2,t\right)\right]^2 + B_2\left[\upsilon\left(l/2,t\right)\right]^2 + \left(2\eta_7\overline{\varphi}_\alpha - \alpha\right)\Xi_4\left(t\right) - EI\lambda_5\delta\left(t\right)\Xi_5\left(t\right) + B_6\left(\zeta\diamondsuit\upsilon\right)\left(t\right) \\
& + \left(\frac{EI}{4} - \frac{\rho Al^4}{4\eta_4}\zeta\left(0\right)\lambda_2\right)\left(\int_0^{l/2}\left(\zeta'\square\upsilon_{xx}^L\right)\left(t\right)dx + \int_{l/2}^{l}\left(\zeta'\square\upsilon_{xx}^R\right)\left(t\right)dx\right) + \frac{EI}{4}\int_0^{l/2}\left(\zeta'\square\upsilon_{xx}^L\right)\left(t\right)dx \\
& + \frac{EI}{4}\int_{l/2}^{l}\left(\zeta'\square\upsilon_{xx}^R\right)\left(t\right)dx + B_3\left(\int_0^{l/2}\left[\upsilon_t^L\left(x,t\right)\right]^2dx + \int_{l/2}^{l}\left[\upsilon_t^R\left(x,t\right)\right]^2dx\right) + \frac{1}{4}\left(\zeta'\diamondsuit\upsilon\right)\left(t\right) \\
& + B_4\left(\int_0^{l/2}\left[\upsilon_{xx}^L\left(x,t\right)\right]^2dx + \int_{l/2}^{l}\left[\upsilon_{xx}^R\left(x,t\right)\right]^2dx\right) + \left[\frac{1}{4} - \lambda_2\left(\frac{\rho Al}{2\eta_4} + \frac{m}{4\eta_4}\right)\right]\zeta\left(0\right)\left(\zeta'\diamondsuit\upsilon\right)\left(t\right) \\
& + \left[\frac{\lambda_1}{2} + \frac{\lambda_2}{2}\left(1-\zeta_\star\right) - \lambda_5\right]EI\int_0^t\zeta\left(t-s\right)\left(\int_0^{l/2}\left[\upsilon_{xx}^L\left(x,s\right)\right]^2dx + \int_{l/2}^{l}\left[\upsilon_{xx}^R\left(x,s\right)\right]^2dx\right)ds \\
& + \left[\lambda_2\left(2\varepsilon_2 + k_p + \frac{k_r}{2\eta_4}\right) + 2\eta_5\right]\kappa\int_{\mathcal{Z}_t}\zeta\left(t-s\right)\left(\upsilon\left(l/2,t\right) - \upsilon\left(l/2,s\right)\right)^2dsdx + \lambda_2\left(\frac{1-\zeta_\star}{4\eta_3} + 2\right)EI\kappa \\
& \times\left(\int_0^{l/2}\int_{\mathcal{Z}_t}\zeta\left(t-s\right)\left(\upsilon_{xx}^L\left(t\right) - \upsilon_{xx}^L\left(s\right)\right)^2dsdx + \int_{l/2}^{l}\int_{\mathcal{Z}_t}\zeta\left(t-s\right)\left(\upsilon_{xx}^R\left(t\right) - \upsilon_{xx}^R\left(s\right)\right)^2dsdx\right) \\
& + B_5\left(\int_0^{l/2}\left(\zeta\square\upsilon_{xx}^L\right)\left(t\right)dx + \int_{l/2}^{l}\left(\zeta\square\upsilon_{xx}^R\right)\left(t\right)dx\right) + B_7\left(\int_0^{l/2}\left[f^L\left(x,t\right)\right]^2dx + \int_{l/2}^{l}\left[f^R\left(x,t\right)\right]^2dx + d^2(l/2,t)\right),
\end{aligned}
\tag{37}
$$

where we have used the following estimates:

$$
k_r\upsilon\left(l/2,t\right)\upsilon_t\left(l/2,t\right) \leqslant \frac{k_r^2}{4\eta_8}\left[\upsilon_t\left(l/2,t\right)\right]^2 + \eta_8\left[\upsilon\left(l/2,t\right)\right]^2,\ \eta_8 > 0,\ t \geqslant 0
$$

$$
\int_0^{l/2}\upsilon_t^L\left(x,t\right)f^L\left(x,t\right)dx \leqslant \varepsilon_3\int_0^{l/2}\left[\upsilon_t^L\left(x,t\right)\right]^2dx + \frac{1}{4\varepsilon_3}\int_0^{l/2}\left[f^L\left(x,t\right)\right]^2dx,\ \varepsilon_3 > 0,
$$

$$
\int_{l/2}^{l}\upsilon_t^R\left(x,t\right)f^R\left(x,t\right)dx \leqslant \varepsilon_3\int_{l/2}^{l}\left[\upsilon_t^R\left(x,t\right)\right]^2dx + \frac{1}{4\varepsilon_3}\int_{l/2}^{l}\left[f^R\left(x,t\right)\right]^2dx
$$

and

$$
d(l/2,t)\upsilon_t\left(l/2,t\right) \leq \frac{1}{4\varepsilon_3}d^2(l/2,t) + \varepsilon_3\left[\upsilon_t\left(l/2,t\right)\right]^2,\ \varepsilon_3 > 0
$$

such that

$$
B_1 := -k_p + \varepsilon_3 + \frac{k_r^2}{4\eta_8} + \left(m + \frac{1}{4\eta_1}\right)\lambda_1 + \left[\frac{k_p}{2} + m\left(\eta_4 - \zeta_\star\right)\right]\lambda_2 + \frac{1}{4\eta_5} + \frac{1}{4\eta_6} + \frac{1}{2\eta_7},
$$

$$
B_2 := \eta_8 + \left(\varepsilon_1(2l+1) + \eta_1k_p^2 - k_r\right)\lambda_1 + k_r\eta_2\lambda_2 + \eta_6,\ B_3 := \varepsilon_3 + \lambda_1\rho A + \lambda_2\left(\eta_4 - \zeta_\star\right)\rho A
$$

$$B_4 := +\left\{\lambda_2(1-\zeta_\star)\left(\eta_3+\frac{3}{2}\int_{\chi_t}\zeta(t-s)\,ds\right)+\lambda_5\varphi_\gamma(0)+\lambda_1\left[\frac{2\varepsilon_1 l^4}{EI}-\left(1-\frac{\kappa}{2}\right)\right]\right\}EI,$$

$$B_5 := \left(2\lambda_2\int_{\chi_t}\zeta(t-s)\,ds-\frac{\lambda_1}{2}+\frac{\varepsilon_2 l^4\kappa\lambda_2}{EI}\right)EI,\ B_6 := \left[\lambda_2\left(2\varepsilon_2+k_p+\frac{k_r}{2\eta_2}\right)+2\eta_5\right]\int_{\chi_t}\zeta(t-s)\,ds-1+2\varepsilon_2 l\kappa\lambda_2$$

and

$$B_7 := \frac{1}{4\varepsilon_3}+\frac{\lambda_1}{4\varepsilon_1}+\frac{\lambda_2}{4\varepsilon_2}.$$

Now, as in reference,[33] we introduce the sets

$$\mathcal{Z}_n := \left\{s\in\mathbb{R}^+ :\ n\zeta'(s)+\zeta(s)\leqslant 0\right\},\ n\in\mathbb{N}^+,$$

and

$$\widetilde{\mathcal{Z}}_{nt} := \left\{s\in\mathbb{R}^+, 0\leqslant s\leqslant t,\ n\zeta'(t-s)+\zeta(t-s)\leqslant 0\right\},\ n\in\mathbb{N}^+.$$

We note that $\underset{n}{\cup}\mathcal{Z}_n=\mathbb{R}^+\setminus\left\{\chi_\zeta\cup N_\zeta\right\}$, where N_ζ is the null set where ζ' is not defined. Furthermore, if we denote that $\chi_n := \mathbb{R}^+\setminus\mathcal{Z}_n$, then $\lim\limits_{n\to+\infty}\widehat{\zeta}(\chi_n)=\widehat{\zeta}\left(\chi_g\right)$ because $\chi_{n+1}\subset\chi_n$ for all n and $\underset{n}{\cap}\chi_n=\chi_\zeta\cup N_n$. We take $\mathcal{Z}_t := \widetilde{\mathcal{Z}}_{nt}$ and $\chi_t := \widetilde{\chi}_{nt}$ in (37). Therefore,

$$\frac{1}{4}\int_0^{l/2}\left(\zeta'\Box v_{xx}^L\right)dx\leqslant-\frac{1}{4n}\int_0^{l/2}\int_{\widetilde{\mathcal{Z}}_{nt}}\zeta(t-s)\left(v_{xx}^L(t)-v_{xx}^L(s)\right)^2 dsdx$$

$$\frac{1}{4}\int_{l/2}^{l}\left(\zeta'\Box v_{xx}^R\right)dx\leqslant-\frac{1}{4n}\int_{l/2}^{l}\int_{\widetilde{\mathcal{Z}}_{nt}}\zeta(t-s)\left(v_{xx}^R(t)-v_{xx}^R(s)\right)^2 dsdx$$

and

$$\frac{1}{4}\left(\zeta'\diamondsuit v\right)(t)\leqslant-\frac{1}{4n}\int_{\widetilde{\mathcal{Z}}_{nt}}\zeta(t-s)\left(v(l/2,t)-v(l/2,s)\right)^2 ds.$$

For small $\varepsilon<\zeta_\star$, we select $\lambda_1=(\zeta_\star-\varepsilon)\lambda_2$, $\lambda_5=(1-\varepsilon)\lambda_2/2$, $\eta_4=\varepsilon/2$, $\varepsilon_3=\varepsilon\lambda_2\rho A/4$, $\eta_2=(\zeta_\star-\varepsilon)/4$, $\eta_5=1/16\kappa n$, $\eta_6=\eta_8=k_r\lambda_1/8$, $\varepsilon_1=k_r/2(2l-1)$, and $\eta_7=\alpha/4\overline{\varphi}_\alpha$ for some positive constant α.

