METHODOLOGICAL ASPECTS OF A DECISION AID FOR TRANSPORTATION CHOICES UNDER UNCERTAINTY

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Submitted to the Department of Civil Engineering
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ABSTRACT

Though urban transportation planning decisions often have to be made with considerable uncertainty surrounding knowledge of their impacts, explicit consideration of this factor is usually not present in the supporting evaluation methodology. Furthermore, characteristics of the complex decision environment generally preclude the use of existing formal decision methodologies. This thesis develops a decision aid for a class of transportation decision problems in which an analyst is assisting a decision maker in selecting from a large finite set of options whose impact is not known with certainty. The features of this decision aid are specified in a set of basic considerations, reflecting characteristics of the problem environment as well as behavioral aspects of individual decision making. Building upon those considerations, an overall decision aiding framework is developed, consisting of three sequential activities: elimination, pairwise comparisons and global comparison.

The main thrust of the methodological development in this thesis consists of the pairwise comparisons procedure (within the above decision aiding framework) including its structure as well as its specific elements. The methodology revolves around the construction of a binary preference-indifference relation, over the set of options, that is consistent with the properties of partial semi-orders (postulated as a model for such a relation, based on the above basic considerations). The first component of the methodology consists of a mechanism for building an initial unrestrictive preference-indifference relation, featuring regret-based risk measures summarizing the implications of a binary choice situation; it is presented for an important problem subclass where the options' impact is expressed along a single attribute and uncertainty represented via alternative scenarios. Violations of the starting postulates by this initial relation are discussed, along with rules for identifying such inconsistencies in a given problem situation, constituting the second component of the methodology. Its third component then consists of approaches for correcting such inconsistencies; three such approaches are presented and their implications discussed. The remaining component consists of obtaining a reduced subset of superior options using the "corrected" relation.

Variations on some of the elements of the core methodology are presented, defining a family of decision rules for use within the above basic structure,

thus increasing the flexibility of the methodology by allowing its details to be fine-tuned to specific decision situations. In addition to analyzing mathematical properties of the above variations, problem situations where each would be appropriate are identified and illustrated via numerical examples.

The methodology is subsequently generalized to the case where uncertainty (in the alternative options' impact) is represented through probability density (or mass) functions. In addition, extension to a problem subclass where options are explicitly evaluated with respect to their uncertain impact along multiple criteria is discussed.

The thesis concludes with a discussion of issues facing (an analyst in) the application of the decision aid, and offers several guidelines and suggestions to that effect.

Thesis Supervisor: Dr. Yosef Sheffi

Title: Associate Professor of Civil Engineering

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CHAPTER 1

INTRODUCTION

Decisions in transportation planning often have to be made with considerable uncertainty surrounding knowledge of their impacts. Indeed, uncertainty in future conditions, both for the long run and increasingly in the short run, has emerged as one of the major factors that transportation systems analysts and planners have to contend with (Manheim, 1981; Visser, 1977; Westerman, 1981). Despite its (widely acknowledged) significance, explicit consideration of this factor is often not present in the evaluation and design of transportation options; when not altogether ignored, its treatment has been through ad hoc approaches usually involving sensitivity analyses and/or alternative scenarios.† A critical gap thus exists in the field in terms of appropriate methodological support for decision-making in the face of uncertainty.

Further complexity is introduced by the need to take a multiplicity of criteria into consideration; indeed, it is widely recognized that many social, economic and environmental impacts in both the short run and the long run result from or are otherwise affected by the implementation of transportation options. Furthermore, the decision to be ultimately implemented depends on a multiplicity of actors/decision-makers, with differing and often conflicting opinions and preferences. This results in another important feature of decision-making in the urban transportation context, namely the ambiguity and impreciseness of the preference or value structure on which decisions should be based.

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†See section 1.2 for a more elaborate survey of previous and existing approaches.

In contrast with the considerable sophistication reached in our prediction methodology,[†] there is a notable absence of explicit treatment of the above complexities in models of evaluation and choice. This situation is largely due to the high input requirements and/or restrictive assumptions of formal decision methodologies,[§] whereby the latter seem irrelevant or unresponsive to the realities of the decision problem and its complex environment (Michaud, 1976).

In the absence of appropriate decision aiding methodology, decisions reached by unstructured, ad hoc, or overly simplified strategies are likely to result in "missing" better options (than the one selected from among the set of possible options). Similarly, but at the other extreme, use of sophisticated but restrictive approaches has the danger of misrepresenting the true preferences of the decision unit(s), and thus also discarding better options. This thesis is, however, motivated by the belief that the use of decision aids for option selection and decision making, if properly designed, can help analysts and decision makers focus their limited attention/integration capabilities, and resources, on essential elements of the evaluation, and as such improve the efficiency of decision making, as well as its effectiveness (in the sense of not missing better options).

In the face of such complexity, Simon (1979) has noted that only one of two approaches are possible: either to build optimal models at the cost

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[†]For example, see Ben-Akiva and Lerman (1982) or Daganzo (1980) for the state-of-the-art and recent advances in travel demand analysis; on the performance or "supply" side, see Larson and Odoni (1981) or Trans. Res. B., Vol. 14, #1/2 (1980).

[§]These methodologies have evolved in related disciplines and fields of application such as operations research, decision theory, management science, economics, water resources planning, mathematical psychology, etc. See review in Mahmassani (1980b).

of simplifying environmental assumptions, or to develop heuristic models that maintain greater environmental realism. Keeping in mind that the definition of optimality in the context of choice among uncertain or multi-attributed options is controversial at best (Keen, 1977; Simon, 1978; Einhorn and Hogarth, 1981), we have opted for the second approach in this research effort. While the overall aim guiding this research has been the development of an integrated structured analytical apparatus for transportation evaluation and decision-making that would be cognizant of the complexity of the decision problems, and operational in terms of being compatible with the realities of the decision environment, this thesis is presented as a bounded contribution towards the achievement of that aim.

In more specific terms, the objectives pursued in this thesis consist of the development of a decision aid, consisting of a family of decision rules embedded within a coherent methodological structure, for a representative (of the urban transport context) class of problems involving selection among a large finite set of options whose impact is not known with certainty; primary emphasis is on cases where the impact is expressed in a single criterion, though special cases where it is expressed along multiple criteria are considered. Furthermore, the methodology is to be consistent with a set of basic considerations which are to be derived from the decision environment as well as behavioral aspects of individual decision-making and judgement.

While we recognize that different decision environments pose different methodological requirements, and that it is rather unlikely that any single unique "off-the-shelf" technique perfectly fits those requirements, it is still possible to identify a common structure to the formulation of the decision problems encountered. It is then possible to pursue the development of the core of the decision aid within this formulation and

obtain a useful product provided sufficient flexibility is allowed to later fine tune it to the specific problem at hand.

The next section provides a more detailed characterization of the decision environment as it relates to the user of the methodology. Section 1.2 subsequently presents a brief discussion of the types and sources of uncertainty of interest to this study and its representation; approaches for dealing with it in the transportation literature are also briefly reviewed. Section 1.3 discusses the treatment of the other dimensions identified earlier in transportation evaluation and decision-making. Section 1.4 then presents a mathematical formulation of the subclasses of problems addressed in this thesis. Section 1.5 summarizes this chapter, and provides an overview of the thesis, highlighting its principal accomplishments.

1.1 Decision Problem Environment

In this section, a characterization is given for the type of decision environment in which the classes of problems addressed in this thesis arise, and thus where the methodology developed herein can come to play a role.

a. It is first assumed that technical analysis and evaluation is the responsibility of a single agency, for which it typically has some form of public mandate. This agency is referred to hereafter as the "lead" agency.

b. In this agency, there exists a single individual (or a group of individuals who "speak with one voice") who is ultimately responsible for that agency's position, and can be considered, for all practical purposes, as the "decision-maker". It is therefore assumed that there exists a sin-- gle source of preferential information, hereafter referred to as the "decision-maker".

c. However, the final outcome of the decision-making activity is not necessarily solely determined by the above "decision maker", but is the result of interaction of a larger set of actors (of which the lead agency is a member) in the political arena, in the absence of formal, strict and known rules.

The user of the methodology is thus an analyst (or team thereof) at the lead agency whose role is to aid the decision-maker in the evaluation process. The task addressed by this study involves selection from among a finite large set of discrete options, or alternatives, $O = \{o_1, \ldots, o_n\}$.

It is additionally recognized that the nature of the inter-agency decision-making process is altogether different from the intra-agency activity, and as such poses different methodological requirements. The analyst in our problem environment is not engaged directly in the inter-

actor interaction process, as his role is confined to aiding a single actor reach decisions consistent with his/her preferences. This, of course, should not be construed to imply that the overall decision environment does not enter as a constraint on implementability of the decisions of the lead agency, but only that the analyst is not formally responsible for bringing out group preferences, nor for directly maneuvering towards group consensus, at least not within the framework of our systematic methodology.

An example of such an environment is the Ministry of Transport in Egypt, in its conduct of urban transport project programming in Cairo (Gakenheimer et al., 1979; Mahmassani, 1980a; Mahmassani and Gakenheimer, 1981a). Other examples are numerous in urban transportation planning, in both the U.S. (and Europe) and developing nations; these include Metropolitan Planning Organizations (MPO's) or Regional Planning Commissions; for example, the Southeastern Wisconsin Regional Planning Commission (SEWRPC), for the Milwaukee region, comes to mind as a prime representative (Beimborn et al., 1979; Schulz, 1979). Transit operators making decisions on service cutbacks or other major service changes are another example. Other examples can conceivably be found outside transportation, including water resources planning commissions, education boards, etc.

This characterization of the decision problem environment is an important element in defining the classes of problems addressed by this study, and it effectively determines the scope of this research. Chapter 2 examines in greater detail the methodological implications and requirements of the above environment, by translating them into the basic considerations underlying the design of the methodology. However, we first discuss the uncertainties of concern to this study and present a brief sur-

vey of related transportation literature, followed by a specific mathematical formulation of the subclasses of problems addressed in this thesis.

1.2. Uncertainty: Types and Treatment in Transportation Planning

In the introduction to this chapter, "uncertainty" was mentioned without further qualification beyond stating that it concerned knowledge of the impacts of the options under consideration. The first part of this section offers further explanation as to what we mean by "uncertainty", how it arises in transport planning situations (in general), and identifies different types of concern to this study. The second part then briefly surveys a variety of approaches that have evolved in transportation planning for dealing with uncertainty.

1.2.1 Sources and Types of Uncertainty

Uncertainty is viewed as a characteristic of the information available about a particular system to an observer of that system. As such, it can arise in conjunction with practically any parameter, variable, or event considered relevant to the evaluation of alternative transport options. Different researchers have thus had cause to be concerned with uncertainties in different items. Neumann (1976) stresses the importance of taking into account uncertainties in resource constraints, implementation timing, expected impacts and political acceptability of alternative transportation plans and programs. Friend and Jessop (1969) on the other hand are concerned with uncertainties surrounding knoweldge of the external planning environment, future intentions in related fields of choice and in appropriate value judgements. More broadly, Manheim (1981) refers to uncertainties in "future growth and evolution of the economic, demographic and social structure of society." In a more problem-oriented analysis, Pecknold (1970) restricted his concern to uncertainty in future demand using the transport system.

Studies conducted in conjunction with the development of a programming procedure for urban transport network investment in Cairo, Egypt, identified an extensive list of uncertainties of relevance to the programming process (Gakenheimer et al., 1977; Keller, 1977). Those were grouped into four categories (not unlike Neumann, 1976): project development related, funding related, delays, and likelihood of implementation. Further work in the same context focussed on uncertainty in one variable of critical significance to programming, namely the availability of funds for capital expenditures on urban transport projects implementation (Mahmassani and Gakenheimer, 1981a).

The above examples merely reaffirm the pervasive nature of uncertainty in transport evaluation and decision-making. While it is outside the scope of this study to investigate the properties of any one specific variable or source of uncertainty,† we can classify these uncertainties into broad types of relevance to the analytical formulation of transport evaluation problems. We first note a distinction commonly made in the literature between descriptive uncertainty and measurement uncertainty (Rowe, 1977). Whereas the first affects the understanding and representation of a particular system, phenomenon, or event, the second is due to inaccuracies in our measurement tools (physical or statistical).§ Those two dimensions of uncertainty are however not mutually exclusive as they can both be present in a given type of uncertainty.

We hereby categorize uncertainties affecting the evaluation of al-

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†Such studies can be quite insightful, especially for practice; the work of Knudsen (1976) on uncertainties in cost estimates of airports should be noted in this context.

§See Mahmassani (1980b) for an elaboration on this distinction and examples in transportation planning.

ternative transportation options in our problem environment into five different types:

1. The unknown, consisting of new and unforeseen situations; these include major unsuspected political upheavals or changes, unanticipated technological breakthroughs, and the like. Representation of this type of uncertainty is a non-issue since the analyst is not even aware of the possibility of such occurrences. Beyond the effort of visionaries, or some forms of technological forecasing (Ayres, 1972; de Neufville, 1976, 41-55), this category is by definition outside the scope of the analysis of the transportation options.

2. Occurrence of exogenous events or states, i.e., independent of the transportation decisions taken, but affecting the environment in which the transportation system operates, including political events (e.g., new administration) and economic or social set of circumstances. Such events are of concern only in as much as they affect the performance of the alternative options on any of the evaluation criteria. For example, a new political administration may have serious implications on budget availability for project implementation; high GNP might imply high auto ownership levels; future development of an outlying suburb would lead to high demand for travel on major connecting facilities, etc. This type of uncertainty can be represented either through discrete "states of nature" or "scenarios"[†] (with analysis of the option's impacts conditional upon their realization), or directly through uncertainty in the variables entering the evaluation (which is the third type, discussed hereafter).

[†]Use of this approach in transport planning is discussed later in this section; a more specific formulation is presented in section 1.4 of this chapter.

3. Uncertainty, or randomness, in the values of measured or predicted impacts, usually the result of a modelling activity. The sources of this type of uncertainty are numerous, including descriptive uncertainty in the knowledge of the phenomena modelled, measurement uncertainty in model parameters as well as in the input data, inaccuracies introduced by approximations in model use, etc. Examples of such variables include estimates of demands, flows on various portions of the transport network under consideration, benefit measures, costs, and many others. Depending on the analysis generating this information, this type of uncertainty can be specified in the form of interval estimates (ranges, confidence intervals), or more completely in the form of (discrete or continuous) probablity density (or mass) functions, defined over the domain of variable(s) of interest.

4. Imprecision or vagueness in the definition of one or more criteria and the description of an option's performance along that criterion. Examples include criteria such as "aesthetics", or "political desirability". Vagueness, or "fuzzines", is a property that concerns the very concept of a variable, and as such has been viewed by many researchers (Zadeh, 1965; Menges and Skala, 1974) as completely distinct from "randomness" (which characterizes the occurrence of well defined events). Because ill-defined, a fuzzy quantity cannot be adequately represented using probabilistic logic; thus the whole area of fuzzy set theory and multi-valued logic has recently emerged in an attempt to provide such a representation (Zadeh, 1965; Kaufmann, 1975). † In this study, we view fuzziness as one type, albeit

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†Further advances along those lines include the concept of "stochastic fuzziness" (Norwich and Turksen, 1981) combining features of both randomness and fuzziness.

an extreme one, of uncertainty, which cannot however be adequately expressed through the sole use of probabilistic notions. Several "engineering" adaptations are generally possible for dealing with such imprecise criteria, such as the use of qualitative descriptions, categorical information, and others which are highly problem specific. These might be needed since the formalism of fuzzy set theory and associated algebra is still of dubious operational value (Kickert, 1978; Nahmias, 1981; see also Mahmassani, 1980b, pp. 55-56, for further comments on its usefulness for transportation decision-making).

5. Uncertainty as to the preferential or normative basis of the evaluation, which ultimately determines the outcome of the decision making process. Examples include uncertainties as to whether or not a particular system attribute ought to be included in the set of evaluation criteria, or as to the appropriate trade-offs between the criteria (or, more generally, the underlying preference structure). In the problem environment described in section 1.1, this type of uncertainty faces the analyst interacting with the decision maker, who in turn might be uncertain about his own preferences.[†] Uncertainties as to which actors have input in the decision, their relative "importance", their underlying preferences and choice rules belong to this category as well.[§] While its representation is a highly complex task, the existence of this type contributes heavily to the specification of the set of basic considerations underlying the

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[†]This is especially true when the options under consideration are characterized by type 3 uncertainty, i.e., randomness in the predicted impacts; the preferential basis of decision making is discussed in greater detail in Chapter 2.

[§]See also Mahmassani (1980b) for further discussion of this aspect.

methodological development in this thesis, as described in Chapter 2. As such, it provides considerable motivation for the use and development of the decision aiding approach presented here.

In summary, the primary types that are explicitly addressed by the methodology developed in this thesis are types 2 and 3, whereby uncertainty in the predicted impacts is represented through discrete states of nature and/or probability density (or mass) functions over the (multi-dimensional) payoff space.† Type 4, or fuzziness in criteria definition, is not of primary concern to this study, as it is primarily an impact representation issue; however, as we mentioned earlier, suggestions for the accomodation of this aspect are made at a number of points in the presentation. Type 5 is naturally of concern, though its implications are addressed in a more fundamental manner than types 2 and 3. Of course, type 1, or the completely unreachable, is outside the scope of our methodological capabilities.

Most if not all of these uncertainties occur to various degrees in urban transportation planning situations. The next section briefly discusses the implications of uncertainty for those situations and surveys approaches recognized in the transportation area for dealing with it.

1.2.2 Implications and Solution Approaches

The fundamental implication of the presence of uncertainty is the following: since selection of an option o* at time t is conditional upon its desirability at the time of selection (desirability resulting from the [structured or unstructured] evaluation of the impacts of the options relative to the normative basis set at that time), the occurence of any changes

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†Specific mathematical formulation of the problems addressed in this thesis is given in section 1.4.

in either knowledge about the impacts or in the normative basis at a future time $t + \Delta t$ might bring the realization that a non-selected option, different from o*, would have been more desirable (relative to $t + \Delta t$) than o* was. In other words, uncertainty can lead to regret at time $t + \Delta t$ for having chosen o* at time t.†

Such situations are known to occur in transportation planning. For instance, when the demand for a facility at time $t + \Delta t$ fails to meet the demand predicted at time t, or indicates a departure from the trend assumed at time t for projections beyond $t + \Delta t$, options discarded at t might seem more attractive (at $t + \Delta t$) than the one selected at t.

A well-known example of a change in the preferential basis of the evaluation that has had serious consequences is the "freeway revolt" (Colcord, 1971), whereby consideration of broader impacts (e.g., environment) resulted in the unacceptability of prior decisions reached according to a different set of criteria (namely user travel costs and facilities costs).

When the partial or complete resolution of the uncertainty present at the time a decision is taken reveals changes in the relative desirability of that decision, the extent of regret depends on the course of action subsequently followed. One of two possibilities are present:

a. no action, in which case one would be paying the opportunity cost of not having had the superior option; or

b. reversing previous decision, usually incurring more or less substantial costs. Such costs are not always monetary, as some impacts or damages cannot be reversed (also known as the "irreversibility effect" [Henry, 1974]), and these are of major importance in transportation invest-

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†The notion of "regret" is central to this thesis and is explicitly incorporated in the methodology (see chapters 3 and 4).

ment decisions, especially those involving the construction of major physical facilities.

We next discuss approaches suggested in transportation to deal with some or all of the above general types of uncertainty. These approaches can be grouped into four categories:

1. Reducing it;
2. Structuring the decision process (over time);
3. Evaluation and design criteria and guidelines; and
4. Explicit evaluation techniques.

1. Reducing Uncertainty

Probably the most obvious way of dealing with uncertainty is to reduce it by acquiring more information of the variable or event of interest.† Different types of uncertainty of course require different reduction strategies, including:

- further measurement (physical or by statistical sampling), as a means of reducing type 3 uncertainties as to the magnitude of different variables. The standard statistical sampling literature is of relevance to the approach (Cochran, 1963), as well as that of statistical decision theory (de Groot, 1970). Efforts to that effect in transportation planning are too numerous to

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†A general point of interest to this discussion is the dichotomy in philosophical and scientific thought as to the possibility of eliminating all uncertainty in propositional truths (White, 1969). One school of thought (including Einstein as a proponent) claims that it is ultimately possible to understand all phenomena in a deterministic, cause to effect, manner. Another school of thought (with Barankin [1956] as a leading advocate) essentially believes that nature is "one huge stochastic process" (White, 1969), which obviously limits our ability to completely eliminate uncertainty in all matters (which is already limited anyway by our tools). Heisenberg's Principle of Uncertainty, fundamental to quantum mechanics, is a reflection of this second school of thought.

mention and outside the scope of this discussion (examples include Johnson, 1969; DiRenzo, 1977; and more recently Sheffi and Tarem, 1981).†

- conducting experiments, expecially when new technology is involved, or new service concepts are contemplated; this is the rationale behind UMTA's Service and Demonstration program. A point to note here is that experimentation allows some type 1 uncertainties to surface.

Reducing uncertainty, as discussed here, is only a general approach, of no direct concern to this study, as it is more of an impact modelling and prediction issue.

2. Structuring the Process

By embedding the information acquisition activity in a sequential decision process, one would take advantage of the possibility of learning about the transportation system as it evolves (adaptive behavior), and avoid overcommitting resources to options designed to meet requirements which might not actualize. Theoretical studies along those lines have been mostly concerned with uncertainty in demand (Pecknold, 1970). Pecknold's time-staged strategic approach for investment decisions was further extended and generalized by Neumann (1976) into an implementation strategy approach, which was actually applied to a region in California (Neumann and Pecknold, 1973).

In general, any process whereby decisions are made sequentially, conditional on prior decisions and the information acquired in the meantime, can avoid the pitfalls and risks of committing irreversible actions, and

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†Of course, at a very general level, all efforts aimed at understanding the phenomena of interest, and at developing "better" prediction models, and the like, can be viewed as efforts to reduce uncertainty.

be more responsive to changes in community values and shifting priorities (see also Manheim, 1979).

3. Design and Evaluation Criteria and Guidelines

As discussed earlier in this section, the occurrence and extent of regret (after the resolution of uncertainties surrounding prior decisions) depends on the ability to reverse or otherwise modify the option that is the target of this regret. Thus flexibility is generally viewed as a "desirable" criterion in evaluating options (and thus in designing them).

Flexibility naturally implies avoiding irreversibility, identified earlier as a major concern in uncertain environments. Examples of irreversibility abound in transportation planning, and are particularly significant in conjunction with community impacts (e.g. neighborhood disruption), landmark demolition, park land loss due to major constructed transportation facilities (Henry, 1974; Pearman, 1977). We prefer the use of the term "guideline", instead of "criterion", to denote such general statements made in the planning literature about the desirable properties of transportation projects and programs. We reserve the term "criterion" for more specific and better defined measures which are unfortunately not present in the transportation literature as far as uncertainty is concerned.

4. Explicit Evaluation Procedures

In contrast with the more general approaches described above, this type of approach refers to actual analytical evaluation tools that explicitly account for the otherwise widely recognized presence of uncertainty. As stated in the introduction, virtually no such explicit procedures have been developed for or used in the transportation context. Rather than

structured, systematic procedures, which are almost totally absent,[†] some unstructured ad hoc approaches have been reported, even though they cannot be considered standard practice.

The rationale behind some of these ad hoc approaches is that since the vector of outcomes (or impacts) of any option cannot be known with certainty, these outcomes should be tested under a variety of assumptions so as to assess the "robustness" of the decision, whereby robustness can be defined as "insensitivity to assumptions" (O'Sullivan and Holtzclaw, 1980). One way of accomplishing this evaluation is by so-called scenario-building (Manheim, 1981), whereby a more or less exhaustive number of mutually exclusive descriptions of future situations are constructed (e.g., "alternative futures" approach described by Bernard [1979]). Alternative scenarios allow for type 2 and 3 uncertainties, and are usually constructed with the aid of assumptions on the values of selected variables and/or the realization of specific discrete events.[§]

Investigating robustness through scenario building can be viewed as one form of conducting more general sensitivity analyses, whereby uncertain parameters (or other elements such as model form) of the prediction models are varied over a likely range and the corresponding figure of merit computed. The usefulness of sensitivity analysis is however severely constrained in major urban tansportation investment studies by the high computational requirements of the model systems which are typically used in that context.