Now, for notation convenience, we shall write all the coefficients in the right-hand side of (37) in the simple form, so for some $0<\sigma<1$, setting

$\tau_1 := -k_p+\frac{\varepsilon\rho A\lambda_2}{4}+\frac{2k_r}{(\zeta_\star-\varepsilon)\lambda_2}+\left(m+\frac{1}{4\eta_1}\right)(\zeta_\star-\varepsilon)\lambda_2+\left[\frac{k_p}{2}+m\left(\frac{\varepsilon}{2}-\zeta_\star\right)\right]\lambda_2+4\kappa n+\frac{2}{k_r(\zeta_\star-\varepsilon)\lambda_2}+\frac{2\overline{\varphi}_\alpha}{\alpha}$,

$\tau_2 := \lambda_2(\zeta_\star-\varepsilon)\left[\varepsilon_1(2l+1)+\eta_1 k_p^2-\frac{k_r}{2}\right]$, $\tau_3 := \frac{1}{4}-\frac{2\rho Al+m}{2\varepsilon}\zeta(0)\lambda_2$, $\tau_4 := \frac{EI}{4}-\frac{\rho Al^4}{2\varepsilon}\zeta(0)\lambda_2$,

$\tau_5 := \left[\lambda_2\left(2\varepsilon_2+k_p+\frac{2k_r}{\zeta_\star-\varepsilon}\right)+\frac{1}{8\kappa n}\right]\kappa\widehat{\zeta}(\chi_n)-1+2\varepsilon_2 l\kappa\lambda_2$,

$\tau_6 := \lambda_2\left\{(1-\zeta_\star)\left[\eta_3+\frac{3}{2}\kappa\widehat{\zeta}(\chi_n)\right]+\frac{(1-\varepsilon)}{2}\varphi_\gamma(0)+[(1-\sigma)+\sigma](\zeta_\star-\varepsilon)\left[\frac{2\varepsilon_1 l^4}{EI}-\left(1-\frac{\kappa}{2}\right)\right]\right\}EI$,

$\tau_7 := \left[\lambda_2\left(\frac{1-\zeta_\star}{4\eta_3}+2\right)\kappa-\frac{1}{4n}\right]EI$, $\tau_8=\lambda_2\left(2\varepsilon_2+k_p+\frac{k_r}{\varepsilon}\right)\kappa-\frac{1}{8n}$,

$\tau_9 := \lambda_2\left(2\kappa\widehat{\zeta}(\chi_n)-\frac{\zeta_\star-\varepsilon}{2}+\frac{\varepsilon_2 l^4\kappa}{EI}\right)EI$ and $\tau_{10} := \frac{(\zeta_\star-\varepsilon)\lambda_2}{4\varepsilon_1}+\frac{\lambda_2}{\varepsilon_2}+\frac{1}{\varepsilon\lambda_2\rho A}$.

Therefore, for $t \geqslant t_\star > 0$, (37) becomes

$$\begin{aligned}\frac{d}{dt}K(t) \leqslant\ & \tau_1\left[v_t(l/2,t)\right]^2 + \tau_2\left[v(l/2,t)\right]^2 - \frac{\alpha}{2}\Xi_4(t) - EI\delta(t)\,\Xi_5(t)\,\lambda_5 + \tau_3\left(\zeta'\diamondsuit v\right)(t) + \tau_5\left(\zeta\diamondsuit v\right)(t)\\ &+ \tau_4\left(\int_0^{l/2}\left(\zeta'\square v_{xx}^L\right)(t)\,dx + \int_{l/2}^{l}\left(\zeta'\square v_{xx}^R\right)(t)\,dx\right) + \tau_9\left(\int_0^{l/2}\left(\zeta\square v_{xx}^L\right)(t)\,dx + \int_{l/2}^{l}\left(\zeta\square v_{xx}^R\right)(t)\,dx\right)\\ &- \frac{\varepsilon}{2}\lambda_2\rho A\left(\int_0^{l/2}\left[v_t^L(x,t)\right]^2dx + \int_{l/2}^{l}\left[v_t^R(x,t)\right]^2dx\right) + \tau_6\left(\int_0^{l/2}\left[v_{xx}^L(x,t)\right]^2dx + \int_{l/2}^{l}\left[v_{xx}^R(x,t)\right]^2dx\right)\\ &+ \tau_7\left(\int_0^{l/2}\int_{\widetilde{\mathcal{Z}}_{nt}}\zeta(t-s)\left(v_{xx}^L(x,t) - v_{xx}^L(x,s)\right)^2dsdx + \int_{l/2}^{l}\int_{\widetilde{\mathcal{Z}}_{nt}}\zeta(t-s)\left(v_{xx}^R(x,t) - v_{xx}^R(s,x)\right)^2dsdx\right)\\ &+ \tau_8\int_{\widetilde{\mathcal{Z}}_{nt}}\zeta(t-s)\left(v(l/2,t) - v(l/2,s)\right)^2dsdx + \tau_{10}\left(\int_0^{l/2}\left[f^L(x,t)\right]^2dx + \int_{l/2}^{l}\left[f^R(x,t)\right]^2dx + d^2(l/2,t)\right).\end{aligned} \tag{38}$$

To achieve our goal, we start selecting the different parameters in the right-hand side of (38) so that all the coefficients in the right-hand side (except τ_{10}). For small ε and large values of n and t_*, if $\widehat{\zeta}(\chi_n)$ is sufficiently small, we have

$$\kappa\widehat{\zeta}(\chi_n) - \frac{\zeta_* - \varepsilon}{2} \leqslant 0$$

and

$$\frac{3}{2}(1-\zeta_\star)\,\kappa\widehat{\zeta}(\chi_n) < \sigma(\zeta_* - \varepsilon)\left(1 - \frac{\kappa}{2}\right) \tag{39}$$

with

$$\sigma = \frac{3\kappa(1-\zeta_\star)}{4\zeta_\star(2-\kappa)}.$$

Observe that, for t_* large enough, we have $0 < \sigma < 1$. For the remaining $1-\sigma$, we require that $\varphi_\gamma(0)$ satisfy

$$\frac{(1-\varepsilon)}{2}\varphi_\gamma(0) < (1-\sigma)(\zeta_* - \varepsilon)\left(1 - \frac{\kappa}{2}\right), \tag{40}$$

then (40) is satisfied if $\varphi_\gamma(0) < 1/4\,[\zeta_*(8-\kappa) - 3\kappa]$ and $\zeta_* > 3\kappa/(8-\kappa)$. Consequently, combining (39) and (40), we select η_3 and ε_1 small enough so that

$$(1-\zeta_\star)\left[\eta_3 + \frac{3}{2}\kappa\widehat{\zeta}(\chi_n)\right] + \frac{(1-\varepsilon)}{2}\varphi_\gamma(0) + [(1-\sigma)+\sigma](\zeta_\star - \varepsilon)\left[\frac{2\varepsilon_1 l^4}{EI} - \left(1 - \frac{\kappa}{2}\right)\right] < 0.$$

Once $\eta_3, \varepsilon_1, n, t_\star$, and ε are fixed, we pick $\eta_1, \varepsilon_2, \lambda_2$ small enough and we select k_r large enough so that

$$\begin{cases} \lambda_2\left(\frac{1-\zeta_\star}{4\eta_3} + 2\right)\kappa - \frac{1}{4n} < 0,\\ \lambda_2\left(2\varepsilon_2 + k_p + \frac{k_r}{\varepsilon}\right)\kappa - \frac{1}{8n} < 0,\\ \frac{EI}{4} - \frac{\rho A l^4}{2\varepsilon}\zeta(0)\,\lambda_2 > 0,\\ \frac{1}{4} - \frac{2\rho A l + m}{2\varepsilon}\zeta(0)\,\lambda_2 > 0,\\ \left[\lambda_2\left(2\varepsilon_2 + k_p + \frac{2k_r}{\zeta_\star - \varepsilon}\right) + \frac{1}{8\kappa n}\right]\kappa\widehat{\zeta}(\chi_n) + 2\varepsilon_2 l\kappa\lambda_2 - 1 < 0\\ 2\kappa\widehat{\zeta}(\chi_n) + \frac{\varepsilon_2 l^4\kappa}{EI} - \frac{\zeta_\star - \varepsilon}{2} < 0,\\ \varepsilon_1(2l+1) + \eta_1 k_p^2 - \frac{k_r}{2} < 0. \end{cases}$$