[†]A possible exception being the modest methodological effort of Pearman (1977).

[§]Such events could be other decisions affecting the system, as in Mahmassani and Gakenheimer (1980), where one such scenario element was whether or not a subway system would be built in a metropolitan area (Cairo, in this particular case).

Scenario building is however an approach to formulating the decision problem, and not of solving it. Thus the evaluation and selection process following impact analysis of the options under alternative scenarios is usually unstructured and left to proceed on an intuitive, ad hoc manner. It is at this area that the methodology developed in this thesis is aimed.

Furthermore, it should be noted that the above ad hoc approaches, notwithstanding their limitations, are more the exception than the rule, as the prevailing way of addressing uncertainty in transportation planning decisions seems to be ignoring it altogether (Pearman, 1977).

In summary, while there is no need to re-emphasize the presence of uncertainty in transportation decision-making, section 1.2.1 has presented a "problem-oriented" typology of the uncertainties encountered, along with examples, indicating the extent to and the mechanism by which they are addressed in the decision aid developed in this thesis. Despite the acknowledged need for explicitly accounting for uncertainty in evaluating transport options, as evidenced by the general decision strategies and guidelines proposed for that purpose in the literature reviewed in section 1.1.2, analytical efforts in that direction have been limited to conducting sensitivity analyses. There are thus virtually no significant reported efforts at developing or using explicit decision models for urban tansportation choices.

The next section briefly discusses the extent of explicit treatment in the transportation literature of the other major sources of complexity of concern to our problem environment, namely the existence of multiple criteria and the dependence of the ultimate decision on a multiplicity of actors.

1.3 Other Dimensions in Transportation Evaluation Methodology

The need to consider a multiplicity of criteria in the evaluation of transportation options is widely acknowledged both in the literature and in practice.[†] A number of efforts in the late sixties and early seventies were aimed at developing analytical representations of the relevant criteria as well as methodologies for the evaluation and choice process (Manheim, 1965; Hall and Manheim, 1967; Follansbee et al., 1968, Pardee et al., 1969; Godard, 1973). In particular, Pardee et al. (1969) compiled a comprehensive list of system performance and impact measures for possible use as evaluation criteria, and proposed a relatively sophisticated analytical approach based on multi-attribute value theory for conducting the evaluation. That approach however has substantial and rigid requirements of a priori preferential information for the decision-making unit, and as such seemed inappropriate for urban transportation decision environments.

Currently, many government agencies seem to have developed lists of criteria and impacts (NCHRP, 1978; Bellomo et al., 1978; Beimborn et al., 1979, Matthias and Wortman, 1980, Neumann and Dresser, 1980, Sinha, 1980). However, systematic evaluation techniques are not generally used in conjunction with them, besides so-called "checklist"-type approches whereby minimum levels or targets are established for a subset of the criteria[§] (NCHRP, 1978). An evaluation concept compatible with such practice is that of the "impact tableau", as described by Manheim (1979a, Ch. 9), which

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[†]For a general introduction to the use of evaluation methods in transportation planning, see Manheim (1979a, Ch. 9) or Morlok (1978, Ch. 14). Other useful discussions can be found in Stopher and Meyburg (1976), Schofer (1976 and Cheslow (1980).

[§]Because it represents an activity that decision-makers feel comfortable with, this feature of existing practice is taken advantage of in the framework of the decision aid proposed in this thesis (cf. Chapter 2).

basically is a format for presenting impact information (on different actors), leaving it up to the analyst to highlight the major trade-offs.[†] Morlok (1978, Ch. 14) also describes a similar approach.

Recently, more formal techniques have appeared in the transportation literature, addressing multi-objective resource allocation problems. Building on the work of Agarwal (1973) on the multi-objective network design problem (assuming a system optimal rule for network flow allocation), Perl (1980) proposed a goal programming formulation to the same problem; Friesz (1980), on the other hand, assumed a user optimal rule (for flow equilibration) and suggested the applicability of Geoffrion's (1970) preference-incorporating technique for solving the resulting vector maximum problem. Friesz et al. (1980) also conducted a comparative assessment of two multi-objective optimization codes for a simplified rural road improvement problem. At a more practical level, Sinha et al. (1980) adapted a goal programming formulation to the allocation of funds for highway maintenance in the state of Indiana. While different from the selection from a finite set of alternatives (in the decision problem environment of section 1.1), the above well-defined optimization problems suggest an increasing interest in explicit treatment of multiple criteria in transportation decision models, as well as the need to be more sensitive to the decision environment and its constraints if such approaches are to be useful in practice.

Moreover, efforts along the latter line include the application of a new multi-criteria evaluation technique (MCQA) to the transportation system in Phoenix (Pfaff et al., 1980) as well as the procedure developed for

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[†]Manheim (1979a) further advocates a participatory open process for ultimately reaching a decision.

programming network improvements in Cairo, Egypt (Mahmassani, 1980c, 1981), based on the ELECTRE method (Nijkamp and Van Delft, 1977).

The presence of multiple actors in the decision process has motivated considerable attention on behalf of those interested in the study of the policy making process. However, virtually no attempts have been made to systematically incorporate this dimension in the evaluation methodology.

As we mentioned in section 1.1, the inter-actor interaction process in the transportation context is not subject to formal, rigid and known rules. Only when the decision-makers involved can be made to agree on a formal interaction process can a systematic normative approach be used. For example, Wilson and Schofer (1976), in one of the very few such efforts, used a Delphi-based process† to achieve what amounted to the calibration of an aggregate group value function over a set of evaluation criteria.

However, a generally accepted practice is for the analyst at the "lead" agency to incorporate concerns of different actors by including impacts or criteria of respective importance to those actors in the set of evaluation criteria; for instance, this is possible in the context of of above mentioned "impact tableau" approach.§

Finally, in terms of explicit techniques for multiple criteria/actors evaluation in the face of uncertainty in future impacts, the gap is even more apparent. Except for an application of Multi-Attribute Utility Theory (to be critically discussed later) to an airport siting problem

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†Another use of the Delphi technique in transportation systems evaluation is described in Leone et al. (1973).

§A somewhat more sophisticated approach, which nevertheless follows the same rationale, is that developed for the Cairo programming procedure (Mahmassani and Gakenheimer, 1981b), whereby an aggregate "political desirability index" was defined and used to screen out institutionally unacceptable options.

(de Neufville and Keeney, 1973), where ad hoc analyst intervention proved more effective than that particular methodology, there is no significant related literature.

The above review thus clearly suggests the lack of adquate treatment of uncertainty both separately and in conjunction with multiple criteria, in the methodology for urban transportation evaluation and decision-making. Where efforts have been attempted (mainly to deal with multiple criteria), steep and rigid a priori input requirements of values/preferences information have detracted from their applicability. This lends further credence to the motivation behind this thesis, as stated in the introduction to this chapter.

Having established the relevance of the aims pursued in this thesis, and defined the prototype decision problem environment for which the decision aiding approach developed herein is intended, we next describe and formulate mathematically the specific subclasses of decision problems under uncertainty pursued in this thesis.

1.4 Specific Problem Subclasses Addressed

The focus of this thesis is on the methodological elements of a decision aid for a broad class of problems involving selection from a given finite large set of transportation options whose impact is not known with certainty. The methodological development is pursued in the context of a particular subclass where the options under consideration consist of univariate random prospects, and where uncertainty is represented through mutually exclusive discrete states of nature.

Problem Formulation 1:

The problem consists of selecting a most preferred option o*, where the decision space consists of a finite set of n non-dominated[†] options, $O = \{o_1, o_2, \ldots, o_n\}$.

Uncertainty is represented through a finite set of m mutually exclusive, collectively exhaustive states of nature (consisting of any suitably defined bundle of joint events) $S = \{S_1, S_2, \ldots, S_m\}$, with known (objectively or subjectively assessed) probabilities of occurrence; the vector $P = \{P_1, P_2, \ldots, P_m\}$, where P_j denotes the probability that S_j occurs, is independent of the option selected.

The impact resulting from the implementation of an option o_i is conditional upon the realization of a state of nature. It is assumed in this subclass of problems that the totality of the conditional impact (of $o_i \in O$) can be expressed by a single numeraire. The impact vector of o_i is denoted by $X_i = \{x_{i1}, x_{i2}, \ldots, x_{im}\}$, where x_{ij} is the conditional payoff from o_i should S_j occur. The n x m matrix of x_{ij}'s is denoted by Λ. □

[†] An option o_i is non-dominated, or pareto-optimal, if there is no option $o_j \in O$ such that $x_{ik} \leq x_{jk}$, for all $S_k \in S$, where the symbols are as defined in this section.

This formulation is characteristic of a widely occurring type of problems whereby alternative scenarios are identified and each option's performance under each scenario determined. As discussed in subsection 1.2.2., such formulation seems to be the most common mechanism used in practice to formulate decision problems under uncertainty. For this reason, and because it is intuitively understandable, it serves as a vehicle to bring out the fundamental issues of choice under uncertainty and to articulate the basics of the methodology developed in this thesis.

The methodology is also generalized to the subclass of problems where the impact of each option is specified as a probability density (or mass) function over the payoff (or impact) set X (which is usually $\mathbb{R}$, the set of real numbers).

Furthermore, the decision aid may in some situations be applicable to a third subclass of problems, where the impact of each option consists of a vector of impacts along p criteria, $C = \{C_1, C_2, \ldots, C_p\}$, conditional upon the realization of an $S_j \in S$. The conditional impact vector of o_i given S_j is then denoted by $X_{ij} = \{x_{ij}^1, x_{ij}^2, \ldots, x_{ij}^p\}$, where x_{ij}^k is the impact o_i along C_k should S_j occur. In particular, problem subclass 3 can effectively reduce to the first formulation if the analyst can structure the decision problem in such a way that all criteria but one are accounted for in the so-called elimination stage of the evaluation process.[†] This is particularly true in many transportation problems, where an economic criterion is used as a basis of comparison subject to meeting social and environmental impact constraints.[§] In addition, we present in chapter 5 an

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[†]This stage is integral to the decision aiding framework developed in this thesis and presented in Chapter 2. It is sufficient to know at this point that the Elimination stage does not involve pairwise comparisons of options.

[§]This is the well-known heuristic for dealing with complex problems by formulation objectives as constraints, to the exclusion of a more prominent one which is then maximized (Simon, 1964).

extension of the methodological elements (developed in conjunction with problem subclass 1) to a special case of problem subclass 3.

The next section summarizes this chapter and presents an overview of the contents of the thesis.

1.5 Overview

This chapter has presented the basic motivation for this research and the aims it set to achieve. Section 1.1 described a prototypical decision environment where the class of decision problems addressed in this thesis arise, and identified the user of the proposed methodology to be an analyst engaged in aiding a decision maker in the evaluation of options with uncertain impacts. An effort at classifying the types of uncertainties of interest, as they affect problem formulation, was presented in section 1.2, indicating the extent to which each is addressed by this thesis. Current approaches for dealing with uncertainty in urban transportation planning were also briefly reviewed, indicating the absence of adequate methodological tools from current practice. Similarly, approaches for dealing with other dimensions (multiple criteria, multiple actors), especially in conjunction with uncertainty, in transportation evaluation were briefly discussed in section 1.3, leading to the same conclusion. Specific formulations for the subclasses of problems addressed in this thesis were presented in section 1.4, leading up to this summary and a chapter-by-chapter overview of the thesis.

In Chapter 2, the basic considerations underlying the methodological development of the decision aid are presented. These are derived from the characteristics of the decision environment as well as behavioral aspects of individual decision making and judgement, including relevant results from the behavioral science literature. In addition to further defining the role of decision aiding for our class of problems, this set of considerations specifies the special features that the decision aiding methodology should possess, and as such constitutes an important output of this research. Based on those considerations, an overall framework for decision

aiding is then developed in section 2.3. The framework consists of three sequential activites: elimination, pairwise comparisons and global compaison; the second of those is the primary focus of the analytical development in the remainder of the thesis.

Chapter 3 addresses the fundamental problem within the pairwise comparisons activity for problem subclass 1 and 2, namely that of choice among single-attributed random prospects. A review of existing normative theories and models for that problem is presented in section 3.1; those are grouped in two categories: 1) those based on Expected Utility Theory, and 2) those which are not. In conjunction with the latter category, shortcomings of existing risk measures are indicated. The axiomatic foundation of Expected Utility Theory is then critically re-examined in section 3.2, with particular emphasis on one of these axioms, known as the independence axiom; the latter's plausibility is questioned using a new argument relying on the implications of that axiom on the risk dimensions underlying a choice situation. Section 3.3. then proposes less restrictive postulates, reflecting the basic considerations of the previous chapter, for the methodology developed in the following chapters. These postulates specify the properties of the preference-indifference relation over the set of options as those of partial semi-orders, wich are defined in that section. This is a behaviorally justified departure from the usual assumption of the existence of a complete order in decision models, and as such is an integral part of the contribution of this research.

Chapter 4 is the centerpiece of this thesis, as it develops the pairwise comparisons methodology, including: 1) its structure, which is applicable to all the problem subclasses addressed in this thesis, and 2) its spe-

cific elements, developed for the discrete states formulation.† The methodology attempts to reflect the basic considerations outlined earlier, and revolves around the construction of a binary preference-indifference relation over the set of options, that is consistent with the postulates of Chapter 3. Following the overview of the structure and introduction of some essential concepts, section 4.1 describes alternative solution concepts for reducing the set of options once a preference-indifference relation is defined.

The first component of the methodology consists of a mechanism for comparing any two given options thus defining an initial binary preference-indifference relation (over the set of options). In section 4.2, such a mechanism is described for the discrete states formulation, consisting of regret-based risk measures and an associated decision rule, requiring preferential input from the decision maker in the form of a pair of threshold levels. The properties of the resulting initial preference-indifference relation are then discussed and shown to be quite unrestrictive (subsection 4.2.4). In particular, the relation may exhibit violations of the starting postulates; the detection of such inconsistencies in a given problem situation is the second component of the methodology. Section 4.2 identified inconsistencies with the properties of partial semi-orders as cycles in the graph theoretic representation of a given preference-indifference relation,§ and presents rules for their detection. The third com-

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†The formulations were given in section 1.4; the discrete states formulation is characteristic of widely occurring situations where the respective payoff of each element of a finite set of options is conditional on the realization of one of a set of mutually exclusive and collectively exhaustive states of nature with known probabilities of occurrence.

§This graph theoretic representation is itself described in section 4.1.

ponent of the methodology then consists of approaches for correcting those inconsistencies in the initial preference-indifference relation; three such approaches are described in section 4.4, one of which requires relation-specific elements (described here in the context of the above initial binary relation). The remaining component of the methodology is to obtain the kernel of the resulting final relation, which is one of the solution concepts defined in section 4.1. A summary of Chapter 4 is given in section 4.5.

Chapter 5 then consolidates the contribution of chapter 4 by extending the methodology in four directions: 1) specification in section 5.1 of alternative functional forms for the risk dimensions used to build the initial preference-indifference relation. Important properties associated with some of the new measures are also derived. 2) Integration, in section 5.2, of the well-known notion of target (or aspiration level) within the methodology, as an elimination rule as well as an alternative mechanism for defining the initial relation over the set of options. Properties and methodological implications of the latter are also described. 3) Generalization of the methodology, in section 5.3, to the subclass of problems were options are characterized by continuous probability density functions over the impact domain. 4) Extension, in section 5.4, to the subclass of problems where impact of each option is expressed along a multiplicity of criteria.

Chapter 6 concludes this thesis, highlighting its principal contributions, and outlines directions for further research.

CHAPTER 2

BASIC CONSIDERATIONS AND FRAMEWORK FOR DECISION AIDING

2.1 Introduction

The subclasses of decision problems addressed in this study were described in the previous chapter, along with a characterization of the prototypical decision environment in which they arise. In that environment, the analyst's role, referred to as decision aiding, was defined as the activity of interacting with the decision maker in order to perform the evaluation and ultimately reach a decision. Our starting point is that it is possible for the analyst to structure the decision aiding activity, and employ analytical tools within this structure; the structure and its analytical components constitute the methodological aspects of the decision aid, which is the principal thrust of this thesis.

Having clarified our terminology, the purpose of this chapter is to present the basic considerations underlying the design of the decision aid, as well as its overall structure. In particular, section 2.2 discusses those basic considerations, including:

- further specification of the role of decision aiding in that particular environment, when facing a problem of the subclasses defined in section 1.4;

- implication of the above on phenomena that the resulting methodology should recognize; in particular, the need to ground the methodology in behavioral principles of individual decision making and judgement is articulated, and those behavioral aspects of concern to us are identified.

Furthermore, in section 2.3, we develop the overall framework for

decision aiding in our problem environment and show its consistency with the above considerations.

It should be noted that those considerations, as well as the framework are applicable to all three problem subclasses formulated in section 1.4. Mathematical details of the representation of the above mentioned principles, and their translation into operational decision rules is avoided in this chapter, and does not begin until the next one with reference to the first two subclasses of problems (i.e., choice among univariate random prospects).

2.2 Basic Considerations

In this section, basic considerations underlying the design of the decision aid are presented. As we mentioned earlier, these are derived from two primary sources:

1. The nature of the problem, its prototypical decision environment in the urban transportation context, and the constraints and requirements it poses;

2. Results in behavioral science pertaining to descriptive aspects of individual decision making and judgement.

Since a review of existing results and theories on human decision making and judgement is in itself a major endeavor, which is clearly beyond the scope of this thesis, we limit our presentation to findings of direct relevance to our methodological development. As such, these results are grouped by the element of the methodology that they affect the most. However, and more importantly, the basic considerations presented hereafter are not a mere collection of empirical findings and contextual observations, but the result of an effort at understanding the intricate interdependence between the above two "sources", and integrating them into a coherent set of features that the methodology should possess. Therefore, in as much as they specify the distinguishing characteristics of the methodology, they collectively form an integral part of the contribution of this thesis.

These considerations are grouped under eight separate headings:

1. Preferences as normative basis;
2. Extreme difficulty of a priori expression of preferences;
3. Decision aiding as preference forming process;
4. Need for behaviorally grounded methodology;
5. Progressive difficulty in preference elicitation;
6. Sequential elimination in methodological structure;
7. Conveying information in comparison measures and criteria;
8. Incomparability of options and threshold phenomena.

The above considerations are by no means independent. For instance, the first two (respectively: no exogenous normative rule, and inability to express preferences a priori) are somewhat "duals" of each other, and together they imply the third proposition, casting decision aiding as a preference forming process. All three then motivate the fourth point, namely the need to ground the methodology in behavioral aspects. The remaining considerations then elaborate on the fourth one, identifying specific behavioral phenomena of concern to the methodological development and defining their implications on that activity.

2.2.1 Preferences as Normative Basis

At several points in Chapter 1, we alluded to the need of having decisions conform to the "preferences" of the decision unit. While this is an accepted precept in decision theory, it might seem somewhat contradictory to prevailing notions of what a normative model is, especially in the engineering and economics tradition, where the professional (analyst in our problem environment) is expected to provide an answer as to what the "best" option is. However, a direct consequence of the presence of uncertainty and/or multi-dimensionality is the loss of the full decidability[†] that often exists when comparison of options is based on respective performance known a priori with certainty, along a single criterion. In the latter case, provided the relevant domain of this single criterion is

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[†]Complete decidability, for a class of problems, is said to hold if decision criteria exist which determine, in a "mechanical" way, a unique option for selection (White, 1969).

unambiguously ordered,[†] a "best" option is automatically implied (by the axioms of ordinary logic, including those of arithmetic).

However, when the impact of each option is uncertain, no such exogenous universally (or at least professionally) accepted rules exist. For instance, Decision Analysis (Raiffa, 1968; Keeney and Raiffa, 1976), viewed by many as the normative tool for decision making under uncertainty is only "normative" to the extent that the resulting decision is made to conform to the decision maker's preferences (which otherwise have to satisfy a set of axioms claimed to define "rationality", as discussed in Chapter 3).

Similarly, when multiple criteria are present, and they are conflicting (e.g., they are negatively correlated: an increase in one is accompanied by a decrease in the other), "objective" normative rules do not exist, and preferential information is required to resolve inescapable tradeoffs.

While it is possible for a given professional community (or governmental entity, through explicit legislation or policies) to evolve some form of agreement as to appropriate normative rules for certain classes of problems, this is generally not the case in transportation.[§] Therefore, in the absence of exogenous agreed upon normative rules for prescribing choices among random prospects, nor among multi-attributed alternatives (nor, a fortiori, among multi-attributed random prospects), the preferences

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[†]Such a complete unambiguous order is not always self-evident, even under certainty. However, most criteria encountered in practice, especially if they are sufficient to serve as a basis for selection (typically monetary or technical criteria), can be unambiguously ordered in the range of interest. The typical example is "money", for which "more is better", i.e., its value is monotonically non-decreasing. We confine our discussion hereafter to such criteria, unless otherwise stated.

[§]And justifiably so, as we have argued throughout this section.

(i.e. value system) of the decision-maker constitute the ultimate normative basis.†

2.2.2 Extreme Difficulty of A Priori Expression of Preferences

For this class of problems, the decision maker typically cannot, a priori, fully express these preferences, due to a number of reasons, including:

a. Political nature of the process, whereby no societal consensus exists on priorities (and more generally social preferences). Since urban transportation decisions primarily occur in the public sector, and the "decision maker" has some form of public mandate/responsibility, he can be viewed as basing decisions on "social preferences". However, in the absence of societal consensus, the political process usually precludes formal prior agreement on an explicit value system (which would then introduce full decidability into the problem). It seems indeed remote to expect a pluralistic, democratic society to adopt a clearly hierarchized and ordered set of priorities, let alone to possess a known complete order over the set of urban transportation alternatives under consideration.

In discussing urban transportation planning in the U.S. context, Altshuler (1979) remarks that "it is particularly misleading to think of the American social and political systems, with their extreme pluralism, as having goals". He further notes that most program objectives in the urban transportation arena are "stated in highly general fashion", adding that "... the objectives of public policy are incredibly diverse and so weakly

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†Further discussion on the notion of "optimality" in decision making under uncertainty as well as in the presence of multiple criteria can be found in Keen (1977), Einhorn and Hogarth (1981) and others.

related as in some circumstances to seem directly at odds with one another". (Altshuler, 1979). A typical example is provided by Levin and Abend (1971), who quote the Intergovernmental Act of 1968[†] to illustrate how such statements "tend to be contradictory or so generalized and hedged with verbiage that no consistent policy can be discerned". The fuzziness in such expressions duly confirms our premise.

b. These preferences are not necessarily fully formed. Even if deciding strictly for himself, regardless of societal responsibilities or constraints, there is very little reason to expect the decision maker to possess, a priori, a complete order over the set of univariate random prospects, let alone that of multivariate random prospects. As indicated by Fischhoff et. al. (1979), people often have poorly formulated, even incoherent, preferences when facing

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[†]The particular paragraph quoted from that act is (U.S. Congress, 1968):

"Such rules and regulations shall provide for full consideration of the concurrent achievement of the following specific objectives and, in to the extent authorized by law, reasoned choices shall be made between such objectives when they conflict:

(1) Appropriate land uses for housing, commercial, industrial, governmental,...

... (2) Wise development and conservation of natural resources, including land, water, minerals, wildlife, and others;

(3) Balanced transportation systems, including highway, air, water, pedestrian, mass transit, and other modes for the movement of people and goods;

(4) Adequate outdoor recreation and open space;

(5) Protection of areas of unique natural beauty, historical and scientific interest;

(6) Properly planned community facilities, including utilities for the supply of power, water, communications, ...

... (7) Concern for high standards of design;

(8) All viewpoints - national, regional, state, and local - should, to the extent possible, be fully considered and taken into account..."

Note that the emphasis in the above text was added by Levin and Abend (1971).

complex problems; March (1978) also discusses ambiguity in preferences held by individuals, and its implications on the "technology question".†

c. Even if such a complete order existed, the state-of-the-art for eliciting, representing and subsequently using this preferential information is not, in our opinion, adequate for a problem environment like ours. The principal operational techniques available for that purpose consist of Decision Analysis, both for the single attribute case, and its multi-attribute developments, both founded on Expected Utility Theory and its extension into Multi-Attribute Utility Theory (or MAUT; cf. Keeney and Raiffa, 1976). A brief critical review of Expected Utility Theory is presented in Chapter 3, and will not be replicated here. It is, however, sufficient at this point to summarize the causes of the above-mentioned inadequacy as being the lack of behavioral realism in the theoretical foundation of those procedures, thus their restrictive character and the resulting dangers of misrepresentation (of the true preferences of the decision maker).§ This opinion is however to a large extent affected by the behavioral considerations described subsequently in this section.

2.2.3 Decision Aiding as Preference Forming Process

It follows directly from the above two points that, in addition to uncovering whatever preferences the decision maker might possess a priori, decision aiding involves supporting the formation of those preferences required to reach a decision. This point constitutes a departure from the

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†This point is discussed further in subsection 2.2.8, in conjunction with incomparability of complex objects.

§Other references on this aspect include Starr and Zeleny (1977) and Schoemaker (1980).

traditional decision analytic viewpoint that assumes the existence of a fully formed underlying preference system which only needs to be uncovered.† In addition, it precludes the possibility of testing for properties assumed a priori to hold for this preference system, since the latter is not fully formed.§

This further role of decision aiding has far reaching implications for the design of the methodology, since the latter itself could become a major force in shaping those preferences (Fishoff et al., 1979), both in terms of their properties as well as content. It becomes even more critical when preferences are as ill-defined as they are in our problem environment, as they can be significantly affected by subtle details of the procedure, such as how and what information is presented in the interaction process (Fishhoff et al., 1979). These concerns are discussed further in the next consideration, which is itself strongly motivated by this point.

2.2.4 Need for Behaviorally Grounded Methodology

A corollary of proposition 2.2.1 above is that any approach for prescribing choice among random prospects (univariate as well as multivariate) has to describe the preferences of the decision-maker. Furthermore, in 2.2.3, we indicated the dependence of preference forming on the procedure followed. Moreover, even elicitation of assumedly existing preferences, in as much as it requires judgements and choices from the decision

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†Zeleny's "Theory of the Displaced Ideal" (1975) also departs from this traditional viewpoint, emphasizing the process of decision making as opposed to the mere act of so doing.

§We are alluding here to the practice in Decision Analysis whereby such strict assumptions as to the properties of the overall preference system of the individual provide the basis for "extrapolating" from responses to simple hypothetical questions to real life complex problems. (Cf. Chapter 3 for further discussion on the restrictive properties of the underlying Expected Utility Theory.)

maker, necessitates consideration of behavioral aspects of individual decision making and judgement. Considerable evidence and insights into the manner in which individual form judgements and incorporate them into their decisions have been presented by behavioral scientists[†] and this can guide the design of decision aids.

Decision aiding methodology should therefore be compatible with operations/activities that individual decision makers feel "comfortable" with, in order to avoid bias and misrepresentation of preferences. This concern is also shared by a segment of researchers in management science and applied decision theory (including Hammond [1975], Keen [1977], Keen and Scott Morton, [1978], Roy [1977], Zeleny [1975], among others), some of whom view it as one prerequiste of the applicability and subsequent successful implementation of the decision tools.[§] Recent efforts at developing interactive "preference-incorporating" multi-criteria optimization methods is a reflection of this concern (surveys of these methods can be found in a number of references, including Cohon [1978] or Hwang and Masud [1979]).

In addition, behavioral scientists are increasingly recognizing the importance of task structure and content, including problem representation and framing of information, on judgement and choice (Newell and Simon, 1972; Pitz, 1977; Fischhoff et al., 1979; Tversky and Kahneman, 1981; Slovic et al., 1981; Einhorn and Hogarth, 1981), thus providing additional impetus as well as information for reflecting these factors in the methodology.

[†]For a historical perspective on Behavioral Decision Theory, the interested reader is referred to the excellent periodic surveys in the Annual Review of Psychology, including Edwards (1961), Becher and Mc Clintock (1967), Rapporport and Wallston (1972), Slovic et al. (1977) and Einhorn and Hogarth (1981).

[§]This recent "descriptively-rooted" viewpoint is strongly influenced by the classic work of Simon (1957), and March and Simon (1959).

For our problem classes, these factors enter the design of the methodology at all its levels, namely:

1. Principles underlying preference elicitation;
2. Structure of approach;
3. Measures and criteria used to convey information to the decision maker; and
4. Properties of specific elements of methodology.

Considerations of relevance to the methodology developed in this thesis for each of the above items are discussed hereafter.

2.2.5 Progressive Difficulty in Preference Elicitation

The process of eliciting preference information in our methodological development follows progressive difficulty principles, namely that:

1. We seek to minimize the need for value judgements and preferential information from the decision maker. In other words, we try to proceed with the evaluation as far as allowed by the existing level of such information.

2. Probes for additional information are delayed until absolutely essential, i.e., until the discriminating power of the already available information is exhausted; this point is a direct consequence of the previous one.

3. When such information is required, a hierarchy of difficulty is followed, going progressively from the least to the most demanding from the decision maker's perspective (in terms of cognitive strain).

A useful perspective to view the difficulty of supplying an item of preferential information is in terms of the risk of making an error, i.e., that the supplied item actually misrepresents the true preferences of the decision maker (Roy, 1973). Conversly, we can think of the degree (or

level) of confidence attached to a particular item by the decision maker (or the analyst, especially if the latter suspects misunderstanding by the former of the exact nature of what is required). Naturally, it is then easier to supply less risky (i.e. more confident) information than more risky and ambiguous items.

The rationale behind these principles is to a large extent self evident. It recognizes that preferential imformation can come in different shapes and forms, depending on the particular decision model used. In addition, some options can be eliminated, and the relative desirability of others determined, without necessarily forming a complete order over the set of options; such information should be used to reduce the problem space which would help the decision maker focus his attention and concentration on the more essential and probably more difficult elements of the evaluation. Furthermore, by proceeding progressively, a "learning effect" develops whereby the decision maker is able to better grasp the true meaning of his preferential input in the specific problem context that he is addressing. By prematurely requesting difficult information, the analyst might induce one or more of the following flaws in representing the decision makers preferences (Fishhoff et al., 1979);

- random error, by confusing the respondent who might then give an answer with a low level of confidence;

- systematic error, by giving the repondent the impression that he (the analyst) is hinting at a "correct" answer, which the respondent then selects as a way out of uncertainty and ambiguity;

- unduly extreme judgements, by implying that the respondent should have a clear opinion on a complex matter where attainment of such clarity is not to be taken for granted.

Therefore, progressive difficulty in preference elicitation is justified in the interest of decision making efficiency as well as faithful representation of decision maker's preferences.

2.2.6 Sequential Elimination in Methodological Structure

Individuals have mental processes and heuristics by which they deal with complex decision situations. For choice, such heuristics usually consist of information processing strategies. Empirical research indicates that individuals typically use more than a single rule in order to reach a decision (Svenson and Montgomery, 1976; Svenson, 1979; Wallsten, 1980). While the nature of the decision task and its environment are important determinants of the strategies used (Slovic et al., 1977; Beach and Mitchell, 1978; Einhorn and Hogarth, 1981), it seems that early in the decision process, when a large number of alternatives are present, individuals tend to use conjunctive rules† on a single attribute (in the case of multi-attributed options) to sequentially eliminate some of these options from further consideration (Slovic et al., 1977).

Tversky's (1972) elimination-by-aspect (EBA) model, combining conjunctive and lexicographic features, describes the choice process as a sequential elimination activity, whereby an aspect is selected and all options unsatisfactory on that aspect are eliminated, then another aspect is selected, etc. To the extent that decision makers feel comfortable with such strategy (and feel confident enough to supply the necessary information), it

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†Conjunctive, or satisficing (Simon, 1955), rules eliminate any option with a less than a minimum acceptable value on any given attribute.

should be taken advantage of in the decision aiding methodology. However, this is not to say that the final decision should be reached by such non-compensatory rules, but only reduction of the options set through partial elimination.

2.2.7 Conveying Information in Comparison Measures and Criteria

This point highlights the whole issue of task representation and framing in conveying information to and eliciting preferences from the decision maker (Fischhoff et al., 1979; Einhorn and Hogarth, 1981; Slovic et al., 1981). It follows from the more general significance of the cognitive representation of the task ("problem space") in determining an individual's problem solving behavior (Newell and Simon, 1972, Pitz, 1977). Indeed, it is well established that an individual's information processing strategies are influenced by the method of presentation of the information (Peters, Hammond and Summers, 1974). Moreover, it appears that decision makers rely almost exclusively on the information that is explicitly displayed to them ("surface information"), ignoring potentially useful information which needs to be inferred from, or otherwise involves mental manipulation of the display (Slovic et al., 1981).†

In the context of risky choice behavior, Lichtenstein and Slovic (1971) showed that individuals respond only to limited aspects of relevant information (see also Payne and Braunstein [1971] and Tversky and Kahneman [1981]). Einhorn (1972) observed that objects tend to be evaluated

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†Slovic et al. (1981) refers to this "concrete thinking" phenomenon as "The tendency for considerations that are out of sight to be out of mind".

according to their best or worst aspects, other aspects being irrelevant (in the context of multi-dimensional choice).

The above observations have clear implications on the measures that we develop to build decision rules for the problem classes addressed in this thesis. It is thus extremely important that the information transmitted to the decision maker explicitly carry the true implications of the solicited response; it is unlikely that he would subject this information to further manipulation, especally when time is limited and a large number of complex options are being evaluated. On the other hand, overloading the decision maker would defeat the very purpose of decision aiding, in addition to introducing errors in judgement and subsequently in response. Therefore, the decision rule should have clear intuitive appeal and be based on measures that capture the implications of the decision situation. Other specific features that should be taken into account with regard to random prospects include:

- characterization of random prospects as multidimensional stimuli, which may be described in terms of basic risk dimensions, based on studies by Slovic and Lichtenstein (1968), Payne (1973) and Coombs (1975).†

- differentiation between gains and losses, in that risk attitudes tend to be different for each case (Tversky and Kahneman, 1981).

- "regret" as a major aspect of random choice situations, as mentioned in section 1.2 in relation to our problem environment, and recognized by Hogarth (1980), Bell (1981), and earlier Festinger (1957).§

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†This point was also behind Zeleny's (1977) "Prospect Rating Vector" approach to portfolio selection based on a multi-dimensional measure of risk. Further discussion of "risk" can be found in Chapter 3.

§Referred to as "cognitive dissonance" by Festinger (1957).

We next discuss two important properties of the preference relations developed in conjunction with the decision aid.

2.2.8 Incomparability of Options and Threshold Phenomena

In subsection 2.2.1, we indicated that the decision maker's preferences might not be fully formed with regard to the options under consideration. He therefore does not necessarily possess a complete order on the set of multivariate random prospects, due to the high complexity of the problem and the ensuing ambiguity in making the required trade-offs. Therefore, when comparing two options o_i and o_j, the decision maker will:

1. prefer o_i to o_j;
2. prefer o_j to o_i;
3. be indifferent between o_i and o_j;
4. be unable to decide.

In the latter case, o_i and o_j are said to be incomparable. In most decision models, incomparability is not allowed, and a complete order is assumed to exist over the set of options. However, as stated by Von Neumann and Morgenstern (1947):

> "It is conceivable ... and may even in a way be more realistic ... to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable."

Roy (1973) has identified three possible situations, within a decision aiding context, where incomparability could occur:

1. Genuine preferential incomparability, whereby the individual "simply cannot form a preference",

2. Technical incomparability, such as that due to poor or imprecise data; and

3. Strategic or tactical incomparability, whereby the individual at a given stage in the evaluation process is not willing to take a position, or would prefer to postpone or defer a difficult and possibly costly decision.

Fishhoff et al. (1979) discuss the serious implications on preference assessment (in the context of decision aiding) of "forcing" a decision maker to state a preference (or indifference) when he is effectively not able nor prepared to do so. Therefore, in as much as decision aiding for our class of problems hinges on a true representation of the decision maker's preferences, it is essential to allow for incomparability in the methodology, and not confound it with preferential indifference.

The second phenomenon of interest here concerns the discriminatory power of the individual decision maker, whereby perceptual threshold phenomena in conjunction with preferential indifference might give rise to intransitivities (Luce, 1956; Fishburn, 1979; Mirkin, 1979). In other words, an individual could be indifferent between o_i and o_j given a choice between the two, and indifferent between o_j and o_k (given a choice between them), yet still prefer o_i to o_k. Such behavior is not necessarily irrational and becomes quite understandable when the options consist of multidimensional and/or random prospects, as noted by Marschak (1975):

> "... To the extent that his indifference between two alternatives is due to their similarity, and thus to the difficulty of distinghishing between them, the decision maker need not regret having practiced intransitivity of the indifference (not of the strict preference!) relation."

The implications for decision aiding methodology are that the above behavioral phenomenon should be explictly represented in the postulates concerning the preference/indifference relation over the set of options, and transitivity should not be forced where it is not warranted. Opera-

tionally, in the methodology developed in this thesis, the above two considerations (incomparability and threshold phenomena respectively) are translated into the representation of the preference/indifference relation between random (and/or multiattributed) prospects by partial semi-orders; this is a behaviorally justified departure from the usual assumption of the existence of a complete order in decision models, which though far more convenient and tractable, is too restrictive for our problem environment.

The formal definition of partial semi-orders is given in Chapter 3. At this stage, we merely point to the link between the above realities (perceptual thresholds, incomparability) and the fact that partial semi-orders are compatible with them.

The set of basic considerations described in this section is in many ways central to the methodological development in this thesis. Other decision models are assessed with respect to their consistency with it. In addition, we refer to it throughout and describe to what extent it is reflected in the structure as well as the specific components of our methodology. We next develop a general framework for decision aiding in the decision environment of section 1.1 for the class of problems addressed in this thesis, and show its consistency with the above basic considerations.

2.3 General Framework for Decision Aiding

In this section, we develop a general framework for conducting the evaluation process consisting of three stages. These are in turn described and consistency with the above basic considerations is discussed. Furthermore, the focus of the methodological development within that framework is indicated for the problem subclasses of section 1.4.

The Framework

The decision aiding framework proposed here consists of three distinct sequential activites: elimination, pairwise comparisons and global comparison, as shown in Figure 2.1 The starting point is a finite set of n mutually exclusive non-dominated options $O = \{o_1, \ldots, o_n\}$, defining the decision space.† The problem is of course to select the most preferred option $o^* \varepsilon O$.

1. Elimination: Each option $o_i \varepsilon O$ is taken individually, and a decision is made as to whether or not it should receive further consideration. This phase attempts to use easiet level preference information from the decision maker, in an EBA-like framework (see subsection 2.2.6,) and conjunctive rules.

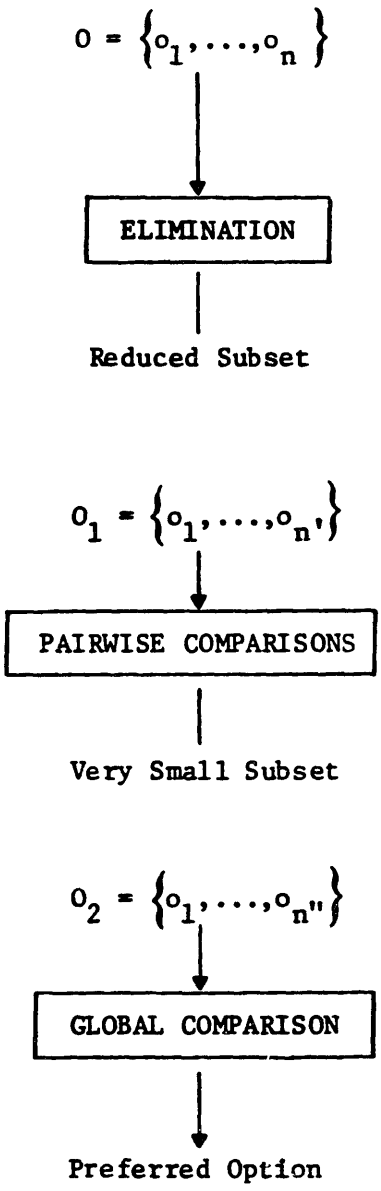
The outcome of this sequential elimination activity is a reduced subset $O_1 \subseteq O$, with $o^* \varepsilon O_1$.

2. Pairwise Comparisons: The elements of o_1 are the object of pairwise comparisons, whereby a binary preference-indifference relation allowing

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†Further characterization of the options is not warranted here, as it depends on the specific problem subclass addressed.

Given: Set of Mutually Exclusive Options



$$o^* \in O_2 \subseteq O_1 \subseteq O$$

Figure 2.1: General Evaluation Framework

for incomparability is induced over O_1. Depending on the properties of this relation, different solution concepts can be used to obtain a further reduced subset $O_2 \subseteq O_1$, with $o^* \in O_2$. The size of this subset depends of course on the relation used, the extent of preferential information that the decision maker is willing to commit and the specific options under consideration; we however envisage a very small subset to result from this stage.

3. Global Comparison: If O_2 contains more than a single element, and cannot be further reduced in the previous stage, all its elements are confronted jointly in a wholistic comparison for a final selection, which only the decision maker is able to make. Alternatively, O_2 could be viewed as a small set of superior options, with final selection left to the political process (see section 1.1).

In the global comparison stage, no direct systematic analyst intervention is assumed; of course, ad hoc support can be expected, in terms of re-display of information generated in the previous phase, or even possible computation of refined impact information. The rationale is as follows: to the extent that the analysis is unable to extract additional discriminating preferential information that would introduce further decidability into the problem, systematic interaction is stopped rather than introducing decidability by forcing consistency with a set of restrictive assumptions on a highly complex and "close" choice situation.

Consistency with Basic Considerations

The above framework is consistent with the set of basic considerations in section 2.2 in a number of ways, some of which are relatively obvious. For instance, note the progressively increasing difficulty of the required

preference information, where each successive stage can be seen to exhaust all information that can be supplied at a given level of confidence (cf. subsection 2.2.5). Thus, at every stage, greater risk is involved in terms of misrepresenting the decision maker's true preferences and eliminating an ultimately superior option.

In addition, the considerations in subsection 2.2.6 are reflected by the use of a combination of rules, starting with conjunction rules in an EBA-like format early in the process, and progressively using more complex context dependent rules. This context dependence culminated with the unstructured global comparison phase. Naturally, other behavioral considerations of section 2.2 map into specific elements of the methodological framework and are thus reflected in the design of these elements (e.g., the preference relation in the second stage).

Focus of Methodological Development

The focus of the methodological development in the remainder of this thesis is primarily on the pairwise comparisons phase, in which the fundamental issues of choice under uncertainty are faced, and thus where the major methodological issues exist. This development is pursued in the context of problem formulation 1 (where options consist of univariate random prospects and uncertainty is represented via discrete states of nature, as described in section 1.4). It is subsequently generalized to problem subclass 2 (i.e., the continuous univariate case). The treatment of the multivariate case, or problem subclass 3 is limited to indicating the applicability of the general methodology to an example preference-indifference relation.

Furthermore, while the elimination phase is recognized as a plausible activity, which should be performed within the decision aiding framework as it can be quite effective at reducing the set of options, it is relatively

straightforward. As such, it is only explicitly treated in two instances; 1) in Chapter 5, to show how the same concept underlying the binary comparison relation can be adapted for elimination and 2) in Appendix A, which specifies this activity in the context of problem subclass 3.† Of course, elimination assumes greater importance for this subclass since the increased preferential complexity of the multivariate case can: 1) allow a wider scope of possibilities in terms of elimination rules and 2) benefit significantly from such elimination (in terms of size reduction).

It should finally be noted that pairwise comparisons (and to a lesser extent elimination) are developed not becuase they are necessarily the most important or the most difficult, but because it is felt that those are the areas where systematic methodology can be most relevant and effective.

Note on Evolutive Set of Options

This note concerns the case where the set of options is evolutive rather than static, as would be the case in the design of alternative transportation options. Though our focus is primarily on evaluation and selection given a fixed set of options, extensions of the framework described in this section, to the design case is possible by allowing the outcome of the global comparison (or any prior stage) to be an empty set, thus requiring the introduction of new elements in O. However, a number of other complex considerations enter the creative design process , and we do not claim to address them here.

The next section summarizes the contents of this chapter.

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†As shown in section 1.4, problem formulation 1 is only a special case of subclass 3.

2.4 Summary

This chapter has presented the set of basic considerations underlying our methodological development. Combining the characteristics of the urban transportation problem environment, and behavioral aspects of individual decision making and judgement, this set of considerations guides the remainder of this thesis. An overall framework for decision aiding consisting of three sequential activities (elimination, pairwise comparisons, global comparison, respectively) was accordingly developed in section 2.3; specific areas of emphasis in this thesis with respect to the subclasses of problems addressed were indicated.

In the next chapter, the set of basic considerations is used to assess approaches for the problem of choice among random prospects and is then translated into specific postulates for our methodological work. In particular, partial semi-orders are defined and proposed as a possible model for a behaviorally oriented representation of the decision maker's preferences over the set of random prospects.

CHAPTER 3

CHOICE AMONG RANDOM PROSPECTS: MODELS AND POSTULATES

In this chapter, we address the fundamental problem of choice among random prospects, which arises in the "pairwise comparisons" stage of the general decision aiding framework presented in the previous chapter. This discussion concerns the first of the two problem subclasses described in section 1.4, where the options consist of univariate random prospects; furthermore, since it is pursued for the general case (i.e., formulation 2 in section 1.4), it is applicable to its "discrete states" version (i.e., formulation 1) as well.

The purpose of this chapter is threefold. First, in section 3.1, it presents a brief review of existing normative theories and models for choice among (univariate) random prospects, with special emphasis on expected utility theory. Second, in section 3.2, the axiomatic foundation of expected expected utility theory is re-examined in light of the basic considerations of chapter 2; in particular, by suitable framing, and with the aid of a simple example, we illustrate the implications of the independence axiom, casting further doubt on its otherwise apparent intuitive appeal. Third, in section 3.3, it proposes less restrictive postulates (again based on the behavioral considerations of Chapter 2) for our methodological development, using the notion of partial semi-orders, which are then described.

As such, this chapter lays the groundwork needed for the methodological development in the following chapters. In addition to placing that methodological work in its proper perspective, sections 3.1 and 3.2 contain the principal ideas underlying the measures used to build the binary relation

in Chapter 4. Furthermore, the postulates in section 3.3 are also fundamental to the innovative character of our methodology.

Note that in this chapter and increasingly in the following ones, we will be using mathematical definitions and concepts that are commonly used in decision theory. For the convenience of the reader who might not be very familiar with these concepts, Appendix B presents a brief introduction to those necessary for the comprehension of the work presented herein.

3.1 Models for Comparing Random Prospects

3.1.1 The Problem

A random prospect consists of a probability function over X, the set of consequences x_1, x_2, ...[†] Let Π denote the set of all such probability functions, with elements f_A, f_B, f_C, ... The elements of Π are assumed to be prospects with objective probabilities (which may be subjectively determined).[§]

The problem of concern is as follows: given a known complete order over the set of consequences X,[&] how should we choose between two random prospects f_A and f_B. We are thus concerned with the normative facet of selection among random prospects.

As stated in subsection 2.2.1, the unambiguous complete order over X (which is sufficient to determine a best choice between elements of X under certainty) does not automatically imply a preference order over Π, nor, a fortiori, an operational choice rule. Approaches for establishing preference between random prospects have thus done one of either of the following:

1. specify summary measures of the respective probability distributions, which then in turn become the basis of comparison (itself performed in a variety of ways, with varying degrees of preference information requirements); such approaches are briefly discussed in subsection 3.1.2, with particular emphasis on the various measures proposed for the "risk" of a distribution.

[†]We consider, in general, that $X \subseteq \mathbb{R}$, the infinite set of real numbers.

[§]The set of options O, for the class of problems 1 formulated in section 1.4, is therefore a subset of Π, i.e. $O \subseteq \Pi$.

[&]In other words, given any x_i, $x_j \in X$, we can always unambiguously state that x_i is preferred to x_j, or vice-versa, or that they are equivalent.

2. postulate a set of properties that preferences over Π should possess (presumably as tenets of rationality), resulting in a structured representation of these preferences, from which a choice rule can be derived.

Of course, this second category essentially consists of the well-known expected utility theory (EUT hereafter), axiomatized by Von Neumann and Morgenstern (1947; VN-M hereafter), which is generally viewed as providing the norm for allegedly "rational" choice among random prospects.[†]

Though VN-M's mathematical derivations have never been questioned, the underlying axiomatic system is subject to considerable disagreement,[§] not about its internal consistency, but regarding the validity of the individual axioms and the extent to which they define retionality. Prior to their critical re-examination in section 3.2, these axioms are presented in subsection 3.1.3, along with some associated operational aspects of EUT.