Finally, we select k_p large enough, such that

$$-k_p+\frac{\varepsilon\rho A\lambda_2}{4}+\frac{2k_r}{(\zeta_\star-\varepsilon)\lambda_2}+\left(m+\frac{1}{4\eta_1}\right)(\zeta_\star-\varepsilon)\lambda_2+\left[\frac{k_p}{2}+m\left(\frac{\varepsilon}{2}-\zeta_\star\right)\right]\lambda_2+4\kappa n+\frac{2}{k_r(\zeta_\star-\varepsilon)\lambda_2}+\frac{2\overline{\varphi}_\alpha}{\alpha}<0.$$

Together with (38), those choices lead to

$$\begin{aligned}\frac{d}{dt}K(t)\leqslant&-c_0\left[\mathcal{E}(t)+\Xi_3(t)+\Xi_4(t)\right]-\lambda_5EI\delta(t)\Xi_5(t)\\&+\varrho\left(\int_0^{l/2}\left[f^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[f^R(x,t)\right]^2dx+d^2(l/2,t)\right),\ t\geqslant t_\star>0\end{aligned}$$

for some positive constants c_0 and ϱ.

If $\lim_{t\to+\infty}\delta(t)=0$, then, for this constant c_0, there exists a $\bar{t}(c_0)\geqslant t_*$ such that $\delta(t)\leqslant c_0$ for $t\geqslant\bar{t}(c_0)$. Therefore, from the right-hand side of Lemma 6, we obtain

$$\frac{d}{dt}K(t)\leqslant-c_1\delta(t)K(t)+\varrho\left(\int_0^{l/2}\left[f^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[f^R(x,t)\right]^2dx+d^2(l/2,t)\right),\ t\geqslant\widehat{t},\tag{41}$$

for some positive constant c_1. Now, applying Lemma 4 with

$$Q(t)=c_1\delta(t),\sigma(t)=0\text{ and }\beta(t)=\varrho\left(\int_0^{l/2}\left[f^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[f^R(x,t)\right]^2dx+d^2(l/2,t)\right),$$

we infer that from (41)

$$\mathcal{E}(t)\leqslant\frac{C}{\Psi(t)},\ t\geqslant0$$

for some $C>0$, on condition $\Psi(0)K(0)<1$.

If $\lim_{t\to+\infty}\delta(t)=\overline{\delta}\neq0$, then there exist a $\widehat{t}\geqslant t_*$ such that $\delta(t)\geqslant c_0$ for $\widehat{t}\geqslant t_*$. Hence,

$$\frac{d}{dt}K(t)\leqslant-c_2K(t)+\varrho\left(\int_0^{l/2}\left[f^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[f^R(x,t)\right]^2dx+d^2(l/2,t)\right),\ t\geqslant\widehat{t},\tag{42}$$

for $c_2>0$, by taking

$$Q(t)=c_2\delta(t),\sigma(t)=0\text{ and }\beta(t)=\varrho\left(\int_0^{l/2}\left[f^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[f^R(x,t)\right]^2dx+d^2(l/2,t)\right).$$

We conclude that

$$\mathcal{E}(t)\leqslant\frac{C}{\Psi(t)},\ t\geqslant0$$

for some positive constant $C>0$, provided that $\Psi(0)K(0)<1$. □

5 | SIMULATION ANALYSIS

In this section, the numerical simulations are carried out to illustrate the effectiveness of the proposed control. The corresponding system parameters are $\rho=2.7\times10^3\,\text{kg/m}^3$, $A=0.12\,\text{m}^2$, $E=6{,}894\times10^{10}\,\text{N m}^{-2}$, $I=1.734\times10^{-7}\,\text{m}^4$, $l/2=8\,\text{m}$, $m=50\,\text{kg}$, $k_p=110$, $k_r=120$, and the initial displacement and velocity are

$$v_0^L(x)=v_0^R(x)=\sin\left(\frac{\pi}{l}x\right)\text{ and }v_1^L(x)=v_1^R(x)=0.$$

The kernel function is chosen to be $\zeta(t) = e^{-\tau t}$ for some $\tau \in \mathbb{R}^+$.

The external disturbances $f^L(x,t)$, $f^R(x,t)$, and $d(l/2,t)$ are given as

$$f^L(x,t) = 0.02\sin(0.002\pi xt)\,,\quad f^R(x,t) = 0.01\sin(0.001\pi xt)\,,\quad \text{and } d(l/2,t) = 0.2 + 0.01\sin(0.001t)\,.$$

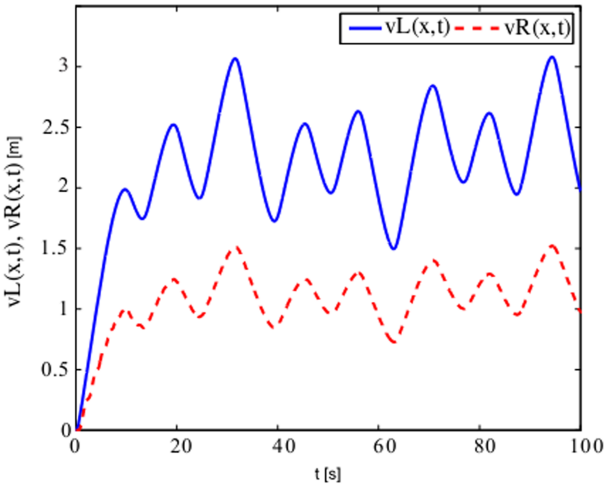


FIGURE 2 Displacement of the flexible satellite without control [Colour figure can be viewed at wileyonlinelibrary.com]

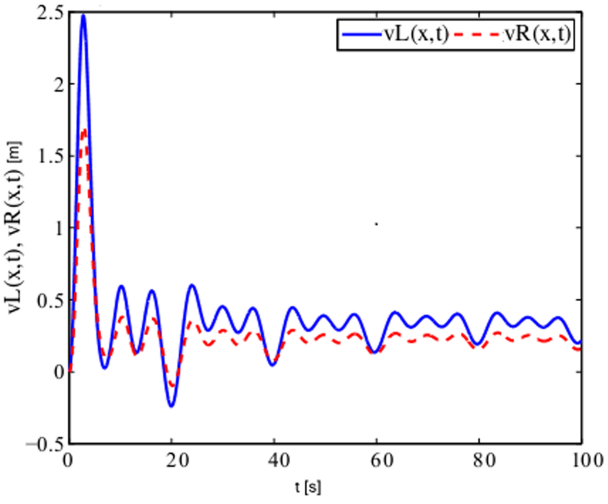


FIGURE 3 Displacement of the flexible satellite with the proposed control [Colour figure can be viewed at wileyonlinelibrary.com]

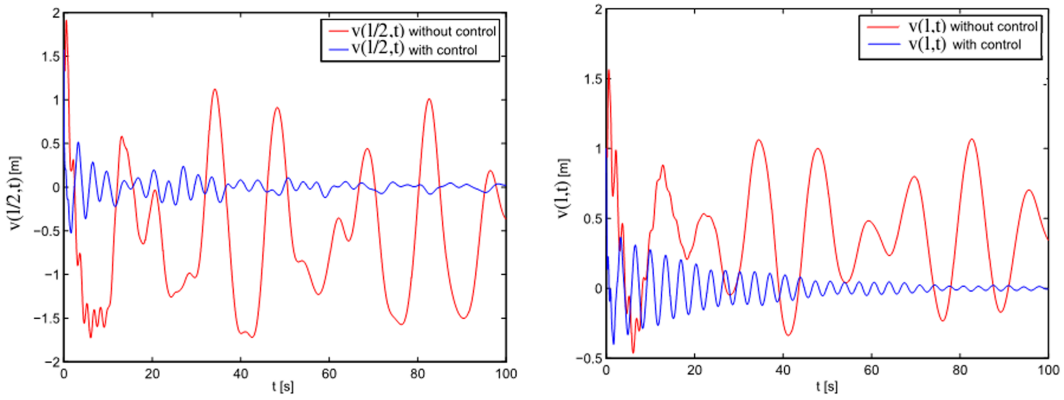


FIGURE 4 Boundary displacement of the center body $\upsilon(l/2,t)$ and the boundary displacement $\upsilon(l,t)$ of flexible satellite [Colour figure can be viewed at wileyonlinelibrary.com]

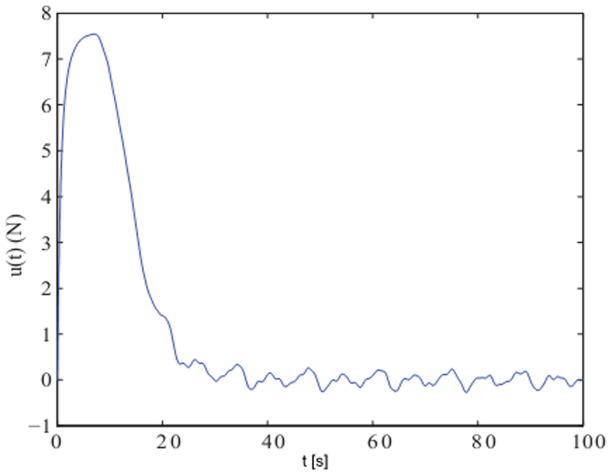


FIGURE 5 Control input [Colour figure can be viewed at wileyonlinelibrary.com]

We have used finite difference method for numerical approximation of the problem (5)–(6). Central difference approximations for both spatial and temporal derivatives in the differential equation (5) and the boundary condition (6) have been adopted.