The following discussion is by necessity very limited. Introduction to the rudiments and models of choice under uncertainty, at a relatively elementary level, can be found in Holloway (1979, Chapters 1, 5), and to varying degrees of difficulty, in any of the numerous textbooks on Decision Analysis. For a historical perspective, see Drèze (1978) or the introduction in Allais and Hagen (1979). A number of surveys can be found in a number of disciplines, notable among them being Fishburn's (1977) thorough and rigourous survey of decision theories.

[†]As such, it is the principal theory on which decision analysis has developed into a full-fledged discipline.

[§]This controversy is most apparent in a recent collection of position papers edited by Allais and Hagen (1979).

3.1.2 Non-EUT based Models

These approaches seek to describe each probability distribution by some "summary" measure(s) which can then be used to compare distributions. Mathematical expectation, or the mean of the distribution, has often been used as a single criterion for that purpose (of course, the higher the mean, the better). It is however well accepted (though not always recognized in practice) that a single central-tendency measure is an inadequate choice rule, especially for unique non-repeatable choice situations.† Thus, existing decision theories in this category typically use the mean in conjunction with a measure of dispersion of the outcomes, or "risk" of a distribution. The most prominent among those is the mean-variance or EV model (Markowitz, 1959), extensively used in portfolio analysis.

Fishburn (1977) views the EV model as part of a large class of mean-risk theories, where each $f_A \in \Pi$ is characterized by its mathematical expectation $\mu(f_A)$, and a real valued measure of risk $R(f_A)$. The comparison between f_A, $f_B \in \Pi$ then becomes a bi-criterion comparison between $[\mu(f_A), R(f_A)]$ and $[\mu(f_B), R(f_B)]$. This class of theories can be further divided depending on 1) the functional form of the risk measure $R(\cdot)$, and 2) the bi-criterion comparison rule used.

Risk Measures

The most commonly used form of $R(\cdot)$ is the variance $\sigma^2(\cdot)$ of a distribution, in the above mentioned EV model. However, in response to criticism of the variance on grounds of its symmetric treatment of high and

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†The best known challenge to the practice of comparing distributions on the basis of their mathematical expection only is the St. Petersburg paradox (see Gorovitz, 1979, for an account of it), which is a hypothetical game with infinite expectation on returns. It was actually this so-called paradox that led Bernouilli (1738) to introduce the concept of utility (Allais and Hagen, 1979, Introduction).

low outcomes, "one-sided" risk measures focussing on low outcomes such as the semivariance have been suggested, primarily for portfolio selection applications (Porter, 1974; Burgess, 1978). Other risk measures focussing on low outcomes include a generalized below target measure (Fishburn, 1977a), the probability of ruin (considered by Allais [1953] as central to the psychology of risk), probability of loss and others. Stone (1973) considered most of those measures as special cases of the following three-parameter risk measure:

$$R(f) = \int_{-\infty}^{\lambda} |c - x|^{\alpha} f(x)dx \qquad (3.1)$$

where
- c is a reference level of payoff from which deviations are computed;
- α is the power to which these deviations are raised;
- λ defines the domain of X of concern in computing these deviations.

Some dispersion measures proposed in the literature consist of composites of the standard deviation σ (or variance σ^2) and mean of a distribution, such as Baumol's (1963) measure $R(\cdot) = K\sigma(\cdot) - \mu(\cdot)$, where K is a real valued positive constant; also Pollatsek and Tversky's (1970) axiomatization of the measure $R(\cdot) = \Theta\sigma^2(\cdot) - (1 - \Theta)\mu(\cdot)$, where $0<\Theta<1$. The only other axiomatization of risk measures is due to Luce (1980), who considers an additive as well as a multiplicative model; its interest however lies mainly in the mathematical derivations rather than in the actual resulting measures.

In addition, note the entropy-based measure proposed by Phillippatos and Wilson (1972), also in the context of protfolio selection.

Having characterized each distribution f_A, $f_B \in \Pi$ separately in terms of $\mu(\cdot)$ and $R(\cdot)$, the next component of a mean-risk model is the bi-criterion choice model, discussed hereafter.

Bi-Criterion Comparison Models

Three types of models have been used in conjunction with the mean-risk class of approaches, each with different preferential information requirements from the decision makers. These are (Fishburn, 1977):

1. mean-risk dominance model, whereby f_A dominates f_B; f_A, $f_B \in \Pi$ if and only if:

$$\mu(f_A) \geq \mu(f_B) \text{ and } R(f_A) \leq R(f_B) \quad .$$

The above dominance is strict if any one of the two associated inequalities is strict. It is clear that the dominance relation is only a partial order.† It does not require any additional preferential information from the decision maker, beyond specification of the parameters of the risk measure (particularly in the case where $R(\cdot)$ is defined with respect to a maximum acceptable loss or ruin). No trade-off information between $\mu(\cdot)$ and $R(\cdot)$ is needed.

The dominance model has been used primarily with $R(\cdot) = \sigma^2$, yielding the EV-dominance model, used extensively in portfolio analysis to identify the efficient set of options (Markowitz, 1959).

2. Compensatory mean-risk trade-off model, which resolves cases incomparable by the dominance model through a mean-risk trade-off curve, thus resulting in a complete order relation. However, the trade-off function has to be supplied by the decision maker, which might prove to be quite a

† The reader is reminded that decision-theoretic terminology is defined in Appendix B.

difficult task. Again, its primary use has been with $R(\cdot) = \sigma^2$ (Markowitz, 1959), though use with "one-sided" low-outcomes oriented risk measure has also been reported (Mao, 1970; Libby and Fishburn, 1977).†

3. Mean-risk lexicographic model, whereby the option with the highest expected return is selected, subject to the risk not exceeding a decision maker sqecified acceptable level. This type of model is typically used in conjunction with "probability of ruin" as a measure of risk (Allais, 1953; Machol and Lerner, 1969; Joy and Barron, 1974). The only preferential input required is the maximum acceptable probability of ruin.

Viewed from the perspective of our overall decision aiding framework (described in section 2.3), the above model consists of an elimination rule (if probability of ruin greater than pre-specified level, then reject), resulting in a subset of options with acceptable probabilities of ruin. The latter are then compared with respect to a single criterion only, namely their respective expected payoff.

Observations on Mean-Risk Model

A number of observations suggest themselves from the above discussion. In particular, concerning risk measures, we note the following:

1. No agreement exists as to a specific definition of "risk", both mathematically and conceptually. This indicates that it is effectively problem context dependent, and might even depend on a particular decision-maker's asset position, as well as his ability to comprehend such information.

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†Discussions on the consistency of EV analysis with EUT can be found in Fishburn and Vickson (1978), Kira and Ziemba (1977), Levy (1979) and others.

2. Risk seems to be viewed as a unidimensional concept or quantity; given the multitude of definitions proposed for it, it seems natural to think of it as a multi-dimensional quantity reflecting various concerns. Such a definition, for a specific problem content (portfolio selection) has been attempted by Zeleny (1977) who defines risk components with respect to minimal outcomes and maximal outcomes respectively.

3. In the above measures, risk is considered as a descriptor of a single distribution, taken individually. As such, they do not characterize the risks relative to a particular choice situation, involving by definition more than a single distribution. In subsection 3.2.2, we develop a choice-related measure of risk in conjunction with the critique of the independence axiom of EUT.

4. The above measures do not seem to have strong intuitive meaning, with the possible exception of the "probability of loss". We articulate this point with subsection 2.2.7 in mind, and in particular with respect to the mean-variance trade-off model, as the latter requires considerable preferential input through non-intuitive (and thus possibly misperceived) information.

The bottom line, of course, is whether or not the decison maker understands those measures and is able to size up the implications of the inputs he supplies. Problem context, type of distributions encountered, and familiarity of the decision maker (with both the problem and the models in question) through repeated decisions are certainly important factors in assessing the validity of such models; in the particular case of portfolio analysis where EV models have been extensively used, it seems that a "calibration" of sorts (of the decision-makers) might have occurred in some

instances. However, transferability to other contexts is rather dubious, especially in light of the limitations of the above-mentioned risk measures.

Furthermore, we note that while the EV-dominance model places virtually no cognitive strain on the decision maker, its discriminatory power is usually quite low, except in relatively trivial situations. On the other hand, the compensatory model suffers from the difficulty of obtaining the needed preferential information, besides being subject to potential assessment hazards. The third approach, featuring the lexicographic decision rule is relatively better grounded behaviorally, yet still suffers from the limitations of using a unidimensional (choice) situation-independent satisficing risk criterion.

As mentioned earlier, in section 3.2, we propose a different, regret-based risk concept reflecting the specific choice situation under consideration. Furthermore, the above ideas are reflected again in the measures, as well as the rationale for the decision rule presented in Chapter 4 which is central to our methodology.

We now proceed with our review and discuss EUT in the next section.

3.1.3 Expected Utility Theory and Related Models

While recognizing that choice among random prospects should conform to the decision maker's preferences, EUT postulates a set of properties that these preferences should possess, resulting in a structured mathematical representation of those preferences and an associated choice rule. A brief exposition of the axiomatic foundation of EUT is presented hereafter.

A binary relation $\succ$ is assumed to exist on the set π of probability functions, where $f_A \succ f_B$ denotes that "f_A is preferred to f_B"; $\succ$ is

thus irreflexive (since $f_A \succ f_A$ is not a true assertion). An indifference relation $\sim$ is associated with $\succ$, such that $f_A \sim f_B \Rightarrow$ not $(f_A \succ f_B)$ and not $(f_B \succ f_A)$; by definition, $\sim$ is reflexive, since $f_A \sim f_A$, $\forall f_A \in \Pi$

The axiomatic foundation† set by VN-M imposes three groups of characteristics on the individual decision maker's preferences; these are (Fishburn, 1970):

A.1. Ordering Axioms, which state the completeness of individual preferences on the set of probability functions Π; in other words, for every f_A, $f_B \in \Pi$, one of the following three conditions holds: $f_A \succ f_B$ or $f_B \succ f_A$ or $f_A \sim f_B$.

Furthermore, the preference relation $\succ$ is postulated to be:

- asymmetric, i.e., $f_A \succ f_B \Rightarrow$ not $(f_B \succ f_A)$
- transitive, i.e., $f_A \succ f_B$ and $f_B \succ f_C \Rightarrow$ $f_A \succ f_C$, $\forall f_A, f_B, f_C \in \Pi$.

The relation $\succ$ is therefore assumed to be a linear order. The associated indifference relations $\sim$ is postulated to be reflexive, symmetric and transitive; it is therefore an equivalence.

A.2. Independence Axiom, which is probably the most crucial to the mathematical derivation and the most controversial. It states that if $f_A \succ f_B$, then $\lambda f_A + (1 - \lambda) f_C \succ \lambda f_B + (1 - \lambda) f_C$, for any $0 \leq \lambda \leq 1$ and and $f_C \in \Pi$.

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†Alternative axiomatizations leading to the same results have been proposed by Marschak (1950), Savage (1954) and Arrow (1971); these sets of axioms are effectively equivalent and are thoroughly discussed and compared in MacCrimmon and Larsson (1979).

In the above axiom, the expression "$\lambda f_A + (1 - \lambda)f_C$" is referred to as a probability mixture of f_A and f_C, and is itself a random prospect. Thus, if we define f_R as $f_R = \lambda f_A + (1 - \lambda)f_C$, then $f_R \in \Pi$ with $f_R(x) = \lambda f_A(x) + (1 - \lambda)f_B(x)$, $\forall x \in X$.

A.3. Archimedean or Continuity Axiom, whereby if $f_A \succ f_B$ and $f_B \succ f_C$, then: $[\alpha f_A + (1 - \alpha)f_C] \succ f_B$ and $f_B \succ [\beta f_A + (1 - \beta)f_C]$ for some α and β such that $0 < \alpha < 1$ and $0 < \beta < 1$.

Axioms A.1, A.2 and A.3 above are necessary and sufficent[†] to prove the existence of a (uniquely defined up to a linear transformation) real valued function $u : X \rightarrow R$ such that:

$$f_A \succ f_B \quad \text{if and only if} \quad E(u, f_A) > E(u, f_B),$$

where $$E(u, f_A) = \int_{-\infty}^{+\infty} u(x) f_A(x) dx.$$

The function $u(\cdot)$ is known as a "utility" function,[§] thus $E(u, f_A)$ is the expected utility of random prospect f_A.

The implications of this result are manifold. First, if a decision maker accepts the axioms (as decision analysts and economists argue that he should), then a utility function defined over X is all that is needed to establish a complete order over the set of probability functions via their respective expected utility. Second, and as a corollary

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[†]Proof for the specific version of the axioms presented here can be found in Jensen (1967) or Fishburn (1970).

[§]We will also refer to $u(\cdot)$ as a "preference curve".

of this last point, a decision maker's utility function can be assessed using his responses to simple hypothetical choice situations,[†] and can subsequently be used to compare more complex options. This of course is the rationale behind Decision Analysis.

What should be re-emphasized concerning EUT, as it has bearing on proposition 2.2.1 in Chapter 2 is that EUT is only normative in that it prescribes choices that supposedly conform to the decision maker's preferences, as represented by his utility function. Furthermore, it is normative in that it requires these preferences to follow the tenets of rationality embodied in the above axioms.

Given that one accepts those tenets, assessment of a utility function for choice under uncertainty can be quite a complex and difficult task.[§] However, in many choice situations, it is possible to reach a decision consistent with EUT knowing only some general properties of the utility function, such as its monotonicity and concavity, thus obviating the need for a full blown preference curve assessment. Such choice situations, characterized by the relative shapes of the probability distributions under consideration, have been identified and discussed by Hammond (1974) and are the subject of Stochastic Dominance (Whitmore and Findlay, 1978).

While stochastic dominance is a technique that can greatly reduce the

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[†]A number of assessment procedures based on this principle have been developed in Decision Analysis; their details are however outside the scope of this presentation, as they can be found in a number of texts, including Raiffa (1968), Keeney and Raiffa (1976) and Holloway (1979), among others.

[§]That such difficulty can be due to the restrictiveness and lack of validity of the axioms is quite a plausible hypothesis, as we see in section 3.2. However, for the time being, let us for the sake of argument, accept that the difficulty is not due necessarily to a violation of the underlying assumptions.

decision maker's congitive strain in a preference assessment session, it requires acceptance of EUT's axioms, and as such does not propose an alternative set of postulates for choice under uncertainty. It is therefore not necessary to discuss it any further here.[†]

Finally, in this section, we note a class of models which have not adopted EUT's axiomatic system (nor do they have a system of their own), but have postulated the existence of a real valued value function $v : X \rightarrow \mathbb{R}$, compatible with the theory of ordered utility differences (Pareto, 1927); presumably reflecting "strength of preference" for an outcome $x \in X$ (see Krzysztofowicz [1981] for a discussion of such a value function and its relation to VN-M's utility function for risky choice).[§] This class of models then defined a criterion function for comparing probability distributions over X (transformed to distributions of value outcomes), with terms consisting of higher moments of the distribution of value outcomes, in addition to the first moment (Hagen, 1979; Munera, 1978). The reason for noting these models here in conjunction with EUT is that they were proposed as an alternative to the expected utility criterion, claiming to explain some of the classical paradoxes associated with EUT. Some of these paradoxes, along with other limitations of EUT are the topic of the next section.

[†]Interested readers are referred to any one of a rapidly growing list of references on the topic, such as Hadar and Russell (1974), Borch (1979), or Whitmore and Findlay (1978); Bawa (1981) recently compiled an extensive research bibliography on the topic.

[§]This topic is also addressed by Sarin (1981), Dyer and Sarin (1981) and Bell (1981a). Note in this respect that the above-mentioned "value function", reflecting "strength of preference" is what economists usually refer to as "utility functions". Following the standard practice in decision theory, we reserve the term "utility function" for a VN-M's function for the risky choice context (Keeney and Raiffa, 1976).

3.2 Critique of Expected Utility Theory

In this section, the axiomatic foundation of EUT is re-examined in terms of its appropriateness for our decision environment, and thus its conformance to the basic considerations of Chapter 2. Subsection 3.2.1 identifies a number of limitations of EUT with respect to these considerations, while subsection 3.2.2 focuses on the independence axiom (axiom A.2). The latter's otherwise apparent intuitive appeal is questioned, using reported evidence in the literature, as well as a new argument articulated through a simple example illustrating the implications of this axiom on the underlying risk dimensions of a given choice situation. These risk dimensions are developed in that subsection to capture the degree of violation of the complete order over the outcome space, which is assumed known and agreed upon.

3.2.1 General Limitations

While EUT is in agreement with proposition 2.2.1, since a VN-M utility function is supposed to encode preferences for risky choice situations, it assumes those preferences to be fully formed a priori (by virtue of axiom A.1). It is thus not compatible with proposition 2.2.2, nor with 2.2.3, which casts decision aiding as a preference forming process. While EUT (as used in Decision Analysis) may claim to "form" preferences over the set of options, using responses to hypothetical gambles and the associated utility construct, this is precisely where it is most susceptible to errors and biases. Indeed, as indicated in subsections 2.2.3 and 2.2.7, the dangers of an assessment methodology shaping and/or misrepresenting preferences are highest when preferences are not well formed (Fishhoff et al., 1979).

The considerations in subsection 2.2.4 to 2.2.8 articulating the need to incorporate behavioral aspects of decision-making and judgement and specifying these aspects, are not generally reflected in EUT. In particular, serious questions have been raised concerning the extent to which EUT reflects some important dimensions of the psychology of choice under uncertainty (Coombs, 1975; Kahneman and Tversky, 1979; Allais, 1979; Bell, 1981a). It is interesting to note in this context that the descriptive power of utility theory is no longer claimed (even by its staunchest supporters, as in Gorovitz [1979], Morgenstern [1979] and Howard [1980]), primarily as a result of considerable evidence gathered and reported by psychologists on the consistent violation of EUT's axioms in observed behavior.†

However, the idiosyncracies of observed behavior of "common" individuals making choices among random prospects do not necessarily provide a sufficient basis for assessing the validity of a supposedly normative theory. On the other hand, observed choices of individuals who are explicitly and consciously trying to act rationally, and who are qualified to understand probabilistic information, should be more valid indicators of the appropriateness of a theory that recognizes the decision-maker's preferences as the basis for choice among random prospects. This argument is invoked again in subsection 3.2.2.

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†A number of alternative descriptive models have been suggested by mathematical psychologists to describe risky choice behavior, including Payne (1973), Schneeweiss (1974), Coombs (1975), Kahneman and Tversky (1979). A review of the empirical research refuting the descriptive validity of EUT is beyond the scope of this discussion. An excellent survey can be found in Slovic et al. (1977); other references include Kahneman and Tversky (1979), MacCrimmon and Larsson (1979) and Allais (1979).

One further point concerns the incompatibility of the ordering axiom (A.1) with the considerations of subsections 2.2.2, 2.2.3 and 2.2.8 in two major aspects: 1) completeness of preferences over the set of random prospects, and 2) the transitivity of the indifference relation $\sim$. Having already amply discussed the unnecessarily restrictive nature of completeness in our decision environment, we briefly illustrate based on Fishburn (1979), why transitivity of $\sim$ is not as obvious as it might first seem. Let f_A and f_B be "sure gambles" with $f_A(35) = 1$ and $f_B(36) = 1$, and f_C a third gamble offering a probability of 0.5 at receiving 100 and 0.5 at receiving nothing (i.e., $f_C(0) = f_C(100) = 0.5$). According to the agreed upon order over $X \subseteq \mathbb{R}$, $[f_B \succ f_{B'}]$ since f_B has a guaranteed greater payoff than f_A. However, as argued by Fishburn, most rational individuals who exhibit $[f_A \sim f_C]$ will also exhibit $[f_B \sim f_C]$. Yet, it is not true that $[f_A \sim f_B]$. Such "anomalies" are typically attributed to perceptual threshold phenomena. In section 3.3, postulates compatible with these phenomena are presented. However, we first present our critique of the independence axiom (A.2)

3.2.2 The Independence Axiom

In the form stated in subsection 3.1.3, the independence axiom is commonly know as Savage's "Sure Thing" principle (Savage, 1954).[†] It essentially says that if $[f_A \succ f_B]$ for a given individual, then rationality requires that he/she should have $[g_A \succ g_B]$, where g_A and g_B consist of the following probability mixtures, respectively:

1) f_A with probability α or f_C with probability $(1 - \alpha)$, i.e.,

$$g_A = \alpha f_A + (1 - \alpha) f_C \quad ;$$

[†]Other independence axioms implied by the expected utility criterion are discussed in Fishburn (1972, 1979).

2) f_B with probability α or f_C with probability $(1 - \alpha)$, i.e.,

$$g_B = \alpha f_B + (1 - \alpha) f_C \quad ;$$

and the above should be true for all f_A, f_B, $f_C \in \Pi$ and $0 \leq \alpha \leq 1$.

The argument typically given in support of the "rationality" of this axiom is as follows: there is a $(1 - \alpha)$ probability that either gamble (g_A or g_B) yields f_C (in that regard, they are thus equivalent), and a probability α that either f_A or f_B are obtained; since g_A offers a probability α of attaining a preferred option, and is otherwise equivalent to g_B, then g_A should be preferred to g_B.

While the logic of the above argument might seem unquestionable, it can be seriously challenged on grounds that g_A and g_B should be viewed and compared wholistically rather than in a "two-stage" fashion (Fishburn, 1979). In other words, g_A should be viewed as the function that takes, for a given x, the value $g_A(x) = \alpha f_A(x) + (1 - \alpha) f_C(x)$, and similarly for g_B, rather than a composite of events.

The "two-stage" argument is an excellent illustration of the framing effect discussed in subsection 2.2.4 and 2.2.7, and recently emphasized by Slovic et al. (1981). By framing the choice situation in a two-stage manner, an illusion is created that f_A and f_B are "tangible" events or objects whose value is known and received with certainty, independently of an exogenous chance occurrence.† Furthermore, even if one could "cash in" f_A or f_B (say by selling it off at a predetermined known price), the out-

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†We are implicitly assuming that the independence axiom is plausible and uncontroversial when the outcomes are certain quantities rather than random prospects. In particular, let f_A, f_B, and f_C be sure gambles yielding \$100, \$50 and \$25, respectively (obviously $[f_A \succ f_B]$). If g_A offers f_A with probability α, and f_C with probability $(1 - \alpha)$, and g_B offers f_B with probability α and f_C with probability $(1 - \alpha)$, then $[g_A \succ g_B]$ is quite plausible.

come received from either mixture g_A or g_B is not a two-stage outcome but a single outcome, which is ultimately evaluated according to the unambiguous order that is known to exist on the outcome space X. Therefore, in as much as preference between random prospects is conditional upon the order accepted over X, and mixtures of elements of Π lead to different pdf's over X, these mixtures should be compared regardless of their constituent elements.

Consistent with our discussion so far, the failure of the relation over Π to satisfy the independence axiom would be a "legitimate" property of the decision-maker's preferences towards random prospects.† It follows from the above that preferences held (and rationally and consciously formed) by a decision maker over a subset of Π, even if they violate the independence axiom, constitute a more valid guide for normative action than the implications of the acceptance of the independence axiom when the latter is presented only in abstract terms or framed in a "two-stage" fashion. This argument is made to further support the conclusions reached in a number of experimental and real-life situations whereby subjects explicitly seeking to act rationally revealed, through their choices, an underlying preference relation over Π which violates the independence axiom.

The best known of these reported instances is Allais' famous so-called "paradox" (Allais, 1953; 1979). However, there have also been "counter-claims" to Allais' paradox, whereby decision makers whose choices violated the independence axiom were later "convinced" to reverse these preferences,

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†In the same sense that "risk aversion" or "risk proneness" are acceptable properties of a decision maker's preferences.

following an "explanation" of that axiom (MacCrimmon, 1968). This particular "counterclaim" illustrates further the point made earlier about framing of explanations and their effect on a respondent's attitude, especially when the latter's preferences are ambiguous. For instance, with regard to MacCrimmon's experiment above, Slovic and Tversky (1974) challenged those results on grounds that the experimenter placed undue pressure on the respondents (to accept the axiom).† The argument of the previous paragraph concurs with that challenge. Other reported instances clearly casting doubt on the plausibility of the independence axiom are discussed in Coombs (1975), Ellsberg (1961) and Moskowitz (1974), among others.

In what follows, the independence axiom is re-examined from the perspective of the accepted order over the outcome space X. The implications of that axiom are in essence "re-framed" by contrasting key risk dimensions of a choice situation between two prospects f_A and f_B to those of a choice between the two mixtures g_A and g_B (where g_A and g_B are as defined at the beginning of this subsection). A simple numerical example is used for that purpose. First, however, the risk dimensions of interest are developed.

In order to be "valid" (from a normative standpoint), any preference relation over Π has to be consistent with the complete unambiguous preference order assumed to exist over the relevant domain of the reward space X. The inherent peculiarity of random prospects, which characterizes preferences over Π, is that a priori preferences (i.e., before resolution of the uncertainty and realization of an outcome) do not necessarily imply ex-post (i.e., after realization of the outcome) consistency with preferences over X. To illustrate this relatively obvious point (yet fundamental to a true

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†According to Slovic et al. (1977), that researcher (MacCrimmon) later acknowledged the inconclusiveness of his own previous findings.

grasp of the implications of preferences between risky prospects), consider the lottery $f_A \in \Pi$ yielding x_A with probability α and x_C with probability $(1 - \alpha)$, and $f_B \in \Pi$ yielding x_B with certainty, where $x_A, x_B, x_C \in X$ and $x_A > x_B > x_C$.[†] Consider an individual who consciously decides to take f_A over f_B in a choice situation where both are present, indicating that, a priori, $[f_A \succ f_B]$. If the realization of the outcome of f_A yields x_C, then, ex-post, because $x_B > x_C$ (unambiguously) the outcome of f_A cannot be preferred to that which would have resulted from the selection of f_B.

It is thus clear that, by definition, preference between random prospects might entail a finite risk of ex-post violation of the underlying order over X, thus resulting in regret. Denoting by x_i^* the outcome of $f_i \in \Pi$, that risk is simply the probability that the outcome x_2^* from the unselected prospect f_2 in a choice situation involving f_1 and f_2 (where $f_1, f_2 \in \Pi$) is such that $x_2^* > x_1^*$; denote this probability by $p(x_2^* > x_1^*)$. For instance, if F_1 and F_2 are cumulative distribution functions corresponding to f_1 and f_2 respectively, we obtain, in the general case:[§]

$$p(x_2^* > x_1^*) = \int_X p(x_2^* > x \mid x_1^* = x) \cdot f_1(x)dx$$

$$= \int_X \{1 - [p(x_2^* \leq x) \mid x_1^* = x]\} \cdot f_1(x)dx$$

$$= 1 - \int_X F_{2|x_1^*=x}(x) \cdot f_1(x)dx \qquad (3.2)$$

[†]Without loss of generality, the preference relation over X is considered to be the same as that over $\mathbb{R}$ and is thus denoted >.

[§]Note that in this discussion, we assume that the conditional probability of occurrence of any outcome from a given option is independent of which option is selected.

where $F_{2|x_1^*=x}$ is the conditional c.d.f. of x_2^* given the value of x_1^*. Of course, if f_1 and f_2 are independent, then expression (3.2) becomes

$$p(x_2^* > x_1^*) = 1 - \int_X F_2(x) \cdot f_1(x)dx \qquad (3.3)$$

The quantities $p(x_2^* > x_1^*)$, $p(x_1^* > x_2^*)$ and $p(x_1^* = x_2^*)$ can thus be viewed as key dimensions of a choice between f_1 and f_2 and are as such critical in determining their relative preference.† We argue here that a change in one or more of these dimensions clearly places the decision maker in an altogether different choice situation which might preclude inferences or extensions (from comparisons among certain prospects) such as the one postulated in the independence axiom. A simple numerical example illustrates this point:

Consider a choice between f_1 and f_2, where

$$f_1(x) = \begin{cases} 0.8 \text{ for } x = 90 \\ 0.2 \text{ for } x = 0 \\ 0 \text{ otherwise} \end{cases} \qquad f_2(x) = \begin{cases} 0.7 \text{ for } x = 100 \\ 0.3 \text{ for } x = 0 \\ 0 \text{ otherwise} \end{cases}$$

This situation can be described by:

$$p(x_1^* > x_2^*) = p(x_1^* = 90 \text{ and } x_2^* = 0) = 0.24$$
$$p(x_2^* > x_1^*) = p(x_2^* = 100, \forall x_1^*) = 0.7$$
$$p(x_2^* = x_1^*) = p(x_1^* = 0 \text{ and } x_2^* = 0) = 0.06$$

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†Note that this characterization of random prospects as multidimensional stimuli defined by basic risk dimensions is consistent with psychological theories and evidence presented by Payne (1973), Payne and Braunstein (1971) Slovic and Lichtenstein (1968) and discussed in subsection 2.2.7.

It is immaterial here which of the two distributions should be preferred, since as emphasize all along, we have no normative basis for comparing all elements of Π without explicit considerations of the decision-maker's preferences. In this particular case, either relative ordering is possible as neither seems to be "obviously" better nor worse than the other. Let us now introduce a third prospect, $f_3 \in \Pi$, defined by:

$$f_3(x) = \begin{cases} 0.25 & \text{for } x = 110 \\ 0.25 & \text{for } x = 85 \\ 0.25 & \text{for } x = 10 \\ 0.25 & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

We can form the following probability mixtures:

$$g_A = \alpha f_1 + (1 - \alpha) f_3 \quad \text{and} \quad g_B = \alpha f_2 + (1 - \alpha) f_3, \quad \text{where } 0 < \alpha < 1.$$

According to the independence axiom, g_A's relative preference with respect to g_B should be the same as that of f_1 relative to f_2. We show here that such an implication should not be taken for granted as substantial changes in the risk dimensions associated with the new choice situation can occur, where the direction of these changes does not necessarily have clear preferential implications (from a normative or "rational" standpoint). Consider the case where $\alpha = 0.2$; g_A and g_B are then:

$$g_A(x) = \begin{cases} 0.2 & \text{for } x = 110 \\ 0.16 & \text{for } x = 90 \\ 0.2 & \text{for } x = 85 \\ 0.2 & \text{for } x = 10 \\ 0.24 & \text{for } x \quad 0 \\ 0 & \text{otherwise} \end{cases}$$

$$g_B(x) = \begin{cases} 0.2 & \text{for } x = 110 \\ 0.14 & \text{for } x = 100 \\ 0.2 & \text{for } x = 85 \\ 0.2 & \text{for } x = 10 \\ 0.26 & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

If we define x_A^* and x_B^* as above but corresponding to g_A and g_B respectively, this choice situation can be described by:

$$p(x_A^* > x_B^*) = p(x_A^* = 110 \text{ and } x_B^* < 110) + p(x_A^* = 90 \text{ and } x_B^* < 100)$$

$$+ p(x_A^* = 85 \text{ and } x_B^* < 85) + p(x_A^* = 10 \text{ and } x_B^* < 10)$$

$$= 0.2 \times 0.8 + 0.16 \times 0.66 + 0.2 \times 0.46 + 0.2 \times 0.26 = 0.41$$

$$p(x_B^* > x_A^*) = 0.48$$

$$p(x_A^* = x_B^*) = 0.182$$

Comparing the above values (0.41, 0.408, 0.182) to the initial set (0.24, 0.7, 0.06), we clearly see that both the absolute and relative risks of having obtained a better outcome from the unchosen option, i.e., of "making the wrong choice", are substantially different in the intial and trans-

formed situations. Figure 3.1 shows the variation of $p(x_A^* > x_B^*)$, $p(x_B^* > x_A^*)$ and $p(x_A^* = x_B^*)$ as a function of α. It clearly reveals the high sensitivity of the relative risks of selecting either prospect g_A or g_B to the value of α, such that the initial relative risks implied by a choice between f_1 and f_2 can vary widely in direction and magnitude when f_1 and f_2 are combined as described above with f_3.

In addition to questioning the independence axiom, the above example has served to introduce the regret-based dimensions of the risk associated with a choice between random prospects, as opposed to the risk measures of section 3.1 which described random prospects taken individually. However, this is not to say that the risk measures presented here are the sole factors that should be considered in the choice. The next chapter builds on this concept by proposing other such risk dimensions and using these in an operational decision aiding methodology for our decision environment.

To summarize section 3.2, the axiomatic foundation of EUT was found to be generally incompatible with the basic considerations described in Chapter 2 and thus of limited use in the decision environment of section 1.1. In particular, the alleged rationality of the independence axiom (A.2) was further questioned through its implications on regret-based risk dimensions of a choice situation between random prospects. Furthermore, the requirements of the ordering axiom (A.1) were found to be unduly restrictive and thus quite problematic for our decision environment where preferences are not formed a priori and characterized by a high degree of ambiguity.†

†We did not even discuss the continuity axiom (A.3), itself the subject of considerable criticism (Fishburn, 1979). However, the same considerations applying to the critique of the completeness of ($\succ$, $\sim$) apply to (A.3)

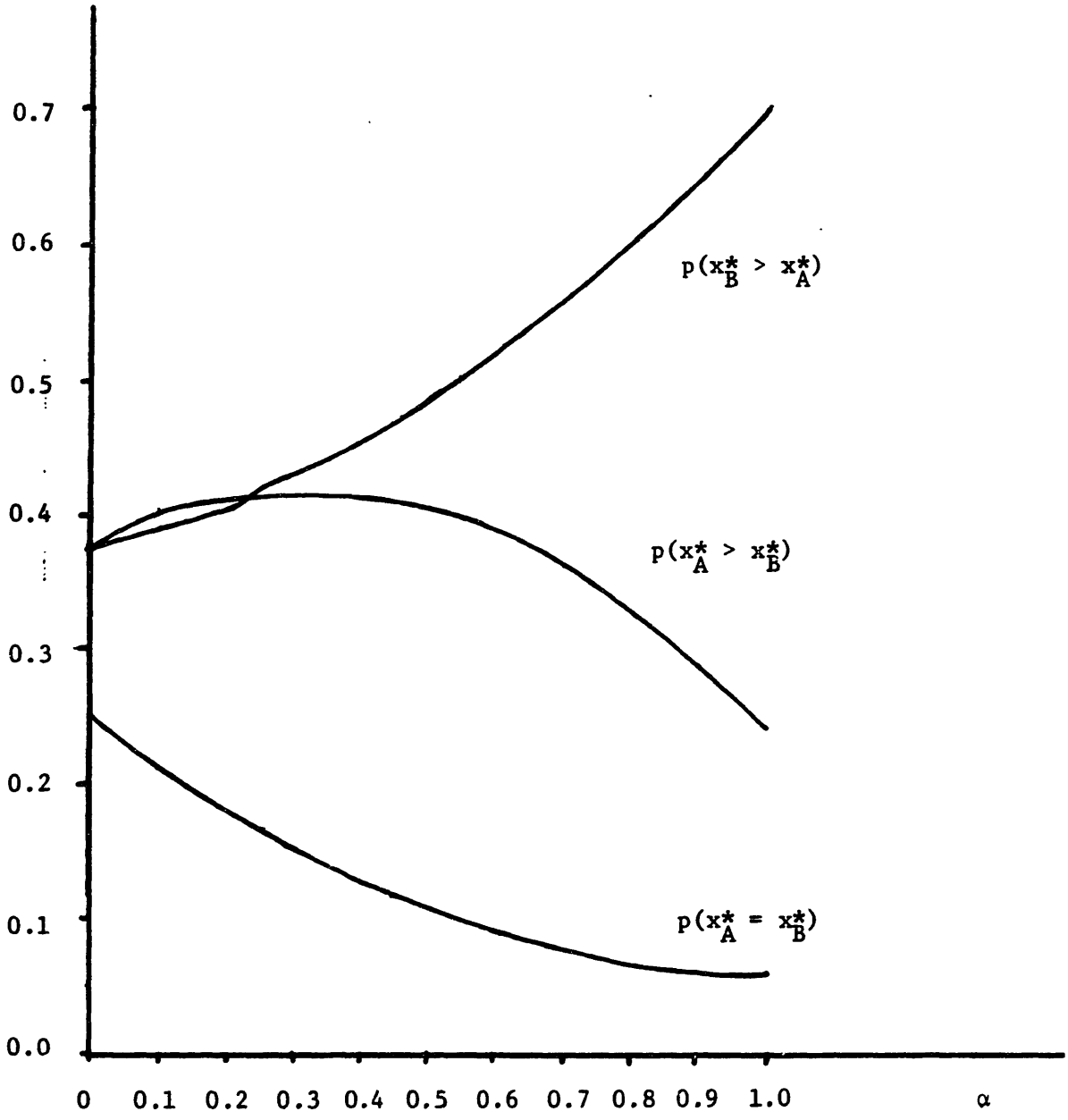


Figure 3.1: Variation of Implied Risks as a Function of α

The reason this critical discussion is necessary in the context of this thesis stems from the prevalence, in related disciplines, of EUT as a model of rational choice under uncertainty. However, as discussed in Chapter 1, its application to transportation planning has been extremely limited. The discussion herein has uncovered the reasons for its poor applicability to that decision environment, and essentially concluded that, despite the need for appropriate decision aiding methodology, efforts do not seem warranted for further attempts at removing these obstacles, as many of these are rooted in its axiomatic foundation.

In the next section, alternative postulates are presented as a possible basis for development of the methodological aspects of the decision aid introduced in the previous two chapters. These postulates are quite unrestrictive (vis-a-vis those underlying EUT) and consistent with the basic considerations of Chapter 2.

3.3 Alternative Starting Postulates

While the postulates underlying this work have been conceptually described in Chapter 2, this section merely translates a number of them into specific mathematical postulates as a basis to the formal derivations in the following chapters. More specifically, properties of the decision maker's preferences over the outcome space X and the subset O of options under consideration (where $O \subseteq \Pi$) are proposed. Those are contrasted with EUT's axioms to more clearly demonstrate their less restrictive character resulting from the basic considerations of Chapter 2.

As a starting point, we maintain the fundamental postulate that an unambiguous known complete order exists over the outcome space X. Furthermore, unless otherwise stated, we implicitly assume that $X \subseteq \mathbb{R}$ the set of real numbers.

The most important postulate concerns the preference-indifference relation over the set of options $O \subseteq \Pi$. It is a direct result of proposition 2.2.8, and is otherwise compatible with the collectivity of propositions of Chapter 2. It also represents a relaxation of the most restrictive and objectionable elements of EUT's ordering axiom, A.1, in particular:

- the completeness of the preference-indifference relation over Π.
- the transitivity of the indifference relation.

Still denoting preference between elements of O by the symbol $\succ$, and indifference by $\sim$, ($\succ$, $\sim$) could be appropriately described by a partial semi-order, though we do not claim such a description to be an absolute inflexible norm of rationality. It does however provide a more realistic representation of preference for the purpose of an operational decision

decision aiding methodology in our problem environment. The properties of partial semi-orders are presented hereafter (Jacquet-Lagrèze, 1975):

P1. for o_i, $o_j \in O$, only one of the following holds:

$o_i \succ o_j$, i.e., o_i is strictly preferred to o_j

$o_j \succ o_i$, i.e., o_j is strictly preferred to o_i

$o_i \sim o_j$, i.e., o_i is preferentially indifferent to o_j

$o_j \; I \; o_j$, i.e., o_i and o_j are incomparable

P2. $[o_i \sim o_i]$ is true $\forall o_j \in O$ (i.e. reflexivity of $\sim$) and $[o_i \succ o_i]$ is false (i.e., irreflexivity of $\succ$).

P3. $([o_i \succ o_j]$ and $[o_j \sim o_k]$ and $[o_k \succ o_\ell]) \Rightarrow [o_i \succ o_\ell]$, where o_i, o_j, o_k, $o_\ell \in O$.

P4. $([o_i \succ o_j]$ and $[o_j \succ o_k]$ and $[o_j \sim o_\ell]) \Rightarrow [o_i \sim o_\ell]$ and $[o_k \sim o_\ell]$ are not both true.

Representation of semi orders in numerical field is discussed in Luce (1956), Fishburn (1973), Jacquet-Lagreze (1975) and Swistak (1980), among others, and is not of direct concern to this discussion.

The first property, P1, reflects the non-completeness of the preference-indifference relation between random prospects. The justification for this property and situations where it occurs have been discussed in subsection 2.2.8. The second property, P2, is straightforward, and merely reflects the basic understanding of what "preference" means versus what "indifference" means.

The third property, P3, needs further clarification. It is effectively a form of transitivity of $\succ$, but not of $\sim$. For instance, if $[o_i \succ o_j]$ and

and $[o_j \sim o_k]$, then it is not necessarily true that $[o_i \succ o_k]$. In other words, while o_i and o_j may be distinct enough to where o_i is clearly preferred to o_j, o_j and o_k may be too similar to cause strict preference, without o_k being distinct enough from o_i for clear preference to exist.

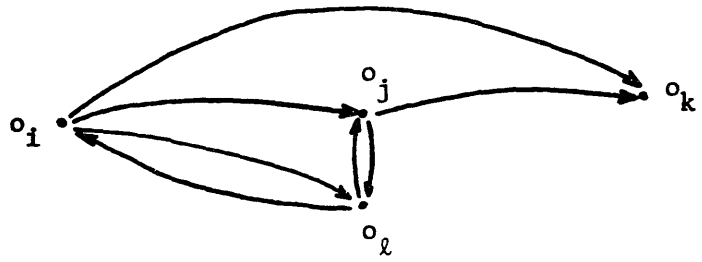
Furthermore, $[o_i \sim o_j]$ and $[o_j \sim o_k]$ do not necessarily imply that $[o_i \sim o_k]$, also because of the perceptual threshold phenomena discussed earlier.

As to the fourth property, P4, the clearest way to explain it is visually using a graph theoretic representation. Thus, in Figure 3.2, case (c) illustrates a situation that is not allowed by P4, while cases (a) and (b) illustrate two permissible situations. Note that in the figure, a directed arc (o_i, o_j) is defined between two nodes representing o_i and o_j respectively, if $[o_i \succ o_j]$ whereas two directed acrs (o_i, o_j) and (o_j, o_i) are defined if $[o_i \sim o_j]$. Therefore, in case (a) for example, we have $[o_i \succ o_j]$, $[o_i \succ o_k]$, $[o_i \sim o_\ell]$, $[o_j \succ o_k]$, $[o_j \sim o_\ell]$.

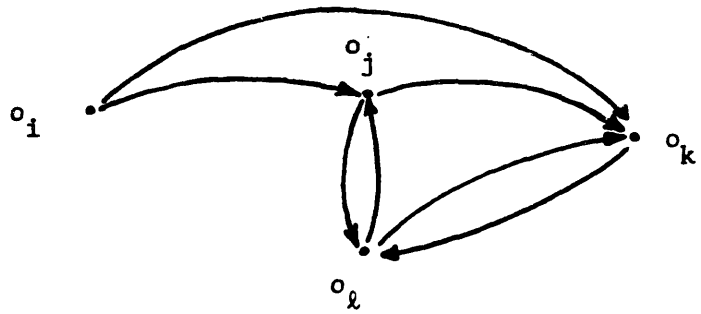
In addition to the above properties, it should be noted that the preference-indifference relation is not required to satisfy the independence axiom, for the reasons presented in the previous section.

Finally, it is re-emphasized here that those postulates concerning the properties of the preference-indifference relation merely translate the basic considerations of Chapter 2 into an operational working basis for methodological development. While these postulates consitute a departure from assumptions made in conjunction with conventional methods and are in themselves an important feature of the decision aiding methodology presented in this thesis, they are only meaningful when viewed within the context of the collective set of basic considerations and propositions of Chapter 2.

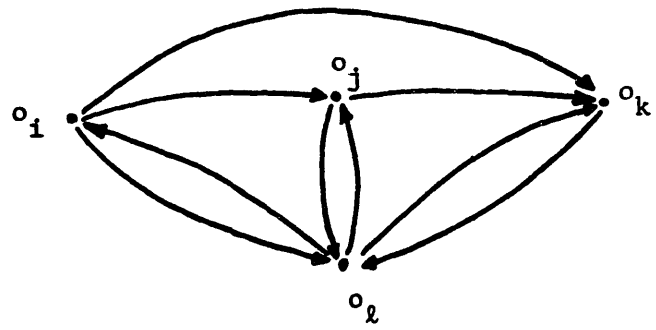
Figure 3.2: Illustration of Property P4 of Partial Semiorders



(a) Valid



(b) Valid



(c) Not Valid

3.4 Summary

This chapter has presented the principal normative theories and models for choice among random prospects, which were grouped in two categories: those which rely on expected utility theory, and those which do not. Both categories were described in section 3.1. In particular, and in conjunction with the latter category, shortcomings of existing risk measures were indicated in view of developing different ones in Chapter 4, reflecting the basic considerations of chapter 2. The axiomatic foundation of expected utility theory was presented in subsection 3.1.3, along with its operational implications. This foundation was critically re-examined in section 3.2, with particular emphasis on the independence axiom, the plausibility of which was questioned via a new argument relying on the implications of that axiom on key risk dimensions underlying a choice situation between random prospects.

The above critical review provided an appropriate setting for introducing alternative postulates adopted in the methodology developed in this thesis. These postulates concern the properties of the preference indifference relation over a subset of the random prospects set, whereby partial semi-orders were suggested as a representation of this relation which is consistent with the basic considerations of Chapter 2. These postulates represent another actualization of those considerations into an operational feature of the methodology.

In the next chapter, we describe the structure as well as the specific elements of the methodology for the pairwise comparisons phase of the decision aid (of section 2.3), where the options consist of random prospects characterized by discrete joint distributions of payoffs, i.e., problem subclass 1, as defined in section 1.4.

CHAPTER 4

STRUCTURE, ELEMENTS AND PROPERTIES OF METHODOLOGY FOR PAIRWISE COMPARISONS

In the previous chapter, the focus was primarily on the fundamental problem of compairing two univariate random prospects. This chapter considers the entire picture, where all elements of the set O of options are compared on a pairwise basis, and it is desired to reduce the set O to a smaller subset. It describes the construction of a binary preference-indifference relation over the set O, which is consistent with the basic considerations of Chapter 2. The methodological development is presented in conjunction with the "discrete states" problem formulation (i.e., problem subclass 1), even though the basic structure is applicable to all formulations described in section 1.4.

General but essential underlying concepts as well as the basic logical and mathematical structure of the methodology, which are applicable to all problem subclasses of interest in this thesis, are presented in section 4.1. Section 4.2 then presents the rationale and the mathematical specification of the decision rule for comparing two options in the discrete states problem (subclass 1); after expending further on that particular formulation, an initial binary preference-indifference relation $(\succ_1, \sim_1)$ which allows for incomparability, is constructed, and its properties thoroughly discussed. While these properties are quite unrestrictive, they might present inconsistencies with our basic postulates, manifested primarily through the existence of cycles in the graph associated with $(\succ_1, \sim_1)$; violations of the properties of partial semi-orders are identified in section 4.3, and rules for their detection specified. In particular, the notion of "P3-closure" of a relation is defined, and obstacles in the way of its imple-

mentation, thus leading to inconsistencies, are presented. Three specific approaches for correcting these inconsistencies are then proposed and developed in section 4.4. Finally, section 4.5 summarizes the chapter, highlighting the way in which the methodology adheres to the basic considerations of Chapter 2; in addition, it describes the link(s) between following chapters and the material contained herein.

This chapter contains the essence of the methodological contribution of this thesis, in that it describes the actualization of the features discussed in Chapter 2. In addition to the premise and basic features of the methodology, the contributions in this chapter include the operational pairwise comparison relation, based on a multidimensional representation of the risk implications of a binary choice situation. While the rationale is adapted from the well-known ELECTRE technique (Roy, 1974; Nijkamp and Van Delft, 1977) for multi-criteria decision problems under certainty, the adaptation to the random prospect case is quite novel, especially since the measures are derived from risk considerations of concern in the application context.†

Furthermore, and directly as a consequence of the features of the methodology and its underlying postulates (which, as we stated earlier, are a departure from previous and existing approaches, and as such an integral part of the contribution), the structure of the methodology, including the definition and sequencing of steps, should be noted, and in particular the correction strategies, described in section 4.5. In addition, and at a

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†As a matter of fact, as shown in section 4.2, perhaps the most objectionable aspect of ELECTRE, when used in multi-criteria decision problems under certainty, which is the arbitrariness of the criteria relative weight vector, is overcome in our problem situation because of the clear risk significance of the comparison measures.

somewhat more theoretical level, the detection rules for inconsistencies with partial semi-orders in graphs of partial relations, for our particular operational definition of inconsistency, is worthy of note. However, as we said throughout, it is the totality of the approach that constitutes the primary contribution in this chapter (albeit this thesis), rather than the sum of the "local" contributions.