For the convolution term, we have used the trapezoid method. We note that the convolution term requires to store the solution from $t = 0$ and use them at each time step. This, together with large values of the model parameters, demands very small values of time steps that increase the computational time. Therefore, one can scale the parameters for computer simulations to overcome the time performance.

In Figure 2, we plot the behavior of the solution of the system (5) to (7) without control, and the vibration in the flexible spacecraft under the distributed disturbances should be effectively suppressed in the practical systems.

The behavior of the solution of the model problem (5)–(7), with the control (23), has been illustrated in Figure 3. We can see that the displacement can be suppressed within the proposed control law. The displacement of the center body and the right boundary displacement of flexible satellite without and with control are shown in Figure 4. The boundary control input imported on the center body is shown in Figure 5.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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APPENDIX A

In this section, we establish some lemmas that will be of help in the main result (i.e., Theorem 2).

Lemma 6. *There exist two positive constants δ_1 and δ_2, such that*

$$\delta_1\left(\mathcal{E}(t)+\sum_{i=3}^{5}\Xi_i(t)\right)\leqslant K(t)\leqslant\delta_2\left(\mathcal{E}(t)+\sum_{i=3}^{5}\Xi_i(t)\right), \tag{A1}$$

for all $t\geqslant 0$.

Proof. By using Cauchy–Schwarz, Young's, and Poincarés inequalities, we obtain

$$\begin{aligned}\Xi_1(t)\leqslant\frac{\rho A}{2}\int_0^{l/2}\left[v^L(x,t)\right]^2dx+\frac{\rho A}{2}\int_0^{l/2}\left[v_t^L(x,t)\right]^2dx+\frac{\rho A}{2}\int_{l/2}^{l}\left[v^R(x,t)\right]^2dx\\+\frac{\rho A}{2}\int_{l/2}^{l}\left[v_t^R(x,t)\right]^2dx+\frac{m}{2}\left[v_t(l/2,t)\right]^2+\frac{m}{2}\left[v(l/2,t)\right]^2,\end{aligned} \tag{A2}$$

and from Lemma 1 and Remark 3, we get

$$\int_0^{l/2}\left[v^L(x,t)\right]^2dx\leqslant l\left[v(l/2,t)\right]^2+l^4\int_0^{l/2}\left[v_{xx}^L(x,t)\right]^2dx, \tag{A3}$$

and

$$\int_{l/2}^{l}\left[v^R(x,t)\right]^2dx\leqslant l\left[v(l/2,t)\right]^2+l^4\int_{l/2}^{l}\left[v_{xx}^R(x,t)\right]^2dx. \tag{A4}$$

Then, we have

$$\begin{aligned}\Xi_1(t)\leqslant\frac{\rho Al^4}{2}\left(\int_0^{l/2}\left[v_{xx}^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[v_{xx}^R(x,t)\right]^2dx\right)+\frac{m}{2}\left[v_t(l/2,t)\right]^2\\+\frac{\rho A}{2}\left(\int_0^{l/2}\left[v_t^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[v_t^R(x,t)\right]^2dx\right)+\left(\rho Al+\frac{m}{2}\right)\left[v(l/2,t)\right]^2,\ t\geqslant 0.\end{aligned} \tag{A5}$$

For the functional $\Xi_2(t)$, we have

$$\begin{aligned}\Xi_2(t)\leqslant\frac{\rho A}{2}\left(\int_0^{l/2}\left[v_t^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[v_t^R(x,t)\right]^2dx\right)+\frac{m}{2}\left[v_t(l/2,t)\right]^2+\frac{m}{2}\left(\int_0^{t}\zeta(s)\,ds\right)\left(\zeta\diamondsuit v\right)(t)\\+\frac{\rho A}{2}\left(\int_0^{t}\zeta(s)\,ds\right)\left(\int_0^{l/2}\left(\zeta\square v^L\right)(t)\,dx+\int_{l/2}^{l}\left(\zeta\square v^R\right)(t)\,dx\right),\ t\geqslant 0,\end{aligned}$$

and making use the Remark 3 again, we get

$$\int_0^{l/2} \left(\zeta \square v^L\right)(t)\, dx \leqslant l\left(\zeta \diamondsuit v\right)(t) + l^4 \int_0^{l/2} \left(\zeta \square v_{xx}^L\right)(t)\, dx,\ t \geqslant 0$$

and

$$\int_{l/2}^{l} \left(\zeta \square v^R\right)(t)\, dx \leqslant l\left(\zeta \diamondsuit v\right)(t) + l^4 \int_{l/2}^{l} \left(\zeta \square v_{xx}^R\right)(t)\, dx,\ t \geqslant 0.$$

Therefore

$$\begin{aligned}\Xi_2(t) \leqslant \frac{\rho A}{2}\left(\int_0^{l/2} \left[v_t^L(x,t)\right]^2 dx + \int_{l/2}^{l} \left[v_t^R(x,t)\right]^2 dx\right) + \kappa\left(\frac{m}{2} + \rho A l\right)\left(\zeta \diamondsuit v\right)(t) \\ + \frac{\rho A l^4 \kappa}{2}\left(\int_0^{l/2} \left(\zeta \square v_{xx}^L\right)(t)\, dx + \int_{l/2}^{l} \left(\zeta \square v_{xx}^R\right)(t)\, dx\right) + \frac{m}{2}\left[v_t(l/2,t)\right]^2.\end{aligned} \tag{A6}$$

Now, by using (17), (A5), and (A6), we conclude that

$$\begin{aligned}K(t) \leqslant \frac{1}{2}\rho A(1+\lambda_1+\lambda_2)\left(\int_0^{l/2} \left[v_t^L(x,t)\right]^2 dx + \int_{l/2}^{l} \left[v_t^R(x,t)\right]^2 dx\right) + \frac{1}{2} m(1+\lambda_1+\lambda_2)\left[v_t(l/2,t)\right]^2 + \Xi_4(t) + \lambda_5 \Xi_5(t) \\ + \left(\frac{EI}{2} + \lambda_2 \frac{\rho A l^4 \kappa}{2}\right)\left(\int_0^{l/2} \left(\zeta \square v_{xx}^L\right)(x,t)\, dx + \int_{l/2}^{l} \left(\zeta \square v_{xx}^R\right)(x,t)\, dx\right) + \frac{1}{2}\left[\lambda_1(2\rho A l + m) + 1\right]\left[v(l/2,t)\right]^2 \\ + \left[\frac{EI}{2}\left(1 - \int_0^t \zeta(s)\, ds\right) + \lambda_1 \frac{\rho A l^4}{2}\right]\left(\int_0^{l/2} \left[v_{xx}^L(x,t)\right]^2 dx + \int_{l/2}^{l} \left[v_{xx}^R(x,t)\right]^2 dx\right) + \frac{1}{2}\left[\lambda_2 \kappa(m + 2\rho A l) + 1\right]\left(\zeta \diamondsuit v\right)(t).\end{aligned}$$

Similarly,

$$\begin{aligned}2K(t) \geqslant \rho A(1-\lambda_1-\lambda_2)\left(\int_0^{l/2} \left[v_t^L(x,t)\right]^2 dx + \int_{l/2}^{l} \left[v_t^R(x,t)\right]^2 dx\right) + m(1-\lambda_1-\lambda_2)\left[v_t(l/2,t)\right]^2 + 2\Xi_4(t) + 2\lambda_5 \Xi_5(t) \\ + \left(EI - \lambda_2 \rho A l^4 \kappa\right)\left(\int_0^{l/2} \left(\zeta \square v_{xx}^L\right)(x,t)\, dx + \int_{l/2}^{l} \left(\zeta \square v_{xx}^R\right)(x,t)\, dx\right) + \left[1 - \lambda_1(2\rho A l + m)\right]\left[v(l/2,t)\right]^2 \\ + \left[EI(1-\kappa) - \lambda_1 \rho A l^4\right]\left(\int_0^{l/2} \left[v_{xx}^L(x,t)\right]^2 dx + \int_{l/2}^{l} \left[v_{xx}^R(x,t)\right]^2 dx\right) + \left[1 - \lambda_2 \kappa(m + 2\rho A l)\right]\left(\zeta \diamondsuit v\right)(t).\end{aligned}$$