4.1 General Concepts and Structure

The purpose of this section is to present an overview of the methodological structure (for the pairwise comparisons stage of the decision aiding framework), and to explain, conceptually, the nature and purpose of its different components. As such, it is applicable to all classes of problems described in section 1.4. In the first subsection, an overview of the structure is presented. Subsection 4.1.2 subsequently presents different solution concepts and their properties for the reduction of the set O of options to a smaller subset.

4.1.1 Overview

Consider a set O of options. Assume that the issue of comparing any two options o_i, o_j in O is resolved through the specification of some mechanism which yields a preferential statement between o_i and o_j, allowing for incomparability. The juxtaposition of preferential statements for all pairs of options in O defines a partial binary relation over O, denoted by $(\succ_1, \sim_1)$,[†] and referred to hereafter as the initial relation.

There are two things of interest about such a binary relation: 1) its properties, in as much as they conform to or violate certain conditions or principles deemed desirable by the decision maker (and/or his counsel) and 2) the manner in which it can be used to reach a reduced subset of superior options (containing the most preferred one), in order to perform its role in the overall decision aiding framework (of section 2.3).

These two points are strongly interdependent, since some solution concepts are not always compatible with certain properties. For instance, if

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[†]Strict preference is represented by $\succ_1$, while $\sim_1$ denotes preferential indifference.

the relation is a complete order, then there exists an element o* in O such that o* is either preferred to or indifferent to any other element $o_i \in O$. However, when the relation has more general and less restictive properties (such as not being complete), the existence of such an element is not guaranteed, and is actually impossible in the case of a partial relation. Therefore different solution concepts can be used, as discussed in subsection 4.1.2.

Relative to the first point of interest, namely the properties of the binary relation, assume that it is desired that the relation be consistent with a set of postulates, such as those proposed in section 3.3. In this case, inconsistencies with those postulates have to be identified. In particular, cycles might be present in the graph of that initial relation, † and these might constitute violations of the starting postulates.

This suggests that a number of operations can be carried out in order to transform the initial relation $(\succ_1, \sim_1)$ into a <u>final</u> relation $(\succ, \sim)$ where inconsistencies are removed. We refer to such operations as "correction" strategies or approaches.

Given the resulting final $(\succ, \sim)$, a solution concept compatible with the properties of $(\succ, \sim)$ can then be solved for. In our case, the kernel of $(\succ, \sim)$, defined below, is such a solution, as it is compatible with the properties of partial semi-orders.

Essentially, given a set of starting postulates, the following highly schematic structure is followed:

1. Build an initial partial binary relation $(\succ_1, \sim_1)$;
2. Identify inconsistencies of $(\succ_1, \sim_1)$ with starting postulates ;

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†Graph theoretic representation of a binary relation is discussed in section 4.3.

3. Perform correcting strategies on $(\succ_1, \sim_1)$ to obtain final relation $(\succ, \sim)$;

4. Solve for reduced subset (kernel or possibly other solution concepts).

Within each of the above steps, a variety of assumptions can be made and a number of alternative approaches followed. There is, of course, no unique best way to proceed, and considerable flexibility is allowed in fine-tuning the methodology to the specific situation at hand. The development in this chapter (and in the remainder of this thesis) elaborates on all these components, and recommends specific procedures primarily derived from the basic considerations of Chapter 2. In the process, alternative procedures, their underlying assumptions and their implications are also discussed.

A key assumption in terms of determining the structure of the approach beyond the initial relation $(\succ_1, \sim_1)$ concerns the properties of the final $(\succ, \sim)$. Three possibilities are identified and discussed in this chapter, namely:

a. accepting the resulting initial $(\succ_1, \sim_1)$ "as is", in which case the second and third steps in the above structure become irrelevant, since $(\succ_1, \sim_1)$ automatically becomes $(\succ, \sim)$;

b. making further preferential assumptions in interpreting the existing $(\succ_1, \sim_1)$. For example, the reduction procedure followed in ELECTRE I and II (Roy, 1973) assumes cycles to indicate preferential indifference, thus defining equivalence classes, as described in section 4.3;

c. performing operations aimed at removing inconsistencies with starting postulates, in particular with partial semi-orders.

It is, of course, the latter which is the primary focus of this chapter,

consistently with the basic considerations of Chapter 2 and the resulting starting postulates of section 3.3.

In relation to the above structure, section 4.2 addresses its first component by specifying the binary comparison measures and rationale used to construct the initial relation $(\succ_1, \sim_1)$ for the single attribute "discrete states" problem formulation (i.e., problem subclass 1), and investigating its mathematical properties. Section 4.3 addresses the second component in the above structure, namely the issue of consistency with the properties of partial semi-orders, and derives rules for detecting such inconsistencies in the graph of $(\succ_1, \sim_1)$. Correction strategies, forming the third component of the structure, are described in section 4.4; while these are generally applicable to all problem subclasses of interest, some of their elements are specific to the binary relation $(\succ_1, \sim_1)$ developed herein.

Having described the logic and structure of the methodology developed in this chapter, the next subsection presents different solution concepts for the reduced subset obtained using the final relation $(\succ, \sim)$ induced over the set O of options. The reason for presenting these concepts at this stage is two fold -- the first being that it helps place things in perspective, and thus clarify the point of the discussion, while the second is that these are general decision theoretic concepts, which are independent of our development, yet essential for its proper understanding.

4.1.2 Solution Concepts

Given a binary preference indifference relation $(\succ, \sim)$ defined over the elements of a set of n options $O = \{o_1, o_2, \ldots, o_n\}$, we are interested in a reduced subset $K \subseteq O$ of "superior" or more preferred options. The purpose of this reduction and its role in the overall decision aiding framework

were described in section 2.3. Of course, if there exists an option $o^* \in O$ such that $([o^* \succ o_i]$ or $[o^* \sim o_i], \forall o_i \in O)$, then this option should be selected and the problem is solved. However, as we said in the previous subsection, the existence of such a solution is by no means guaranteed under the general properties of $(\succ, \sim)$.

A commonly used solution concept for the reduced subset is the kernel of the relation $(\succ, \sim)$.†

Definition 4.1: A subset $K \subseteq O$ is a kernel if and only if:

(i) $o_i \in O$ and $o_i \notin K$, then $\exists o_j \in K$ such that $(o_j \succ o_i)$, and

(ii) for all $o_j, o_k \in K$, then not $[o_j \succ o_k]$. □

In other words, property (i) requires that for every option excluded from the kernel, there is at least one better option retained in the kernel; condition (ii) excludes the possibility of having elements in the kernel such that one is preferred to the other.

A less strict concept accepts to eliminate a given element when there exists at least one option that is as good retained in the reduced subset; a weak kernel is thus defined by changing condition (i) in Definition 4.1 as follows:

Definition 4.2: A subset $K \subseteq O$ is a weak kernel if and only if

(i') $o_i \in O$ and $o_i \notin K$, then $\exists o_j \in K$ such that $([o_j \succ o_i]$ or $[o_j \sim o_i])$ and condition (ii) above is true for K. □

Definition 4.3: A kernel (or weak kernel) $K \subseteq O$ is minimal if and only if

(ii') for all $o_j, o_k \in K$, then (not $[o_j \succ o_k]$ and not $[o_j \sim o_k]$) and condition (i) (or (i'), if K is a weak kernel) is true for K. □

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†The definitions given in this section are adapted from White (1976).

In other words, a minimal (weak) kernel differs from a (weak) kernel in that condition (ii') replaces condition (ii) in its definition, meaning that a minimal (weak) kernel does not contain any two elements that are either preferentially indifferent or such that one is strictly preferred to the other.

It should be noted that these concepts developed in the graph theoretic literature (Berge, 1970) the connection of course being that a graph representation can be associated with any binary relation, as described hereafter.

Graph Theoretic Representation of $(\succ, \sim)$. The binary relation $(\succ, \sim)$ defined over the set O of options can be represented by a graph consisting of a set of nodes, where each node corresponds to an element o_i of O, connected by a set of arcs defined as follows. If $o_i \succ o_j$, then a directed arc (o_i, o_j) is defined, with o_i as its origin and o_j its end node. When $o_i \sim o_j$, then two directed arcs (o_i, o_j) and (o_j, o_i) are defined. No arc is defined between o_i and o_j if they are incomparable by $(\succ, \sim)$.

The resulting graph is denoted by $G(\succ, \sim)$. The kernel K of $G(\succ, \sim)$ would thus consist of the set of nodes such that:

(i) no two nodes $o_i, o_j \in K$ are connected by an arc (o_i, o_j) without (o_j, o_i) existing as well, and

(ii) every node in $\{O - K\}$[†] is the end node of an arc the origin of which is in K, with no return arc going from that origin node back to o_ℓ.

For illustrative purposes, consider the following:

[†]$\{O - K\}$ denotes the complement in O of the subset K, i.e., $o_i \in \{O - K\} \iff o_i \in O$ and $o_i \notin K$.

Example 4.1: Let $O = \{o_1, o_2, o_3, o_4, o_5, o_6\}$, and assume a relation $(\succ, \sim)$ is defined such that $o_1 \succ o_2$; $o_1 \succ o_3$; $o_2 \sim o_4$; $o_2 \succ o_3$; $o_4 \succ o_6$ and $o_4 \succ o_3$. The graph $G(\succ, \sim)$ is shown in Figure 4.1; its kernel is $\{o_1, o_4, o_5\}$, and it is also minimal. □

However, the existence of a kernel for any graph is not guaranteed, though it is well known that an acyclic graph (i.e., with no circuits, or directed paths originating at and returning to the same node) always has one and only one kernel (Richardson, 1946; Roy, 1971). In the general case where cycles are possible in $(\succ, \sim)$, a kernel may or may not exist (Hansen et al., 1975). In the event a kernel does not exist, and assuming that the decision maker/analyst no longer wish to transform the relation $(\succ, \sim)$,† then a different solution concept may be used, namely that of a quasi-kernel, defined hereafter.

Definition 4.4: A subset $K \subseteq O$ is a quasi-kernel if and only if

(i) $o_i \in O$ and $o_i \notin K$, then $\exists o_j \in K$ such that $(o_j \succ o_i)$. □

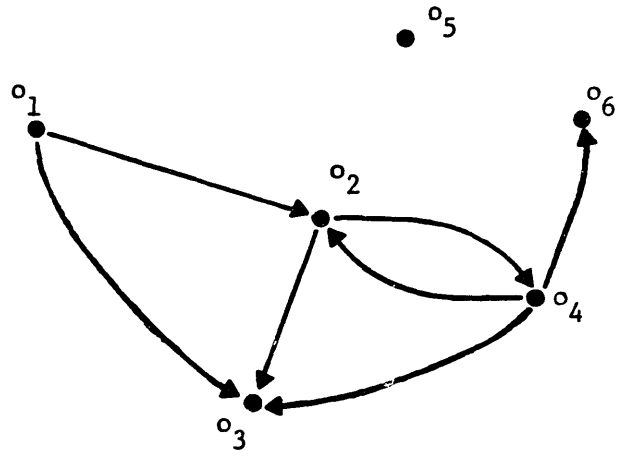
The difference between a quasi-kernel and a kernel can be seen to be the absence of condition (ii) in Definition 4.1 above. A weak quasi-kernel can be similarly defined by replacing property (i) in Definition 4.4 by property (i') in Definition 4.2 above.

While kernels might not always exist, every graph has at least one quasi-kernel (Lovasz and Chvatal, 1974), as illustrated in the simple example below.

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†See discussion in section 4.1.1 on assumptions concerning the final relation $(\succ, \sim)$ and approaches to obtain it from the initial relation $(\succ_1, \sim_1)$.

Figure 4.1: Graph Representation of $(\succ, \sim)$ in Example 4.1



Kernel here is $\{o_1, o_4, o_5\}$.

Example 4.2: Let $O = \{o_1, o_2, o_3\}$, with associated $(\succ, \sim)$ defined as follows: $o_1 \succ o_2$; $o_2 \succ o_3$ and $o_3 \succ o_1$. A quasi-kernel for this case is $K = \{o_1, o_2\}$, since for every point outside K (in this case o_3) there exists at least one element in K that is preferred to it. However, K is not a kernel because $o_1 \succ o_2$ violates property (ii) in Definition 4.1; furthermore, it is obvious that no other subset of O can be a kernel. □

In general, since a kernel is a smaller reduced subset than a quasi-kernel, it is a more desirable solution concept from the perspective of decision aiding. In early efforts by Roy (1971, 1973) in conjunction with the ELECTRE I and II techniques for multi-criteria decision problems under certainty, the motivation behind making further preferential assumptions in the interpretation of an initial general relation was the reduction of cycles in order to guarantee the existence of a kernel.† In the case of the methodology described herein, the properties of the final relation $(\succ, \sim)$ that we seek to construct over the set O of options, as described in Section 3.3., are compatible with kernels as a solution concept.

Having described how to use a final binary relation $(\succ, \sim)$ to reduce a set O of options in the context of our decision aid, we begin the construction of $(\succ, \sim)$ in the next section, which defines the initial relation $(\succ_1, \sim_1)$.

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†These assumptions are described in section 4.3.1 in conjunction with further operations on the initial relation $(\succ_1, \sim_1)$.

4.2 Specification and Properties of Initial Binary Relation

In this section, the rationale and mathematical specification of a decision rule for comparing two options in the "discrete states" problem (i.e., subclass 1) are presented. To facilitate the reader's ability to follow the discussion herein, subsection 4.2.1 presents the formulation of problem subclass 1, highlighting some of its special features. An outranking relation R, which uses a multidimensional representation of the risk implications of a binary choice situation to convey to and seek information from the decision maker, is then developed in subsection 4.2.2, and its properties discussed. Rules for using the above relation and its associated risk measures to build an initial binary relation ($\succ_1$, $\sim_1$) are specified in subsection 4.2.3. The properties of this initial relation are shown to be quite unrestrictive in subsection 4.2.4, which concludes this section.

4.2.1 Decision Problem Formulation

We merely restate problem subclass 1, first presented in section 1.4 as a very important special case where alternative options can be represented by their respective impacts (or payoffs) under discrete states of nature. Such situations, broadly referred to as "strategic planning" problems (where alternative future scenarios are identified and the performance of each of the options predicted under each scenario) were recognized in Chapter 1 as being of particular importance in transportation planning, as well as other fields such as facility location (Schilling, 1981) and others. These can be generically formulated as follows.

The problem is to select the most preferred element o^* of a given finite set of n options $O = \{o_1, o_2, \ldots, o_n\}$. The impact or payoff resulting

from the implementation of an option o_i is conditional on the realization of an element of a set of m mutually exclusive and collectively exhaustive states of nature[†] $S = \{S_1, S_2, \ldots, S_m\}$. It is assumed that the (possibly subjective) probability of occurrence P_j of state S_j is known, for all $S_j \in S$, yielding the vector of m probabilities $P = \{P_1, P_2, \ldots, P_m\}$; of course $\sum_{j=1}^{m} P_j = 1$. It is further assumed that these probabilities are independent of which option in O is implemented.

The conditional payoff from action o_i should S_j occur is denoted by x_{ij}, while the n x m matrix of x_{ij}'s is denoted by Λ. By definition, it is assumed that the totality of this conditional impact can be expressed in terms of a single numeraire (X-domain, where X is typically a subset of $\mathbb{R}$), over which the decision maker's preferences can be characterized by a complete order, as postulated in section 3.3.

In addition to all the decision models discussed in the previous chapter, such discrete states problems have been tackled, in practice, by some of the following approaches:

- use of a single aggregate desirability index, such as the expected value of the payoffs computed over the possible states of nature. This approach suffers from the well-known limitations of comparing probability distributions based on a single central tendency measure, as discussed in section 3.1 and 3.2.

- basing comparison solely on most likely scenario, i.e., using x_{ik},

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[†]The reader is reminded that a "state of nature", as used here, may indicate any suitably defined bundle of joint events.

for S_k such that $P_k = \max_{\ell | S_\ell \varepsilon S} P_\ell$, as a single figure of merit for any $o_i \varepsilon O$. Of course, such an approach is inefficient as it discards valuable information on the impact of alternative options.

- basing comparison on worst case scenario, i.e., using $\min_{k | S_k \varepsilon S} (x_{ik})$ for for each $o_i \varepsilon O$ as figure of merit, this is the well-known mini-max rule (MacCrimmon, 1968a; Foerster, 1979); the same comment as in the previous approach is applicable here as well.

- other ad hoc approaches, usually combining two or more of the above approaches, such as using an expected value criterion in combination with a worst case scenario, not unlike the mean-risk models discussed in section 3.1. However, such combination is usually made in an ad hoc manner as well and as such may become quite unmanageable or dangerously restricted in the presence of a large number of alternative options.

In this section, we describe an alternative comparison mechanism that reflects the basic considerations of Chapter 2 and the critical discussion, based on those considerations, of existing risk measures in the previous chapter.

It is important to note the special characteristic of this problem, which is that we are able to establish a conditional order over the set O under each of the possible states of nature, since the joint possible realizations of the conditional payoffs resulting from the elements of O are fully specified and assumed to be known with certainty. Note also that we cannot treat $o_i \varepsilon O$, for all i, as probabilistically independent random prospects, as the covariance pattern is essential to the nature of the problem. The unidimensional random prospect choice problem thus formulated can be viewed as a multiple criteria (MC) decision problem where the conditional payoffs under each state of nature constitute independent criteria known with certainty, with one fundamental difference between the two

classes of problems: in the MC problem, the payoff on each criterion occurs with certainty if an option is implemented, whereas here one and only one of the states will occur, and we do not know which one it is, regardless of which option is implemented.

As such, concepts developed in conjunction with the MC problem have been found useful here, in particular the notion of "outranking relation", and its underlying rationale, used in conjunction with the ELECTRE method (Roy, 1971; Jacquet-Lagreze, 1974). The next subsection develops such an outranking relation between any two options in O which recognizes the special nature of the choice situation and associated risk dimensions.

4.2.2 Definition and Properties of Outranking Relation R

This subsection presents the measures summarizing the implications of a binary comparison, the evaluation rationale or decision rule for accepting that one option outperforms another, along with the accompanying preferential information needed from the decision maker. The measures used to convey information to and elicit information from the decision maker have to be considered jointly with the decision rule. Following the discussion of the behavioral considerations in Chapter 2 in general and proposition 2.2.7 in particular,† the measures used should be intuitively meaningful and convey the implications of the choice situation of concern to the decision maker and the decision problem at hand. It was also noted there that the decision rule, in as much as it has to be communicated to and

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†The reader is reminded of the comments made in subsections 2.2.3 and 2.2.7 concerning the issue of framing of information.

understood by the decision maker, should avoid using more than two or three measures,[†] and not require further "hidden" manipulations (Fischhoff et al., 1979; Slovic et al., 1981).

Furthermore, in our problem context, where irreversibility was found (in Chapters 1 and 2) to be an important issue when taking decisions in the face of uncertainty, as well as in our discussion of risk measures in Chapter 3, regret was identified as a major concern in comparing uncertain options. However, unlike existing risk measures,we are interested in a multidimensional risk representation that summarizes the risk implications of a binary choice situation (as opposed to summarizing a single option). Therefore, a regret-based two-dimensonal risk measure is presented hereafter. One of its dimensions was introduced in section 3.2 as the risk of selecting the option which ultimately yields the less preferred outcome once the uncertainty is resolved (by the occurrence of one of the states of nature);[§] in other words, it can be viewed as the occurrence of regret. The second dimension considered is the magnitude of the potential regret (or opportunity cost) in the event of its occurrence. We express the above representation mathematically and show its use to define an outranking relation R hereafter. In doing so, we keep with the terminology of the above-mentioned ELECTRE technique from which the evaluation rationale was adapted, even though our definition and interpretation of the risk measures is entirely specific to the class of decision problems under uncertainty

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[†]As a matter of fact, as shown in Chapter 3, most existing models rely on a single measure.

[§]As discussed in section 3.2, this risk is that of violating the accepted complete order over the outcome space X, thus causing the occurence of regret.

of interest here (and not the MC problem for which ELECTRE is intended). Note that Figure 4.2 summarizes the notation used in this section.

Given two options o_i, $o_j \in O$, the set S of states of nature is partitioned into two subsets: the concordance set C_{ij}, where

$$C_{ij} = \{S_k \in S \mid x_{ik} \geq x_{jk}\} \quad ,$$

and its complement, the discordance set D_{ij}, where

$$D_{ij} = \{S_k \in S \mid x_{ik} < x_{jk}\} \quad .$$

By definition, $C_{ij} \cap D_{ij} = \{\emptyset\}$ and $C_{ij} \cup D_{ij} = S$. The concordance set C_{ij} is further partitioned into two subsets: The dominance set B_{ij}, where

$$B_{ij} = \{S_k \in C_{ij} \mid x_{ik} > x_{jk}\} \quad ,$$

and its complement in C_{ij}, the indifference set E_{ij}, where

$$E_{ij} = \{S_k \in C_{ij} \mid x_{ik} = x_{jk}\} \quad .$$

The first risk dimension c_{ij} is defined as:

$$c_{ij} = \sum_{k \mid k \in C_{ij}} P_k \qquad (4.1)$$

which is the probability of occurrence of a state where the impact of o_i is not inferior to that resulting from o_j. In other words, it is the probability that no regret occurs if o_i is selected over o_j.† Using the ELECTRE analogy, c_{ij} can be viewed as a concordance index, indicating the degree to

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†Using some of the notation of section 3.2, c_{ij} is identical to $p(x_i^* \geq x_j^*)$.

Figure 4.2: Summary of Notation for Section 4.2

S = Set of states of nature

S_k = k-th state of nature, k=1, ..., m

P_k = Probability of occurrence of state S_k

O = Set of options

o_i = i-th option, i=1, ..., n

x_{ij} = impact of o_i should S_j occur

C_{ij} = concordance set; $C_{ij} = \{S_k \in S \mid x_{ik} \geq x_{jk}\}$

D_{ij} = discordance set; $D_{ij} = \{S_k \in S \mid x_{ik} < x_{jk}\}$

B_{ij} = dominance set; $B_{ij} = \{S_k \in S \mid x_{ik} > x_{jk}\}$

E_{ij} = indifference set; $E_{ij} = \{S_k \in S \mid x_{ik} = x_{jk}\}$

c_{ij} = concordance index

d_{ij} = discordance index

$\bar{c}$ = critical concordance level

$\bar{d}$ = critical discordance level

which o_i dominates or outperforms option o_j. The range of values taken by c_{ij} is the interval [0,1] with $c_{ij} = 0$ (implying that $C_{ij} = \{\emptyset\}$) indicating complete dominance of o_j over o_i; while $c_{ij} = 1$, implying that $C_{ij} = S$, indicates the reverse.† Values of c_{ij} between these two extremes indicate the existence of at least one non-null element in each of C_{ij} and D_{ij} respectively. While C_{ij} depends exclusively on the number of states where o_i outperforms o_j and their associated probability of occurrence, the second risk dimension of selecting o_i over o_j consists of the magnitude of the worst case loss (vis a vis o_j); a "discordance" index d_{ij} is thus defined as follows:

$$d_{ij} = \max_{k|S_k \epsilon D_{ij}} \left[\frac{(x_{jk} - x_{ik})}{(d^{max})} \right] \tag{4.2}$$

where d^{max} is the maximum difference in payoff between any two options for a given state of nature, i.e.,

$$d^{max} = \max_{i,j|o_i,o_j \epsilon O} \left[\max_{k|S_k \epsilon S} |x_{jk} - x_{ik}| \right] \tag{4.3}$$

The term d^{max} is used here as a normalizing constant which constrains d_{ij} to be in the interval [0,1].§ This discordance index d_{ij} can be interpreted as the maximum regret or loss that could result from the implementation of option o_i instead of o_j. Note that alternative forms for d_{ij} are discussed later in Chapter 5.

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†It is implicitly assumed here and throughout this thesis that the trivial case where two options have identical payoffs for every state, i.e. where $E_{ij} = S$, does not arise.