Thus, there exists $\delta_i > 0, i = 1, 2$ such that $\delta_1\left(\mathcal{E}(t) + \sum_{i=3}^{5} \Xi_i(t)\right) \leqslant K(t) \leqslant \delta_2\left(\mathcal{E}(t) + \sum_{i=3}^{5} \Xi_i(t)\right)$, where

$$\lambda_1 < \min\left[1, \frac{1}{2\rho A l + m}, \frac{EI(1-\kappa)}{\rho A l^4}\right] \text{ and } \lambda_2 < \min\left[1 - \lambda_1, \frac{1}{(2\rho A l + m)\kappa}, \frac{EI}{\rho A l^4 \kappa}\right].$$

□

Lemma 7. *Let (v^L, v^R) be the solution of (5)–(7). Then, the functional $\Xi_1(t)$ satisfies for any positive η_1 and ε_1, the estimate*

$$\begin{aligned}\frac{d}{dt}\Xi_1(t) \leqslant \rho A\left(\int_0^{l/2}\left[v_t^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[v_t^R(x,t)\right]^2dx\right)+\left(m+\frac{1}{4\eta_1}\right)\left[v_t(l/2,t)\right]^2\\ -\frac{EI}{2}\left(\int_0^{l/2}\left(\zeta\square v_{xx}^L\right)(x,t)\,dx+\int_{l/2}^{l}\left(\zeta\square v_{xx}^R\right)(x,t)\,dx\right)+\left[\varepsilon_1(2l+1)+\eta_1k_p^2-k_r\right]\left[v(l/2,t)\right]^2\\ +\frac{EI}{2}\left(\int_0^t\zeta(t-s)\int_0^{l/2}\left[v_{xx}^L(x,s)\right]^2dxds+\int_0^t\zeta(t-s)\int_{l/2}^{l}\left[v_{xx}^R(x,s)\right]^2dxds\right)\\ +EI\left[\frac{2\varepsilon_1l^4}{EI}-\left(1-\frac{\kappa}{2}\right)\right]\left(\int_0^{l/2}\left[v_{xx}^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[v_{xx}^R(x,t)\right]^2dx\right)\\ +\frac{1}{4\varepsilon_1}\left(\int_0^{l/2}\left[f^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[f^R(x,t)\right]^2dx+d^2(l/2,t)\right)\end{aligned}\tag{A7}$$

for $t\geq 0$.

Proof. Direct computations, using (23), we get

$$\begin{aligned}\frac{d}{dt}\Xi_1(t)=I_1(t)+I_2(t)+\rho A\int_0^{l/2}\left[v_t^L(x,t)\right]^2dx+\rho A\int_{l/2}^{l}\left[v_t^R(x,t)\right]^2dx+m\left[v_t(l/2,t)\right]^2+EIv(l/2,t)\,v_{xxx}^L(l/2,t)\\ -EIv(l/2,t)\int_0^t\zeta(t-s)\,v_{xxx}^L(l/2,s)\,ds+EIv(l/2,t)\int_0^t\zeta(t-s)\,v_{xxx}^R(l/2,s)\,ds-EIv(l/2,t)\,v_{xxx}^R(l/2,t)\\ -k_pv(l/2,t)\,v_t(l/2,t)-k_r\left[v(l/2,t)\right]^2+\int_0^{l/2}v^L(x,t)\,f^L(x,t)dx+\int_{l/2}^{l}v^R(x,t)\,f^R(x,t)dx+d(l/2,t)v(l/2,t),\end{aligned}\tag{A8}$$

where

$$I_1(t)=-EI\int_0^{l/2}v^L(x,t)\,v_{xxxx}^L(x,t)\,dx-EI\int_{l/2}^{l}v^R(x,t)\,v_{xxxx}^R(x,t)\,dx,$$

and

$$I_2(t)=EI\int_0^{l/2}v^L(x,t)\int_0^t\zeta(t-s)\,v_{xxxx}^L(x,s)\,dsdx+EI\int_{l/2}^{l}v^R(x,t)\int_0^t\zeta(t-s)\,v_{xxxx}^R(x,s)\,dsdx.$$

Integrating by parts in I_i, $i=1,2$, and in view of the boundary (6), we have

$$I_1(t)=-EIv(l/2,t)\,v_{xxx}^L(l/2,t)-EI\int_0^{l/2}\left[v_{xx}^L(x,t)\right]^2dx+EIv(l/2,t)\,v_{xxx}^R(l/2,t)-EI\int_{l/2}^{l}\left[v_{xx}^R(x,t)\right]^2dx,\ t\geqslant 0\tag{A9}$$

and

$$I_2(t) = EI\upsilon(l/2,t)\left(\int_0^t \zeta(t-s)\,\upsilon_{xxx}^L(l/2,s)\,ds - \int_0^t \zeta(t-s)\,\upsilon_{xxx}^R(l/2,s)\,ds\right) + EI\left(\int_0^{l/2} \upsilon_{xx}^L(x,t)\int_0^t \zeta(t-s)\,\upsilon_{xx}^L(x,s)\,dsdx + \int_{l/2}^{l} \upsilon_{xx}^R(x,t)\int_0^t \zeta(t-s)\,\upsilon_{xx}^R(x,s)\,dsdx\right),\ t \geqslant 0. \tag{A10}$$

Substituting the estimates (A9) and (A10) in (A8), we get

$$\begin{aligned}\frac{d}{dt}\Xi_1(t) = {}& \rho A\left(\int_0^{l/2}\left[\upsilon_t^L(x,t)\right]^2dx + \int_{l/2}^{l}\left[\upsilon_t^R(x,t)\right]^2dx\right) - EI\left(\int_0^{l/2}\left[\upsilon_{xx}^L(x,t)\right]^2dx + \int_{l/2}^{l}\left[\upsilon_{xx}^R(x,t)\right]^2dx\right)\\ &+ EI\left(\int_0^{l/2} \upsilon_{xx}^L(x,t)\int_0^t \zeta(t-s)\,\upsilon_{xx}^L(x,s)\,dsdx + \int_{l/2}^{l} \upsilon_{xx}^R(x,t)\int_0^t \zeta(t-s)\,\upsilon_{xx}^R(x,s)\,dsdx\right) - k_r\left[\upsilon(l/2,t)\right]^2\\ &+ m\left[\upsilon_t(l/2,t)\right]^2 - k_p\upsilon(l/2,t)\,\upsilon_t(l/2,t) + \int_0^{l/2} v^L(x,t)\,f^L(x,t)dx + \int_{l/2}^{l} v^R(x,t)\,f^R(x,t)dx + d(l/2,t)\upsilon(l/2,t).\end{aligned} \tag{A11}$$

Now, we will estimate some terms in (*A*11). For $\eta_1 > 0$ and $\varepsilon_1 > 0$, we have

$$k_p\upsilon(l/2,t)\,\upsilon_t(l/2,t) \leqslant \eta_1 k_p^2\left[\upsilon(l/2,t)\right]^2 + \frac{1}{4\eta_1}\left[\upsilon_t(l/2,t)\right]^2, \tag{A12}$$

$$\int_0^{l/2} v^L(x,t)\,f^L(x,t)dx \leq \varepsilon_1 l\left[\upsilon(l/2,t)\right]^2 + \varepsilon_1 l^4\int_0^{l/2}\left[\upsilon_{xx}^L(x,t)\right]^2dx + \frac{1}{4\varepsilon_1}\int_0^{l/2}\left[f^L(x,t)\right]^2dx, \tag{A13}$$

$$\int_{l/2}^{l} v^R(x,t)\,f^R(x,t)dx \leq \varepsilon_1 l\left[\upsilon(l/2,t)\right]^2 + \varepsilon_1 l^4\int_{l/2}^{l}\left[\upsilon_{xx}^R(x,t)\right]^2dx + \frac{1}{4\varepsilon_1}\int_{l/2}^{l}\left[f^R(x,t)\right]^2dx, \tag{A14}$$

and

$$d(l/2,t)\upsilon(l/2,t) \leq \frac{1}{4\varepsilon_1}d^2(l/2,t) + \varepsilon_1\left[\upsilon(l/2,t)\right]^2 \tag{A15}$$

for all $t \geq 0$, and by using Lemma 3, we obtain (A7). □

Lemma 8. *Let (v^L, v^R) be the solution of (5)–(7). Then, for some positive constants η_i, $i = 2, 3, 4$, the functional $\Xi_2(t)$ satisfies the estimate*