§Note that in this case, the discordance index is different from the ELECTRE index as the latter is meant for evaluation in the presence of non-commensurable attributes or cirteria, thus requiring the use of criteria-specific normalizing parameters.

We now define the outranking relation R over the set of options O. We say that $[o_k \; R \; o_j]$ if $c_{ij} \geq \bar{c}$ and $d_{ij} \leq \bar{d}$, where $\bar{c}$ and $\bar{d}$ are two threshold values, set in a "satisficing" manner, expressing the preferences of the decision maker for the specific problem at hand. In other words, in comparing o_i and o_j, we accept that o_i outperforms o_j if the probability of receiving a higher payoff from o_i exceeds a minimum acceptable level, while at the same time the maximum potential regret or loss (vis a vis o_j) is less than a tolerable level. Of course, one would want $\bar{c}$ to be as high as possible (max = 1) and $\bar{d}$ as low as possible (min = 0), reflecting a high degree of dominance (or low risk of regret) accompanied by low level of potential regret.

The indices c_{ij} and d_{ij} thus summarize the impact information along two risk dimensions, providing complementary information which forms the basis of an intuitively appealing heuristic decision rule, which can be expressed as follows: "option o_i performs better than o_j on a number of states ($c_{ij} \geq \bar{c}$, which means that it is more likely than o_j to be a candidate for ultimate selection), so let us look at where it is not doing so well; consider the worst performance (of o_i relative to o_j) under any of the possible states (i.e., $d_{ij} \leq \bar{d}$), then o_i can be safely considered to outperform o_j". Acceptance of the above logic is a prerequisite for the usefulness of this decision aiding approach.

When assessing whether or not o_i outranks o_j, for all o_i, $o_j \in O$, one and only one of the two following situations can arise:

1. $(c_{ij} < \bar{c}$ or $d_{ij} > \bar{d}) \Rightarrow$ not $(o_i \; R \; o_j)$, in other words, we can only state that o_i does not outrank o_j, or

2. $(c_{ij} \geq \bar{c}$ and $d_{ij} \leq \bar{d}) \Rightarrow o_i \; R \; o_j$.

The properties of R, as defined here are:

- reflexivity, since $c_{ii} = 1$ and $d_{ii} = 0$ guarantee that $[o_i \; R \; o_i]$ since 1 and 0 are the upper and lower bounds respectively on $\bar{c}$ and $\bar{d}$.
- R is not symmetric (since $[o_i \; R \; o_j]$ does not necessarily imply $[o_j \; R \; o_i]$), not antisymmetric nor asymmetric (as it is possible for both $[o_i \; R \; o_j]$ and $[o_j \; R \; o_i]$ to be true without necessarily having $o_i = o_j$).
- R is not necessarily transitive, because of the presence of threshold levels. R is therefore not an "order" relation in the strict sense of the term.
- R is obviously not complete, as it is possible to have both not $[o_i \; R \; o_j]$ and not $[o_j \; R \; o_i]$.

As such, R serves its purpose of establishing pairwise preferences between options outside of a priori restrictive assumptions as to the resulting relation on O, consistently with the behavioral considerations of Chapter 2.† However, further assumptions are needed in order to proceed from the pairwise assessments $[o_i \; R \; o_j]$ or not $[o_i \; R \; o_j]$, for all $o_i, o_j \in O$, to the selection of a preferred option o*, or at least to the identification of a reduced subset $K \subseteq O$ which contains o*. We specify in the next subsection rules for using R and its associated risk dimensions to build the

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†Note also, that in terms of requirements from the decision maker, satisficing levels are usually the easiest type of preference information to supply.

the initial binary relation ($\succ_1$, $\sim_1$) over O, then discuss the properties of the latter relation, in particular its consistence with the properties of partial semi-orders.

4.2.3 Rules to Establish ($\succ_1$, $\sim_1$) on a Pairwise Basis

For any given pair o_i, $o_j \in O$, $i \neq j$, we can distinguish one of the following cases:

1. $([o_k \; R \; o_j]$ and not $[o_j \; R \; o_k]) \Longrightarrow [o_k \succ_1 o_j]$
2. (not $[o_i \; R \; o_j]$ and $[o_j \; R \; o_i] \Longrightarrow [o_j \succ_1 o_i]$
3. $([o_i \; R \; o_j]$ and $[o_j \; R \; o_i])$, which can be further refined into a number of subcases, depending on the relative magnitudes of c_{ij}, c_{ji}, d_{ij} and d_{ji}. Note that we have in this case that $c_{ij} \geq \bar{c}$ and $d_{ij} \leq \bar{d}$, as well as $c_{ji} \geq \bar{c}$ and $d_{ji} \leq \bar{d}$. The subcases identified here are:

   3.1. $c_{ij} = c_{ji}$, and

   3.1.a. $d_{ij} < d_{ji}$, in which case $[o_i \succ_1 o_j]$, since for the same level of concordance, o_i offers a lower maximum potential regret than o_j.

   3.1.b. $d_{ji} < d_{ij}$, in which case $[o_j \succ_1 o_i]$

   3.1.c. $d_{ji} = d_{ij}$, which we define as implying $[o_i \sim_1 o_j]$, i.e., o_i and o_j are preferentially indifferent.

   3.2 $c_{ij} > c_{ji}$, and

   3.2.a. $d_{ij} < d_{ji}$, implying $[o_i \succ_1 o_j]$ since o_i offers a higher degree of dominance (and a guarantee that $B_{ij} \neq \{\emptyset\}$) as well as a lower maximum potential loss with respect to o_j.

3.2.b. $d_{ji} < d_{ij}$; in this case $B_{ij} \neq \{\emptyset\}$, and o_i has a higher probability of dominance over o_j, whereas o_j offers a lower maximum potential regret with respect to o_i. We view this situation as reflecting preferential indifference, i.e., $[o_i \sim_1 o_j]$, given that the outranking relation R holds in both directions.

3.2.c. $d_{ij} = d_{ji}$, implying that $[o_i \succ_1 o_j]$.

3.3. $c_{ij} < c_{ji}$, and

3.3.a. $d_{ij} < d_{ji}$, which is similar to situation 3.2.b. for which $[o_i \sim_1 o_j]$ was concluded.

3.3.b. $d_{ij} > d_{ji}$, implying that $[o_j \succ_1 o_i]$ in an analogous manner to 3.1.a.

3.3.c. $d_{ij} = d_{ji}$, implying that $[o_j \succ_1 o_i]$ (analogous to 3.2.c).

4. (not $[o_i R o_j]$ and not $[o_j R o_j]$) $\Rightarrow [o_i I o_j]$, where I denotes the incomparability relation, indicating that no preferential assertion can be made concerning o_i and o_j at that point in the decision process.

Figure 4.3 provides a summary of the situations arising in the comparison of two options o_i and $o_j \in O$.

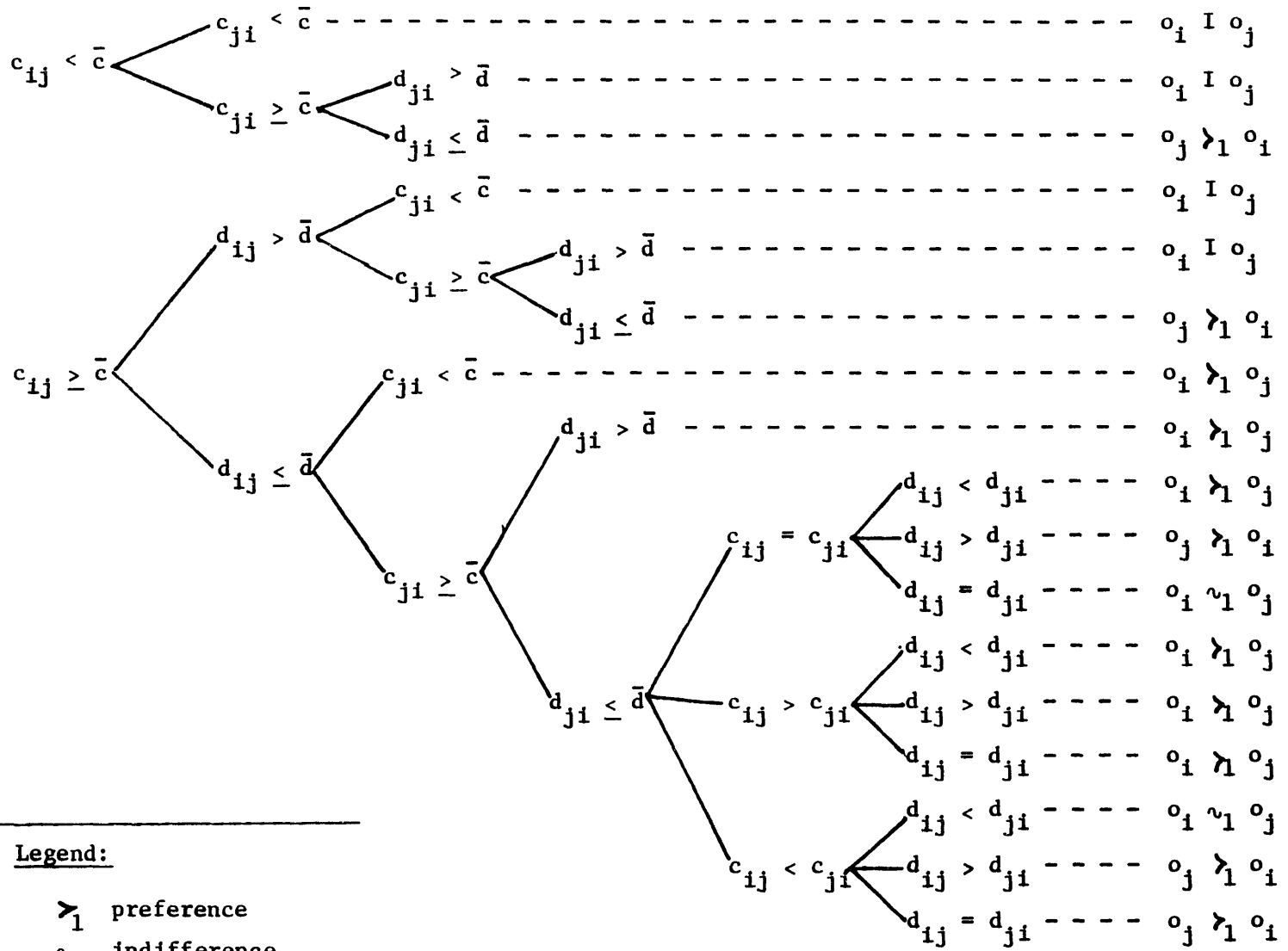


Figure 4.3: Summary of Preferential Situations When Comparing o_i and o_j

Having specified the mechanism for a "first-pass" comparison of any two options in O, we investigate in the next subsection the properties of the resulting initial binary relation $(\succ_1, \sim_1)$ over O.

4.2.4 General Properties of Initial Binary Relation $(\succ_1, \sim_1)$

Investigation of the properties of $(\succ_1, \sim_1)$ in this subsection is primarily concerned with consistency of such a relation, in the general case (i.e., in any problem context), with the basic considerations of Chapter 2 and resulting starting postulates. Completeness, reflexivity, symmetry, transitivity as well as properties P3 and P4 (of partial semi-orders, described in section 3.3) are discussed hereafter.†

Completeness. By definition, $(\succ_1, \sim_1)$ is not necessarily complete as it allows for incomparability, consistently with proposition 2.2.8. It can thus be seen in Figure 4.3 that, $\forall o_i, o_j \in O$, only one of the following holds: $[o_i \succ_1 o_j]$, $[o_j \succ_1 o_i]$, $[o_i \sim_1 o_j]$ or $[o_i \; I \; o_j]$.

Reflexivity and Symmetry. Since $[o_i \succ_1 o_i]$ cannot be true, $\forall o_i \in O$, then $\succ_1$ is irreflexive; $\succ_1$ is also asymmetric, since $[o_i \succ_1 o_j] \Rightarrow \underline{\text{not}}\, [o_j \succ_1 o_i]$. Both reflexivity and asymmetry are usual properties of preference relations. Figure 4.3 further reveals that the indifference relation $\sim_1$ is both reflexive and symmetric.

Transitivity. It is relatively easy to produce counter examples to prove that neither $\succ_1$ nor $\sim_1$ are transitive, as the following example illustrates for $\succ_1$.

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†The reader is reminded that most of the properties and concepts used in this section are summarized in Appendix B.

Example 4.3: Consider a situation with 5 equiprobable states of nature (Thus $P_j = 0.2$, $j = 1, \ldots, 5$), and where $O = \{o_1, o_2, o_3\}$ has the following impact matrix Λ:

$$\Lambda = \begin{bmatrix} 100 & 150 & 175 & 125 & 200 \\ 130 & 110 & 160 & 125 & 150 \\ 170 & 70 & 150 & 100 & 150 \end{bmatrix}$$

Using Equations (4.1) and (4.2), we compute:

$$c_{12} = 0.8 \qquad d_{12} = \frac{30}{80} = 0.375$$

$$c_{23} = 0.8 \qquad d_{23} = \frac{40}{80} = 0.5$$

$$c_{13} = 0.8 \qquad d_{13} = \frac{70}{80} = 0.875$$

If $\bar{c} = 0.7$ and $\bar{d} = 0.6$, then $o_1 \succ_1 o_2$ and $o_2 \succ_1 o_3$. However, even though $c_{13} = 0.8 > 0.7$, we have $d_{12} = 0.875 > \bar{d}$, indicating that possible loss should S_1 occur might be sufficiently significant to where we cannot affirm (at least prior to further assessment) that $[o_1 \succ_1 o_3]$. The rules of Fig. 4.3 here yield $[o_1 \; I \; o_3]$, in a typical example of operational or tactical incomparability (see section 2.2). □

While the non-transitivity of the indifference relation is consistent with proposition 2.2.8 and the postulates of section 3.3, the non-transitivity of the preference relation $\succ_1$ is a more serious concern. Because transitivity is a key property in further developing the use of the preference relation $\succ_1$ for decision aiding, it is explored more carefully hereafter. In particular, we address the question of whether there are conditions on the values of the thresholds $\bar{c}$ and $\bar{d}$ that would ensure that the

resulting preference relation $\succ_1$ is transitive in the general case. In addition, the definition of the transitive closure of a binary relation is presented; and violations of the transtive closure of $\succ_1$ are discussed.

Definition 4.5: The transitive closure of a binary relation $\succ$ over a set O, denoted by $\succ^t$, is defined as follows:

$[o_i \succ^t o_j]$ if and only if $[o_i \succ o_j]$ or there are $o_1, o_2, \ldots, o_m \in O$ such that $o_i \succ o_1$ and $o_1 \succ o_2$ and ... and $o_m \succ o_j$ (Fishburn, 1973a). □

Let us see what building the transitive closure of a relation implies. Consider three options $o_i, o_j, o_k \in O$, with $[o_i \succ_1 o_j]$ and $[o_j \succ_1 o_k]$. Intransitivity can arise in one of three ways:

1. $[o_k \; I \; o_i]$, as in Example 4.3 above;
2. $[o_k \succ_1 o_i]$; or
3. $[o_k \sim_1 o_i]$.

While the first of these (i.e., $[o_i \; I \; o_k]$) indicates the absence of sufficient basis for asserting a more positive association or relation between o_i and o_k, it does not go as far as saying that there exists sufficient basis for establishing the reverse of what transitivity would have implied. As such, either of $[o_k \succ_1 o_i]$ or $[o_k \sim_1 o_i]$ constitutes a more serious obstacle in the way of building the transitive closure of $\succ_1$.†

For the same reason, preference cycles, defined hereafter, when present in $\succ_1$, constitute an obstacle to building the transitive closure of $\succ_1$.

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†The transitive closure relation thus obtained would not be asymmetric nor irreflexive, and would thus not be a preference relation.

Definition 4.6: A preference cycle connecting any two options o_i, $o_\ell \in O$ is a finite sequence of elements $\{o_1, o_2, \ldots, o_p\}$, such that $o_1 = o_i$ and $o_p = o_i$ with $o_\ell = o_k$ for some $1 < k < p$ and $[o_j \succ_1 o_{j+1}]$ for all $1 \leq j < p$ (Fishburn, 1973a). □

Note that in the next section the cycle defined above is considered as a special type of more general cycles that occur in the graph of $(\succ_1, \sim_1)$.

Having identified the different forms of intransitivities that may occur in $(\succ_1, \sim_1)$, an interesting question concerns whether conditions on the values of $\bar{c}$ and $\bar{d}$ can be found such as to prevent such intransitivites from occurring, regardless of the specific options under consideration. As it turns out, no such general conditions exist (except for the trivial case where $\bar{c} = 1.0$ which reduces to the definition of a complete dominance relation). The occurrence of intransitivities may be controlled or presented with appropriate values of $\bar{c}$ and $\bar{d}$ only within specific problem situations. We have however, been able to establish some interesting properties of transitive behavior when the concordance index c_{ij} is defined only over those states where strict dominance exists (i.e., over the dominance set B_{ij}), as seen in Chapter 5.

Since $\succ_1$ is not necessarily transitive, the "first pass" relation $(\succ_1, \sim_1)$ is not in general compatible with property P3 of partial semi-orders (described in section 3.3), as discussed hereafter, along with compatibility with property P4.

Property P3. As defined in section 3.3., P3 says that $\{[o_i \succ_1 o_j]$ and $[o_j \sim_1 o_k]$ and $[o_k \succ_1 o_\ell]\} \Rightarrow [o_i \succ_1 o_\ell]$, for all o_i, o_j, o_k, $o_\ell \in O$. Of course, transivity of $\succ_1$ is implied by this property, since, by the reflexivity of $\sim_1$, $\{[o_i \succ_1 o_j]$ and $[o_j \succ_1 o_k]\}$ is equivalent to

$\{[o_i \succ_1 o_j]$ and $[o_j \sim_1 o_j]$ and $[o_j \succ_1 o_k]\}$, which, by P3, implies that $[o_i \succ_1 o_k]$. Therefore violations of transitivity of $\succ_1$ are violations of P3 for $(\succ_1, \sim_1)$.

Property P4. Also stated in section 3.3, P4 says that if $\{[o_i \succ_1 o_j]$ and $[o_j \succ_1 o_k]$ and $[o_j \sim_1 o_\ell]\}$, then $[o_i \sim_1 o_\ell]$ and $[o_k \sim_1 o_\ell]$ are not both true, for all $o_i, o_j, o_k, o_\ell \in O$. This property does not always hold for $(\succ_1, \sim_1)$ as shown in the following counter example.

Example 4.4: Consider a situation with 6 equiprobable states of nature (then $P_j = \frac{1}{6}$, $j = 1, \ldots, 6$), and where $O = \{o_1, o_2, o_3, o_4\}$ has the following impact matrix Λ_1:

$$\Lambda_1 = \begin{bmatrix} 105 & 80 & 80 & 100 & 100 & 140 \\ 100 & 100 & 100 & 100 & 100 & 100 \\ 100 & 120 & 60 & 95 & 100 & 120 \\ 120 & 100 & 80 & 80 & 100 & 120 \end{bmatrix}$$

Using Equations (4.1) and (4.2), we compute:

$$\begin{aligned} c_{12} &= \tfrac{2}{3} & d_{12} &= \tfrac{20}{40} = 0.5 \\ c_{21} &= \tfrac{2}{3} & d_{21} &= 1.0 \\ c_{23} &= \tfrac{2}{3} & d_{23} &= 0.5 \\ c_{32} &= \tfrac{2}{3} & d_{32} &= 1.0 \\ c_{24} &= \tfrac{2}{3} & d_{24} &= 0.5 \\ c_{42} &= \tfrac{2}{3} & d_{42} &= 0.5 \\ c_{14} &= \tfrac{2}{3} & d_{14} &= 0.5 \\ c_{41} &= \tfrac{2}{3} & d_{41} &= 0.5 \\ c_{34} &= \tfrac{2}{3} & d_{34} &= 0.5 \\ c_{43} &= \tfrac{2}{3} & d_{43} &= 0.5 \end{aligned}$$

If we set $\bar{c} = \frac{2}{3}$ and $\bar{d} = 0.55$, then $o_1 \succ_1 o_2$, $o_2 \succ_1 o_3$, $o_2 \sim_1 o_4$, yet both $[o_1 \sim_1 o_4]$ and $[o_3 \sim_1 o_4]$ are true, using the rules of Figure 4.3. Again, the principal causes for the occurrence of such cases is the "veto power" of the discordance threshold, even though options o_1, o_2 and o_3 are otherwise relatively "close" to each other. □

It is thus apparent that $(\succ_1, \sim_1)$ is quite unrestrictive in its properties, and does not subject the outcome of the binary comparisons using the intuitively meaningful risk measures, or notions of concordance and discordance, to a priori constraints on the resulting preferences in option-domain. In that, it is consistent with the behavioral considerations of Chapter 2. However, as explained in section 4.1, there is no reason to claim $(\succ_1, \sim_1)$ as an inflexible representation of the decision-maker's preferences. Rather, a case can be made for operating on $(\succ_1, \sim_1)$ and affecting adjustments towards whatever tenets the decision maker and his analyst can accept as essential to their sense of rationality. Three alternative assumptions were discussed in section 4.1, each leading to a different set of methodological steps for building the final relation $(\succ, \sim)$ given $(\succ_1, \sim_1)$. Of particular interest is the third one, proposing consistency of $(\succ, \sim)$ with the postulates of section 3.3, namely the properties of partial semi-orders.

In what follows we develop possible operations by which adjustment can be affected in $(\succ_1, \sim_1)$ towards obtaining a final $(\succ, \sim)$ in conformance with the properties of partial semi-orders. The next section first describes rules for detecting inconsistencies with those propoerties in a given $(\succ_1, \sim_1)$, while section 4.4 describes possible correction strategies.

4.3 Inconsistencies in Initial Relation $(\succ_1, \sim_1)$

This section addresses the issue of proceeding beyond the initial relation $(\succ_1, \sim_1)$ in view of obtaining over the set O a relation $(\succ, \sim)$ which is subsequently reduced to a subset K containing the most preferred alternative, as explained in section 4.1. In particular, three possible assumptions on the resulting preferences were mentioned in subsection 4.2.1:

1. accept $(\succ_1, \sim_1)$ as is and equate it to $(\succ, \sim)$; should a kernel prove not to exist, then one could settle for a quasi-kernel (Hansen et al., 1975).

2. affect further assumptions of $(\succ_1, \sim_1)$ as in the ELECTRE technique.

3. perform operations to make the final $(\succ, \sim)$ conform to the starting postulates of section 3.3.

While all three are "viable" possibilities that the analyst could follow once the initial $(\succ_1, \sim_1)$ is obtained, our primary concern here is with the third of these assumptions, thus the need to identify violations with partial semi-orders in a particular $(\succ_1, \sim_1)$.

In this section, the approach corresponding to the second assumption above is first described in subsection 4.3.1 along with its preferential implications so as to provide a "base of reference" for our approach. The latter begins in subsection 4.3.2 where we introduce the notion of "P3-closure", in a manner parallel to that of the "transitive closure" of a binary relation, as a useful construct to analyze the consistency of a binary relation with the properties of partial semi-orders. We then identify the types of violations of those properties and define rules for their detection in subsection 4.3.3.

4.3.1 Assumptions for Reduction of $(\succ_1, \sim_1)$

This subsection describes the reduction assumptions of the ELECTRE technique (Roy, 1973) and discusses its preferential implications.

As mentioned in section 4.1, the primary motivation for those assumptions was to eliminate cycles in the graph of the binary preference-indifference relation so as to guarantee the existence of a kernel in the final $(\succ, \sim)$. This is done by assuming cycles to indicate preferential indifference, thus defining equivalence classes as follows:

1. Two elements $o_i, o_j \in O$ are defined as $[o_i \sim o_j]$ if and only if there exists in the graph of $(\succ_1, \sim_1)$ a cycle connecting o_i and o_j.† The resulting binary indifference relation $\sim$ is then an equivalence relation (reflexive, symmetric and transitive), which defines a partition on the set O into mutually exclusive and collectively exhaustive equivalence classes $Q = \{Q_1, Q_2, \ldots, Q_p, \ldots, Q_{n_c}\}$, such that:

$$\{[o_i \in Q_p] \text{ and } [o_j \in Q_p]\} \text{ if and only if } [o_i \sim o_j], \forall o_i, o_j \in O \ .$$

Of course, some elements of Q can be singletons. Effectively then, these assumptions render any individual element in each of the clasess $Q_i \in Q$, representative (preferentially) of all the elements in the same class.