$$\frac{d}{dt}\Xi_2(t) \leqslant (\eta_4-\zeta_\star)\rho A\left(\int_0^{l/2}\left[v_t^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[v_t^R(x,t)\right]^2dx\right)+\left[\frac{k_p}{2}+m(\eta_4-\zeta_\star)\right]\left[v_t(l/2,t)\right]^2$$
$$-\left[\frac{\rho Al}{2\eta_4}+\frac{m}{4\eta_4}\right]\zeta(0)\left(\zeta'\diamondsuit v\right)(t)-\frac{\rho Al^4}{4\eta_4}\zeta(0)\left(\int_0^{l/2}\left(\zeta'\square v_{xx}^L\right)(t)\,dx+\int_{l/2}^{l}\left(\zeta'\square v_{xx}^R\right)(t)\,dx\right)$$
$$+EI(1-\zeta_\star)\left[\eta_3+\frac{3}{2}\left(\int_{\chi_t}\zeta(t-s)\,ds\right)\right]\left(\int_0^{l/2}\left[v_{xx}^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[v_{xx}^R(x,t)\right]^2dx\right)$$
$$+k_r\eta_2\left[v(l/2,t)\right]^2+\left[\frac{EI(1-\zeta_\star)}{4\eta_3}+2EI\right]\kappa\left(\int_0^{l/2}\int_{A_t}\zeta(t-s)\left(v_{xx}^L(t)-v_{xx}^L(s)\right)^2dsdx\right.$$
$$\left.+\int_{l/2}^{l}\int_{\mathcal{Z}_t}\zeta(t-s)\left(v_{xx}^R(t)-v_{xx}^R(s)\right)^2dsdx\right)+\left(2\varepsilon_2+k_p+\frac{k_r}{2\eta_4}\right)\kappa\int_{\mathcal{Z}_t}\zeta(t-s)\left(v(l/2,t)-v(l/2,s)\right)^2dsdx$$
$$+\frac{EI}{2}(1-\zeta_\star)\left(\int_{\chi_t}\zeta(t-s)\int_0^{l/2}\left[v_{xx}^L(x,s)\right]^2dxds+\int_{\chi_t}\zeta(t-s)\int_{l/2}^{l}\left[v_{xx}^R(x,s)\right]^2dxds\right)+\left(\int_{\chi_t}\zeta(t-s)\,ds\right)$$
$$\times 2EI\left(\int_0^{l/2}\int_{\chi_t}\zeta(t-s)\left(v_{xx}^L(x,t)-v_{xx}^L(x,s)\right)^2dsdx+\int_{l/2}^{l}\int_{\chi_t}\zeta(t-s)\left(v_{xx}^R(x,t)-v_{xx}^R(x,s)\right)^2dsdx\right)$$
$$+\left(2\varepsilon_2+k_p+\frac{k_r}{2\eta_2}\right)\left(\int_{\chi_t}\zeta(t-s)\,ds\right)\int_{\chi_t}\zeta(t-s)\left(v(l/2,t)-v(l/2,s)\right)^2dsdx+\varepsilon_2 l^4\kappa\left(\int_0^{l/2}\left(\zeta\square v_{xx}^L\right)(t)\,dx\right.$$
$$\left.+\int_{l/2}^{l}\left(\zeta\square v_{xx}^R\right)(t)\,dx\right)+2\varepsilon_2 l\kappa\left(\zeta\diamondsuit v\right)(t)+\frac{1}{4\varepsilon_1}\left(\int_0^{l/2}\left[f^L(x,t)\right]^2dx+\int_{l/2}^{l}\left[f^R(x,t)\right]^2dx+d^2(l/2,t)\right),$$

for all $t\geq t_\star>0$ *where* $\varepsilon_2>0$.

Proof. The differentiation of the functional $\Xi_2(t)$ yields

$$\frac{d}{dt}\Xi_2(t)=-\rho A\int_0^{l/2}v_{tt}^L(x,t)\int_0^t\zeta(t-s)\left(v^L(x,t)-v^L(x,s)\right)dsdx-\rho A\left(\int_0^t\zeta(s)\,ds\right)\int_0^{l/2}\left[v_t^L(x,t)\right]^2dx$$
$$-\rho A\int_0^{l/2}v_t^L(x,t)\int_0^t\zeta'(t-s)\left(v^L(x,t)-v^L(x,s)\right)dsdx-\rho A\left(\int_0^t\zeta(s)\,ds\right)\int_{l/2}^{l}\left[v_t^R(x,t)\right]^2dx$$
$$-\rho A\int_{l/2}^{l}v_{tt}^R(x,t)\int_0^t\zeta(t-s)\left(v^R(x,t)-v^R(x,s)\right)dsdx-mv_{tt}(l/2,t)\int_0^t\zeta(t-s)\left(v(l/2,t)-v(l/2,s)\right)ds$$
$$-\rho A\int_{l/2}^{l}v_t^R(x,t)\int_0^t\zeta'(t-s)\left(v^R(x,t)-v^R(x,s)\right)dsdx-mv_t(l/2,t)\int_0^t\zeta'(t-s)\left(v(l/2,t)-v(l/2,s)\right)ds$$
$$-m\left(\int_0^t\zeta(s)\,ds\right)\left[v_t(l/2,t)\right]^2,\ t\geqslant 0.$$

By using Equation (5), boundary condition $(6)_3$, and (23), we find

$$\begin{aligned}\frac{d}{dt}\Xi_2(t) &= I_3(t)+I_4(t)+I_5(t)+I_6(t)-\rho A\int_0^{l/2} v_t^L(x,t)\int_0^t \zeta'(t-s)\left(v^L(x,t)-v^L(x,s)\right)dsdx\\ &-\rho A\int_0^t \zeta(s)\,ds\int_0^{l/2}\left[v_t^L(x,t)\right]^2dx-\rho A\int_{l/2}^{l} v_t^R(x,t)\int_0^t \zeta'(t-s)\left(v^R(x,t)-v^R(x,s)\right)dsdx\\ &-\rho A\int_0^t \zeta(s)\,ds\int_{l/2}^{l}\left[v_t^R(x,t)\right]^2dx-\left(EIv_{xxx}^L(l/2,t)+d(l/2,t)\right)\int_0^t \zeta(t-s)\left(v(l/2,t)-v(l/2,s)\right)ds\\ &+EI\left(\int_0^t \zeta(t-s)\,v_{xxx}^L(l/2,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(v(l/2,t)-v(l/2,s)\right)ds\right)\\ &-EI\left(\int_0^t \zeta(t-s)\,v_{xxx}^R(l/2,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(v(l/2,t)-v(l/2,s)\right)ds\right)\\ &+\left(EIv_{xxx}^R(l/2,t)+k_pv_t(l/2,t)+k_rv(l/2,t)\right)\int_0^t \zeta(t-s)\left(v(l/2,t)-v(l/2,s)\right)ds\\ &-mv_t(l/2,t)\int_0^t \zeta'(t-s)\left(v(l/2,t)-v(l/2,s)\right)ds-m\left(\int_0^t \zeta(s)\,ds\right)\left[v_t(l/2,t)\right]^2\\ &-\int_0^{l/2} f^L(x,t)\int_0^t \zeta(t-s)\left(v^L(x,t)-v^L(x,s)\right)dsdx-\int_{l/2}^{l} f^R(x,t)\int_0^t \zeta(t-s)\left(v^R(x,t)-v^R(x,s)\right)dsdx,\end{aligned}\tag{A16}$$

where

$$I_3(t)=EI\int_0^{l/2} v_{xxxx}^L(x,t)\int_0^t \zeta(t-s)\left(v^L(x,t)-v^L(x,s)\right)dsdx,$$

$$I_4(t)=EI\int_{l/2}^{l} v_{xxxx}^R(x,t)\int_0^t \zeta(t-s)\left(v^R(x,t)-v^R(x,s)\right)dsdx,$$

$$I_5(t)=-EI\int_0^{l/2}\left(\int_0^t \zeta(t-s)\,v_{xxxx}^L(x,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(v^L(x,t)-v^L(x,s)\right)ds\right)dx,$$

and

$$I_6(t)=-EI\int_{l/2}^{l}\left(\int_0^t \zeta(t-s)\,v_{xxxx}^R(x,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(v^R(x,t)-v^R(x,s)\right)ds\right)dx.$$

Integrating by parts twice and using the boundary condition (6), we get

$$\begin{aligned}I_3(t)&=EIv_{xxx}^L(l/2,t)\int_0^t \zeta(t-s)\left(v^L(l/2,t)-v^L(l/2,s)\right)ds\\ &+EI\int_0^{l/2} v_{xx}^L(x,t)\int_0^t \zeta(t-s)\left(v_{xx}^L(x,t)-v_{xx}^L(x,s)\right)dsdx,\ \ t\geq 0\end{aligned}$$

$$I_4(t) = -EIv_{xxx}^R(l/2,t)\int_0^t \zeta(t-s)\left(v^R(l/2,t)-v^R(l/2,s)\right)ds + EI\int_{l/2}^{l} v_{xx}^R(x,t)\int_0^t \zeta(t-s)\left(v_{xx}^R(x,t)-v_{xx}^R(x,s)\right)dsdx,\ t\geq 0$$

$$I_5(t) = -EI\left(\int_0^t \zeta(t-s)\,v_{xxx}^L(l/2,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(v^L(l/2,t)-v^L(l/2,s)\right)ds\right) - EI\int_0^{l/2}\left(\int_0^t \zeta(t-s)\,v_{xx}^L(x,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(v_{xx}^L(x,t)-v_{xx}^L(x,s)\right)ds\right)dx,\ t\geq 0$$

and

$$I_6(t) = EI\left(\int_0^t \zeta(t-s)\,v_{xxx}^R(l/2,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(v^R(l/2,t)-v^R(l/2,s)\right)ds\right) - EI\int_{l/2}^{l}\left(\int_0^t \zeta(t-s)\,v_{xx}^R(x,s)\,ds\right)\left(\int_0^t \zeta(t-s)\left(v_{xx}^R(x,t)-v_{xx}^R(x,s)\right)ds\right)dx,\ t\geq 0.$$

Collecting the estimates $I_3(t)$, $I_4(t)$, $I_5(t)$, and $I_6(t)$ into (A16), we obtain

$$\begin{aligned}\frac{d}{dt}\Xi_2(t) = &-\int_0^{l/2} f^L(x,t)\int_0^t \zeta(t-s)\left(v^L(x,t)-v^L(x,s)\right)dsdx - \int_{l/2}^{l} f^R(x,t)\int_0^t \zeta(t-s)\left(v^R(x,t)-v^R(x,s)\right)dsdx\\ &- d(l/2,t)\int_0^t \zeta(t-s)\left(v(l/2,t)-v(l/2,s)\right)ds + \left(k_p v_t(l/2,t)+k_r v(l/2,t)\right)\int_0^t \zeta(t-s)\left(v(l/2,t)-v(l/2,s)\right)ds\\ &+ EI\left(1-\int_0^t \zeta(s)\,ds\right)\int_0^{l/2} v_{xx}^L(x,t)\int_0^t \zeta(t-s)\left(v_{xx}^L(x,t)-v_{xx}^L(x,s)\right)dsdx\\ &+ EI\left(1-\int_0^t \zeta(s)\,ds\right)\int_{l/2}^{l} v_{xx}^R(x,t)\int_0^t \zeta(t-s)\left(v_{xx}^R(x,t)-v_{xx}^R(x,s)\right)dsdx\\ &+ EI\int_{l/2}^{l}\left(\int_0^t \zeta(t-s)\left(v_{xx}^R(x,t)-v_{xx}^R(x,s)\right)ds\right)^2 dx + EI\int_0^{l/2}\left(\int_0^t \zeta(t-s)\left(v_{xx}^L(x,t)-v_{xx}^L(x,s)\right)ds\right)^2 dx\\ &- \left(\int_0^t \zeta(s)\,ds\right)\left(\rho A\int_0^{l/2}\left[v_t^L(x,t)\right]^2 dx + \rho A\int_{l/2}^{l}\left[v_t^R(x,t)\right]^2 dxm\left[v_t(l/2,t)\right]^2\right)\end{aligned} \tag{A17}$$

$$\begin{aligned}&- \rho A\int_0^{l/2} v_t^L(x,t)\int_0^t \zeta'(t-s)\left(v^L(x,t)-v^L(x,s)\right)dsdx - \rho A\int_{l/2}^{l} v_t^R(x,t)\int_0^t \zeta'(t-s)\left(v^R(x,t)-v^R(x,s)\right)dsdx\\ &- mv_t(l/2,t)\int_0^t \zeta'(t-s)\left(v(l/2,t)-v(l/2,s)\right)ds,\ t\geq 0.\end{aligned}$$

Now, using Lemma 1, Remark 3, and Young's and Cauchy–Schwarz inequalities, we estimate the terms on the right-hand side of expression (A17). First, we start with the first, second, third, fourth, and fifth terms; for $\varepsilon_2 > 0$, we have

$$\int_0^{l/2} f(x,t) \int_0^t \zeta(t-s)\left(v^L(x,t) - v^L(x,s)\right) ds dx \leq \varepsilon_2 l\kappa \left(\zeta \diamondsuit v\right)(t) + \varepsilon_2 l^4 \kappa \int_0^{l/2} \left(\zeta \square v_{xx}^L\right)(t)\, dx + \frac{1}{4\varepsilon_2} \int_0^{l/2} f^2(x,t) dx, \tag{A18}$$

$$\int_{l/2}^{l} f(x,t) \int_0^t \zeta(t-s)\left(v^R(x,t) - v^R(x,s)\right) ds dx \leq \varepsilon_2 l\kappa \left(\zeta \diamondsuit v\right)(t) + \varepsilon_2 l^4 \kappa \int_{l/2}^{l} \left(\zeta \square v_{xx}^R\right)(t)\, dx + \frac{1}{4\varepsilon_2} \int_{l/2}^{l} f^2(x,t) dx \tag{A19}$$

and for all measurable sets $\mathcal{Z}$ and χ such that $\mathcal{Z} = \mathbb{R}^+ \setminus \chi$, we have

$$\begin{aligned} d(l/2,t) \int_0^t \zeta(t-s)\left(v(l/2,t) - v(l/2,s)\right) ds &\leqslant \frac{1}{4\varepsilon_2} d^2(l/2,t) + 2\kappa\varepsilon_2 \int_{\mathcal{Z}_t} \zeta(t-s)\left(v(l/2,t) - v(l/2,s)\right)^2 ds \\ &+ 2\varepsilon_2 \left(\int_{\chi_t} \zeta(t-s)\, ds\right) \int_{\chi_t} \zeta(t-s)\left(v(l/2,t) - v(l/2,s)\right)^2 ds, \end{aligned} \tag{A20}$$

$$\begin{aligned} v_t(l/2,t) \int_0^t \zeta(t-s)\left(v(l/2,t) - v(l/2,s)\right) ds &\leqslant \frac{1}{2}\left[v_t(l/2,t)\right]^2 + \kappa \int_{\mathcal{Z}_t} \zeta(t-s)\left(v(l/2,t) - v(l/2,s)\right)^2 ds \\ &+ \left(\int_{\chi_t} \zeta(t-s)\, ds\right) \int_{\chi_t} \zeta(t-s)\left(v(l/2,t) - v(l/2,s)\right)^2 ds, \end{aligned} \tag{A21}$$

and

$$\begin{aligned} v(l/2,t) \int_0^t \zeta(t-s)\left(v(l/2,t) - v(l/2,s)\right) ds &\leqslant \eta_2\left[v(l/2,t)\right]^2 + \frac{\kappa}{2\eta_2} \int_{\mathcal{Z}_t} \zeta(t-s)\left(v(l/2,t) - v(l/2,s)\right)^2 ds \\ &+ \frac{1}{2\eta_2} \left(\int_{\chi_t} \zeta(t-s)\, ds\right) \int_{\chi_t} \zeta(t-s)\left(v(l/2,t) - v(l/2,s)\right)^2 ds \end{aligned} \tag{A22}$$

for $\eta_2 > 0$, where we have used the notation $\mathcal{Z}_t = \mathcal{Z} \cap [0,t]$.

Similar to Tatar,[33] the sixth, seventh, eighth, and ninth terms can be handled in the following manner:

$$\begin{aligned} \int_0^{l/2} v_{xx}^L(x,t) \int_0^t \zeta(t-s)\left(v_{xx}^L(x,t) - v_{xx}^L(x,s)\right) ds dx &\leqslant \left(\eta_3 + \frac{3}{2} \int_{\chi_t} \zeta(t-s)\, ds\right) \int_0^{l/2} \left[v_{xx}^L(x,t)\right]^2 dx \\ &+ \frac{\kappa}{4\eta_3} \int_0^{l/2} \int_{\mathcal{Z}_t} \zeta(t-s)\left(v_{xx}^L(x,t) - v_{xx}^L(x,s)\right)^2 ds dx + \frac{1}{2} \int_{\chi_t} \zeta(t-s) \int_0^{l/2} \left[v_{xx}^L(x,s)\right]^2 dx ds, \end{aligned} \tag{A23}$$

$$\begin{aligned}\int_{l/2}^{l} v_{xx}^{R}(x,t)\int_{0}^{t}\zeta(t-s)\left(v_{xx}^{R}(x,t)-v_{xx}^{R}(x,s)\right)dsdx &\leqslant \left(\eta_3+\frac{3}{2}\int_{\chi_t}\zeta(t-s)\,ds\right)\int_{l/2}^{l}\left[v_{xx}^{R}(x,t)\right]^2dx\\ &+\frac{\kappa}{4\eta_3}\int_{l/2}^{l}\int_{\mathcal{Z}_t}\zeta(t-s)\left(v_{xx}^{R}(x,t)-v_{xx}^{R}(x,s)\right)^2dsdx+\frac{1}{2}\int_{\chi_t}\zeta(t-s)\int_{l/2}^{l}\left[v_{xx}^{R}(x,s)\right]^2dxds,\quad \eta_2>0\end{aligned} \tag{A24}$$

$$\begin{aligned}&\int_{0}^{l/2}\left|\int_{0}^{t}\zeta(t-s)\left(v_{xx}^{L}(t)-v_{xx}^{L}(s)\right)ds\right|^2dx\\ &\leqslant 2\kappa\int_{0}^{l/2}\int_{\mathcal{Z}_t}\zeta(t-s)\left(v_{xx}^{L}(t)-v_{xx}^{L}(s)\right)^2dsdx+2\left(\int_{\chi_t}\zeta(t-s)\,ds\right)\int_{0}^{l/2}\int_{\chi_t}\zeta(t-s)\left(v_{xx}^{L}(t)-v_{xx}^{L}(s)\right)^2dsdx\end{aligned} \tag{A25}$$

and

$$\begin{aligned}&\int_{l/2}^{l}\left|\int_{0}^{t}\zeta(t-s)\left(v_{xx}^{L}(t)-v_{xx}^{R}(s)\right)ds\right|^2dx\\ &\leqslant 2\kappa\int_{l/2}^{l}\int_{\mathcal{Z}_t}\zeta(t-s)\left(v_{xx}^{R}(t)-v_{xx}^{R}(s)\right)^2dsdx+2\left(\int_{\chi_t}\zeta(t-s)\,ds\right)\int_{l/2}^{l}\int_{\chi_t}\zeta(t-s)\left(v_{xx}^{R}(t)-v_{xx}^{R}(s)\right)^2dsdx.\end{aligned} \tag{A26}$$

Finally, using Lemma 1, hypothesis (H_2), and Remark 3, the last three terms can be estimated as follows:

$$\begin{aligned}&\int_{0}^{l/2}v_t^{L}(x,t)\int_{0}^{t}\zeta'(t-s)\left(v^{L}(x,t)-v^{L}(x,s)\right)dsdx\\ &\leqslant\eta_4\int_{0}^{l/2}\left[v_t^{L}(x,t)\right]^2dx+\frac{1}{4\eta_4}\left(\int_{0}^{t}\left|\zeta'(s)\right|ds\right)\int_{0}^{l/2}\left(\left|\zeta'\right|\square v^{L}\right)(t)\,dx\\ &\leqslant\eta_4\int_{0}^{l/2}\left[v_t^{L}(x,t)\right]^2dx-\frac{l}{4\eta_4}\zeta(0)\left(\zeta'\diamondsuit v\right)(t)-\frac{l^4}{4\eta_4}\zeta(0)\int_{0}^{l/2}\left(\zeta'\square v_{xx}^{L}\right)(t)\,dx,\end{aligned} \tag{A27}$$

$$\begin{aligned}&\int_{l/2}^{l}v_t^{L}(x,t)\int_{0}^{t}\zeta'(t-s)\left(v^{R}(x,t)-v^{R}(x,s)\right)dsdx\\ &\leqslant\eta_4\int_{l/2}^{l}\left[v_t^{R}(x,t)\right]^2dx+\frac{1}{4\eta_4}\left(\int_{0}^{t}\left|\zeta'(s)\right|ds\right)\int_{l/2}^{l}\left(\left|\zeta'\right|\square v^{R}\right)(t)\,dx\\ &\leqslant\eta_4\int_{l/2}^{l}\left[v_t^{R}(x,t)\right]^2dx-\frac{l}{4\eta_4}\zeta(0)\left(\zeta'\diamondsuit v\right)(t)-\frac{l^4}{4\eta_4}\zeta(0)\int_{l/2}^{l}\left(\zeta'\square v_{xx}^{R}\right)(t)\,dx,\end{aligned} \tag{A28}$$

and

$$\begin{aligned}
&\upsilon_t(l/2,t)\int_0^t \zeta'(t-s)\left(\upsilon(l/2,t)-\upsilon(l/2,s)\right)ds\\
&\leqslant \eta_4\left[\upsilon_t(l/2,t)\right]^2+\frac{1}{4\eta_4}\left(\int_0^t\left|\zeta'(s)\right|ds\right)\int_0^t\left|\zeta'(t-s)\right|\left(\upsilon(l/2,t)-\upsilon(l/2,s)\right)^2ds\\
&\leqslant \eta_4\left[\upsilon_t(l/2,t)\right]^2-\frac{1}{4\eta_4}\zeta(0)\left(\zeta'\diamondsuit\upsilon\right)(t),\ \eta_4>0\ t\geq 0.
\end{aligned}\tag{A29}$$

Collecting the previous estimates (A18) to (A29) in (A17), Lemma 8 is established. □

Lemma 9. *The derivative of the functional $\Xi_3(t)$ is estimated as follows:*

$$\begin{aligned}
\frac{d}{dt}\Xi_3(t)\leqslant\frac{1}{2}\left(\zeta'\diamondsuit\upsilon\right)(t)+\eta_6\left[\upsilon(l/2,t)\right]^2+2\kappa\eta_5\int_{\mathcal{Z}_t}\zeta(t-s)\left(\upsilon(l/2,t)-\upsilon(l/2,s)\right)^2ds\\
+\left(\frac{1}{4\eta_5}+\frac{1}{4\eta_6}\right)\left[\upsilon_t(l/2,t)\right]^2+2\eta_5\left(\int_{\chi_t}\zeta(t-s)\,ds\right)\int_{\chi_t}\zeta(t-s)\left(\upsilon(l/2,t)-\upsilon(l/2,s)\right)^2ds,
\end{aligned}\tag{A30}$$

for all $t\geqslant 0$ and some constants $\eta_5,\ \eta_6>0$.

Proof. Clearly, we have

$$\frac{d}{dt}\Xi_3(t)=\frac{1}{2}\left(\zeta'\diamondsuit\upsilon\right)(t)+\omega_t(l/2,t)\int_0^t\zeta(t-s)\left(\upsilon(l/2,t)-\upsilon(l/2,s)\right)ds+\upsilon_t(l/2,t)\,\upsilon(l/2,t),\ t\geqslant 0,\tag{A31}$$

then, using Lemma 1, we can write

$$\begin{aligned}
&\upsilon_t(l/2,t)\int_0^t\zeta(t-s)\left(\upsilon(l/2,t)-\upsilon(l/2,s)\right)ds\leqslant\frac{1}{4\eta_5}\left[\upsilon_t(l/2,t)\right]^2+2\kappa\eta_5\int_{\mathcal{Z}_t}\zeta(t-s)\left(\upsilon(l/2,t)-\upsilon(l/2,s)\right)^2ds\\
&+2\eta_5\left(\int_{\chi_t}\zeta(t-s)\,ds\right)\int_{\chi_t}\zeta(t-s)\left(\upsilon(l/2,t)-\upsilon(l/2,s)\right)^2ds,\quad\eta_5>0,\ t\geqslant 0,
\end{aligned}\tag{A32}$$

and

$$\upsilon_t(l/2,t)\,\upsilon(l/2,t)\leqslant\frac{1}{4\eta_6}\left[\upsilon_t(l/2,t)\right]^2+\eta_6\left[\upsilon(l/2,t)\right]^2,\ \eta_6>0,\ t\geqslant 0.\tag{A33}$$

This completes the proof. □

Lemma 10. *For all $t\geq 0$, we have*

$$\frac{d}{dt}\Xi_4(t)\leqslant\left(2\eta_7\overline{\varphi}_\alpha-\alpha\right)\Xi_4(t)-\left(\zeta\diamondsuit\upsilon\right)(t)+\frac{1}{2\eta_7}\left[\upsilon_t(l/2,t)\right]^2,\tag{A34}$$

where $\overline{\varphi}_\alpha=\int_0^\infty\varphi_\alpha(s)\,ds$ and $\eta_7>0$ is a constant.

Proof. The differentiation of $\Xi_4(t)$ yields

$$\frac{d}{dt}\Xi_4(t) = -\alpha\Xi_4(t) - \int_0^t \zeta(t-s)(\upsilon(l/2,t)-\upsilon(l/2,s))^2 ds + 2\upsilon_t(l/2,t)\int_0^t \varphi_\alpha(t-s)(\upsilon(l/2,t)-\upsilon(l/2,s))\,ds, \tag{A35}$$

then, the last term in the right-hand side of (A35) will be estimated as

$$\upsilon_t(l/2,t)\int_0^t \varphi_\alpha(t-s)(\upsilon(l/2,t)-\upsilon(l/2,s))\,ds \leqslant \frac{1}{4\eta_7}\left[\upsilon_t(l/2,t)\right]^2 + \eta_7\overline{\varphi}_\alpha\Xi_4(t),\ \eta_7>0,\ t\geqslant 0. \tag{A36}$$

This completes the proof. □

Lemma 11 (See reference[33]). *For the functional $\Xi_5(t)$, we have*

$$\begin{aligned}\frac{d}{dt}\Xi_5(t) \leqslant & EI\varphi_\gamma(0)\left(\int_0^{l/2}\left[\upsilon_{xx}^L(x,t)\right]^2 dx + \int_{l/2}^{l}\left[\upsilon_{xx}^R(x,t)\right]^2 dx\right) - EI\delta(t)\Xi_5(t)\\ & - EI\int_0^t \zeta(t-s)\left(\int_0^{l/2}\left[\upsilon_{xx}^L(x,s)\right]^2 dx + \int_{l/2}^{l}\left[\upsilon_{xx}^R(x,s)\right]^2 dx\right) ds,\end{aligned} \tag{A37}$$

for all $t \geq 0$.