2. The preference relation $\succ$ is then defined over Q, such that:

$$[Q_i \succ Q_j] \text{ if and only if } \{\exists o_\ell \in Q_i \text{ and } \exists o_k \in Q_j \mid o_\ell \succ_1 o_k\} \ .$$

In other words, since each equivalence class is represented by any of its elements, the existence of "initial" preference (via $\succ_1$) between an element

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†Note in this regard that $o_i \sim_1 o_j$ would effectively define a cycle since arcs (o_i, o_j) and (o_j, o_i) exist in $G(\succ_1, \sim_1)$. Thus the transitive closure of $\sim_1$ is a subset of $\sim$.

of any equivalence class $Q_i \in Q$ and another element of another equivalence class $Q_j \neq Q_i$ is taken to imply the "final" preference (via $\succ$) of Q_i over Q_j. Note that if $G(\succ_1, \sim_1)$ has no cycles (including no two distinct elements for which $\sim_1$ is true), then $\succ$ is identical to $\succ_1$.

The graph associated with the above defined $\succ$ (with $Q_1, Q_2, \ldots, Q_{n_c} \in Q$ as its nodes) clearly contains no cycles, as it is a strict partial order.

The preferential implications of this approach are quite strong. Violations of transitivity of the preference relation $\succ_1$ are construed as indicating indifference, and the transitivity of the preferential indifference relation is imposed. These assumptions may be interpreting too freely the preferential information elicited from the decision maker (in the form of $[\succ_1, \sim_1]$), thus running the risk of misrepresenting these preferences while trying to have them conform to a priori restrictive properties.†

The alternative approach we propose tolerates the presence of some types of cycles resulting from the non-transitivity of the indifference relation, while preventing violations of the normatively desirable transitivity of the preference relation, as stipulated in our starting postulates in section 3.3. In the next subsections, we seek to identify possible inconsistencies with those basic postulates in the graph of a given initial relation $(\succ_1, \sim_1)$.

4.3.2 Violations of Basic Postulates

In the investigation of the general properties of the initial relation $(\succ_1, \sim_1)$ defined in section 4.2, two important properties of partial semi-

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†The reader is referred to Chapter 2, and in particular, subsections 2.2.3 and 2.2.8, where concerns about such restrictive assumptions are expressed.

orders that could be violated by a given $(\succ_1, \sim_1)$[†] were identified; these are the two properties P3 and P4, defined in section 3.3. In order to identify how such violations might occur, we first introduce the notion of P3-closure, in an analogous manner to the transitive closure of a binary relation (in subsection 4.2.4).

Definition 4.7: The P3-closure of a binary relation $(\succ_1, \sim_1)$ over a set O, denoted by $(\succ_1^P, \sim_1^P)$, is defined as follows:

- $[o_i \sim_1^P o_j]$ if and only if $[o_i \sim_1 o_j]$

- $[o_i \succ_1^P o_j]$ if and only if

$\{[o_i \succ_1 o_j]\}$

or $\{\exists o_\ell \in O \mid ([o_i \succ_1 o_\ell] \text{ or } [o_i \succ_1^P o_\ell]) \text{ and } ([o_\ell \succ_1 o_j] \text{ or } [o_\ell \succ_1^P o_j])\}$

or $\{\exists o_\ell, o_k \in O \mid ([o_i \succ_1 o_\ell] \text{ or } [o_i \succ_1^P o_\ell]) \text{ and } ([o_\ell \sim_1 o_k]) \text{ and } ([o_k \succ_1 o_j] \text{ or }]o_k \succ_1^P o_j])\}$.□

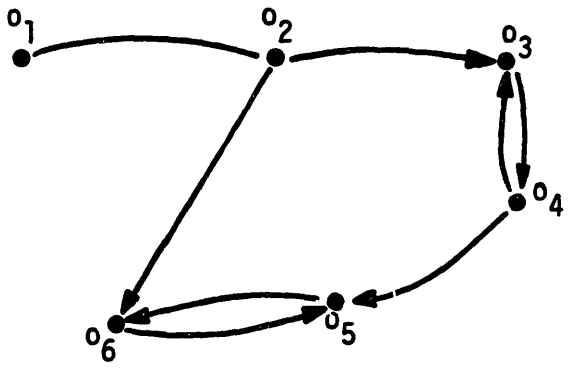
Example 4.5: Consider a relation $(\succ_1, \sim_1)$ defined over the set $O = \{o_1, o_2, o_3, o_4, o_5, o_6\}$ and shown in Figure 4.4.a. Figure 4.4.b illustrates its P3-closure $(\succ_1^P, \sim_1^P)$.□

By definition, property P3 holds for the P3-closure of a given $(\succ_1, \sim_1)$ much in the same way that the transitive closure of a binary relation is transitive. However, it may violate one or more of the other properties of

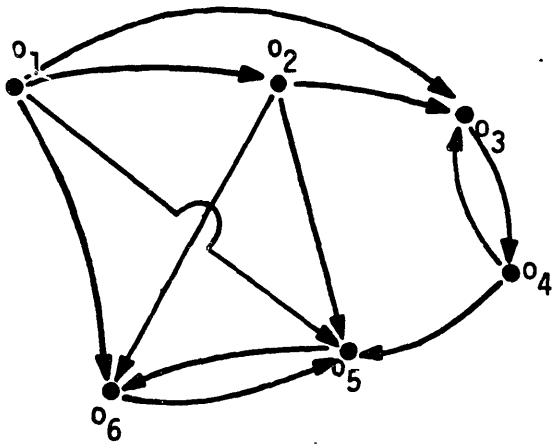
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[†]Note that a given $(\succ_1, \sim_1)$ is the result of a particular problem configuration (i.e., the set O) and the threshold levels supplied by the decision maker. As such it may or may not violate properties P3 and/or P4.

Figure 4.4: Example of P3-Closure



a) $(\succ_1, \sim_1)$



b) P3-closure of $(\succ_1, \sim_1)$

partial semi-orders, again in a similar manner to the obstacles identified is subsection 4.2.4 in conjunction with the transitive closure of $\succ_1$. In particular, obstacles might arise in the face of the P3-closure of $(\succ_1, \sim_1)$ due either to the existence of "opposite" preference or indifference where P3-closure would have indicated preference in a particular direction. Formally, if $[o_1 \succ_1 o_2]$ and $[o_2 \sim_1 o_3]$ and $[o_3 \succ_1 o_4]$, then $[o_1 \succ_1^P o_4]$; however, if either $[o_4 \succ_1 o_1]$ or $[o_4 \sim_1 o_1]$ is true, then $[o_4 \sim_1^P o_1]$ or $[o_4 \succ_1^P o_1]$, both being clearly inconsistent with $[o_1 \succ_1^P o_4]$ if $\succ_1^P$ is to be an asymmetric preference relation (as we postulated it to be). Furthermore, $(\succ_1^P, \sim_1^P)$ might violate property P4.

Note that incomparability of two elements (by $[\succ_1, \sim_1]$) that should be connected by $\succ_1^P$ is not considered an obstacle in the face of the P3-closure of $(\succ_1, \sim_1)$. As such, we consider a partial binary relation $(\succ, \sim)$ to be consistent with our basic postulates if its P3-closure $(\succ^P, \sim^P)$ does not violate properties P1, P2 and P4 (defined in section 3.3)[†] As a corollary, given an initial relation $(\succ_1, \sim_1)$, we seek to identify obstacles to its eventual P3-closure that lead to violations of those properties. All those obstacles appear in the form of directed cycles or circuits in the graph of $(\succ_1, \sim_1)$. The next subsection identifies special cases of such cycles, and determines whether or not they are inconsistent with partial semi-orders, developing a general rule to that effect.

4.3.3. Cycles and Inconsistencies with Partial Semi-orders

Given a partial binary relation $(\succ_1, \sim_1)$ defined over the set O of

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[†]As we said, by definition, $(\succ^P, \sim^P)$ conforms to P3.

options, we seek to identify those cycles, in the assciated graph $G(\succ_1, \sim_1)$,† which would produce violations of the properties P1, P2 or P4 by the relation $(\succ_1^P, \sim_1^P)$, the P3-closure of $(\succ_1, \sim_1)$. As seen hereafter, not all cycles in $(\succ_1, \sim_1)$ are inconsistent with our basic postulates. We identify hereafter four (collectively exhaustive) distinct types of cycles that might be encountered in $G(\succ_1, \sim_1)$. The first two of these are always obstacles in the face of P3-closure, while the third and fourth types are each respectively subdivided into one subtype that is (consistent) and another subtype that is not consistent with partial semi-orders. We also present in conjunction with the fourth type a general systematic rule for distinguishing between cycles that constitute violations and those that do not, based on the number and nature of the preferential connections along those cycles. As such, all different types distinguished hereafter are but special cases of that general rule, with the primary reason for their definition being clarity of exposition as well as the need to give a stronger intuitive feeling as to the implications of the properties of partial semi-orders.

Type 1: cycles with no difference relations, i.e., no two elements along the cycle are preferentially indifferent, as defined hereafter.

Definition 4.8: A directed cycle $C = \{o_1, o_2, \ldots, o_1\}$ in $G(\succ_1, \sim_1)$ is a type 1 cycle if and only if

$$\{o_i, o_j \in C \text{ and } [o_i \succ_1 o_j]\} \Rightarrow \{\text{not } [o_j \succ_1 o_i] \text{ and not } [o_j \sim_1 o_k]\} \cdot \square$$

---

†The reader is reminded that the graph theoretic representation of $(\succ_1, \sim_1)$ was described in subsection 4.1.2.

Figure 4.5 illustrates a type 1 cycle. These cycles are straightforward , and are clear obstacles in the face of the P3-closure and its implied transitivity.

Type 2: cycles with non-consecutive indifference relations.

Definition 4.9: A directed cycle, $C = \{o_1, o_2, \ldots, o_\ell\}$, with $o_\ell = o_1$, in $G(\succ_1, \sim_1)$ is a type 2 cycle if and only if, for $o_{i-1}, o_i, o_{i+1}, o_{i+2} \in C$,

$$\{[o_{i-1} \succ_1 o_i] \text{ and } [o_i \sim_1 o_{i+1}]\} \Rightarrow [o_{i+1} \succ_1 o_{i+2}] \quad \cdot\square$$

Figure 4.6.a shows a type 2 cycle in its simplest form, with $[o_1 \succ_1 o_2]$, $[o_2 \sim_1 o_3]$ and $[o_3 \succ_1 o_1]$. It is evident that such a configuration implies by P3, that $[o_1 \succ_1^P o_1]$, which is of course a violation of the irreflexivity of $\succ_1^P$.

Figure 4.6.b. shows a relatively more intricate example of a type 2 cycle. Its P3-closure would lead to a multitude of violations, including simple type 2 cycles in $G(\succ_1^P, \sim_1^P)$, such as $\{o_1, o_4, o_5\}$.

Type 2 cycles are always inconsistent with the properties of partial semi-orders, and constitute obstacles in the face of P3-closure.

Type 3: Cycles with two (but not more) successive indifference relations; this type of cycle is quite intricate to deal with, as the presence of only two successive indifference relations does not in all cases lead to violations of the properties of partial semi-orders. We thus identify, hereafter a number of subtypes, differentiated by the number of nodes and presence and distribution of other indifference pairs in the cycle:

Type 3.a: 3-node cycles with only two consecutive indifference relations.

Definition 4.10: A directed cycle $C = \{o_1, o_2, o_3, o_1\}$ in $G(\succ_1, \sim_1)$ is a type 3.a cycle if and only if

$$[o_i \succ_1 o_j] \Rightarrow \{[o_i \sim_1 o_k] \text{ and } [o_j \sim_1 o_k]\}, \text{ for } i,j,k = 1.2,3 \text{ and } i \neq j \neq k \cdot \square$$

Figure 4.7 exhibits a type 3.a cycle. Such cycles do not violate the properties of partial semi-orders and are invariant with respect to P3-closure.

Type 3.b: Cycle with more than 3 nodes and only two consecutive indifference relations.

Defintion 4.11: A directed cycle $C = \{o_1, o_2, o_3, \ldots, o_\ell\}$ with $o_\ell = o_1$ in $G(\succ_1, \sim_1)$ is a type 3.b cycle if and only if, for $o_{i-1}, o_i, o_{i+1}, o_{i+2} \in C$,

$$\{[o_{i-1} \succ_1 o_i] \text{ and } [o_i \sim_1 o_{i+1}] \text{ and } [o_{i+1} \sim_1 o_{i+2}]\} \Rightarrow \{\text{not } [o_{i+2} \sim_1 o_{i+3}]$$

$$\text{and } (\nexists o_j, o_{j+1} \in C \text{ and } j \neq i \neq i+1 \mid [o_j \sim_1 o_{j+1}])\}. \square$$

Figure 4.8.a shows a type 3.b cycle in its basic form. This type of cycle is not consistent with the properties of partial semi-orders, both in $G(\succ_1, \sim_1)$ and its P3-closure $G(\succ_1^P, \sim_1^P)$. This violation is however not self evident and requires further explanation. Consider the subgraph of $G(\succ_1, \sim_1)$ shown in 4.8.a, where o_1 and o_3 are incomparable by $(\succ_1, \sim_1)$ as well as by $(\succ_1^P, \sim_1^P)$. Noting that the use of incomparability in our approach to decision aiding is primarily operational and "strategic" (as discussed in subsection 2.2.8), the incomparability between o_1 and o_3 would have to be resolved ultimately. However, an inconsistency is obtained regardless of the preferential relation between o_1 and o_3; this assertion is proved hereafter.

Figure 4.5: Type 1 Cycle

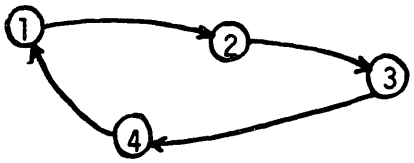
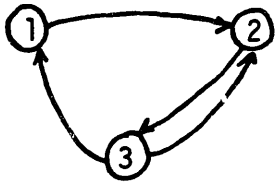
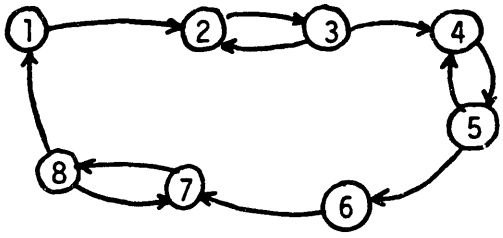


Figure 4.6: Type 2 cycle



a - Simplest Form



b - More Complicated Form

Figure 4.7: Type 3.a cycle

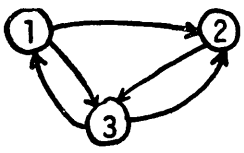
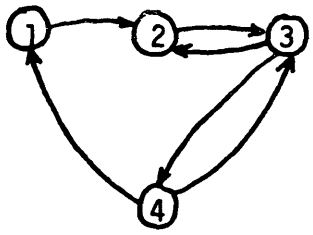
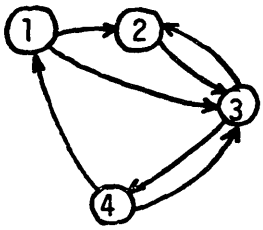


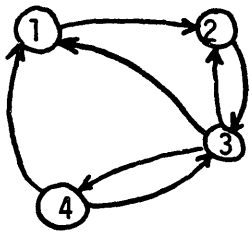
Figure 4.8: Type 3.b cycle



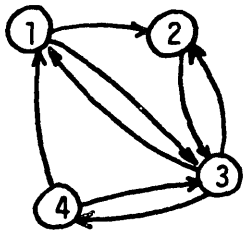
a. Basic Form



b. Violation 1: $o_1 \succ_1 o_3$



c. Violation 2: $o_3 \succ_1 o_1$



d. Violation 3: $o_1 \sim_1 o_3$

PROOF. If incomparability is resolved, one and only one of the following would be true: $[o_1 \succ_1 o_3]$, $[o_3 \succ_1 o_1]$ or $[o_1 \sim_1 o_3]$. If $[o_1 \succ_1 o_3]$, then $\{o_1, o_3, o_4, o_1\}$ is a type 2 cycle. Moreover, if $[o_3 \succ_1 o_1]$, then $\{o_1, o_2, o_3, o_1\}$ is a type 2 cycle. Finally, if $[o_1 \sim_1 o_3]$, then we obtain a straightforward and clear violation of property P4.† These three possible cases are illustrated in Figures 4.8.b, 4.8.c and 4.8.d respectively.□

It is very easy to extend this proof to the case where there are more than four nodes, since the same configuration as the one in the above proof is obtained in the P3-closure of $(\succ_1, \sim_1)$.

Type 4: Cycles with 3 or more indifference pairs, at least two of which being consecutive. Though such cycles may at first seem to be compatible with partial semi-orders (primarily because the non-completeness of the relation may conceal violations), situations may arise, depending on the number and nature of the preferential connections between the elements of those cycles, where violations to the desired properties might occur. Fortunately, we can show that a relatively simple systematic rule exists to tell whether any give type 4 cycle is a violation or not. As for type 3 cycles, we refer to those non-violating types as type 4.a cycles and to the others as type 4.b cycles. Let us first consider different examples of type 4.a and type 4.b cycles respectively.

Example 4.6: Figure 4.9a shows a type 4.a cycle in $G(\succ_1, \sim_1)$; we can check that it is not an obstacle in the face of the P3-closure of $(\succ_1, \sim_1)$.□

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†Property P4 was defined and illustrated in section 3.3. As a convenience for the reader, we restate it here: Given $o_1, o_2, o_3, o_4 \in O$, P4 says that $\{[o_1 \succ_1 o_2]$ and $[o_2 \succ_1 o_3]$ and $[o_2 \sim_1 o_4] \Rightarrow \{not([o_1 \sim_1 o_4]$ and $[o_3 \sim_1 o_4])\}$ A graphical illustration is given in Figure 3.2.

Example 4.7: Figure 4.9.b illustrates a type 4.b cycle in $G(\succ_1, \sim_1)$, which differs from the previous example only in that it has one additional node o_6, such that $[o_5 \succ_1 o_6]$ and $[o_6 \succ_1 o_1]$. □

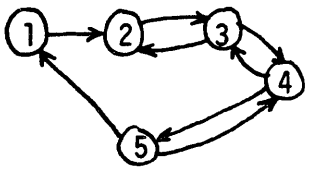
We contend that the cycle in Example 4.7 is not compatible with the properties of partial semi-orders, and support this contention hereafter, in a manner similar to the one used above to demonstrate the incompatibility of type 3.b cycles. In that cycle $\{o_1, o_2, o_3, o_4, o_5, o_6, o_1\}$, o_3 and o_6 are incomparable by $(\succ_1, \sim_1)$ as well as by $(\succ_1^P, \sim_1^P)$. We prove hereafter that regardless of the relation between o_3 and o_6, we obtain an inconsistency.

PROOF. If incomparability is resolved then one and only one of the following would hold: $[o_3 \succ_1 o_6]$, $[o_6 \succ_1 o_3]$ or $[o_6 \sim_1 o_3]$. If $[o_3 \succ_1 o_6]$, as shown in Figure 4.9.c, then $\{o_1, o_2, o_3, o_6, o_1\}$ would be a type 2 cycle, which is unacceptable. If $[o_6 \succ_1 o_3]$, as shown in Figure 4.9.d, then $\{o_3, o_4, o_5, o_6, o_3\}$ is a type 3.b cycle, also unacceptable. Finally, if $[o_3 \sim_1 o_6]$, as shown in Figure 4.9.e, then $\{o_6, o_1, o_2, o_3, o_6\}$ is a type 3.b cycle. □

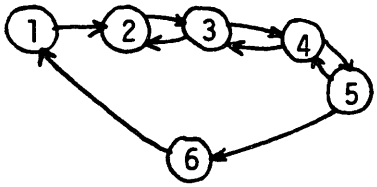
The basic difference between the graphs of Figure 4.9.a and 4.9.b respectively, is that the number of strict preference arcs connecting o_5 to o_2 has increased. This brings us to the above mentioned rule separating type 4.a from type 4.b cycles, and more generally determing if any cycle can be consistent with the properties of partial semi-orders.

Consider a cycle in the graph of a partial relation; let n_i denote the total number of indifference pairs, and n_p the total number of strict preference pairs along this cycle. For example, the cycle in Figure 4.9.a has $n_i = 3$ and $n_p = 2$, while in the more complicated cycle of Figure 4.10, $n_i = 7$ and $n_p = 6$. Whenever $n_p > n_i - 1$, then the corresponding cycle

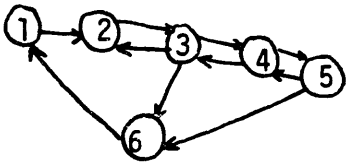
Figure 4.9: Examples of Type 4 Cycles and Associated Violations



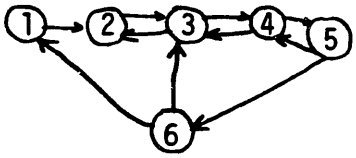
(a) Type 4.a



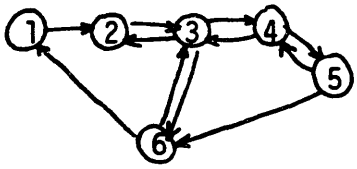
(b) Type 4.b



(c) o_3 o_6'

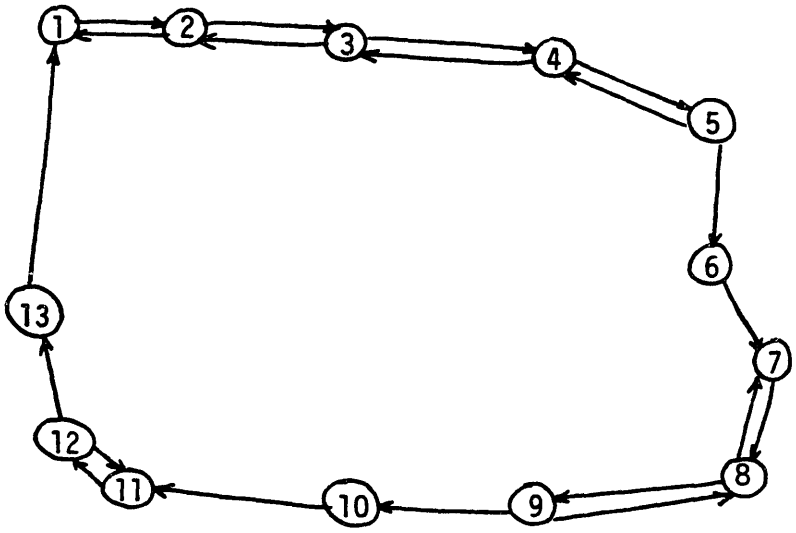


(d) o_6 o_3



(e) $o_6 \sim o_3$

Figure 4-10: Cycle with $n_i = 7$ and $n_p = 6$



cannot be compatible with the properties of semi-orders, and is thus a type 4.b cycle; otherwise, i.e., if $n_p \leq n_i - 1$, it is a type 4.a cycle which is not necessarily a violation. This rule can be shown to hold in the above mentioned examples of Figure 4.9, whereby in Figure 4.9.a, $n_p = 2 \leq n_i - 1$, whereas in Figure 4.9.b, for the same value of n_i (=3), $n_p = 3 > n_i - 1$. A notable special case of type 4.a is the one depicted in Figure 4.11, where $n_p = 0$, i.e., the cycle consists of consecutive indifference pairs, highlighting the non-transitivity of $\sim_1$.

Similarly, the properties of all previous types of cycles discussed above can be obtained as special cases of this general rule. For instance, type 1 cycles have $n_i = 0$ and as such any value of n_p would exceed $n_i - 1$. The non-consecutiveness requirement for indifference pairs in type 2 cycles implies that $n_p \geq n_i$, which is why that type is always inconsistent with our postulates. The same rule applies to type 3 cycles as well.

The above rule can be derived in a straightforward manner using the numerical representation properties of semi-orders mentioned in Section 3.3 in particular the "constant representation" property. However, so as not to complicate this discussion any further, we will not present this derivation as it is not, in itself essential to the further development of this discussion.†

Definition 4.12: Cycles of types 1, 2, 3.b and 4.b, in the graph of a partial binary preference and indifference relation, are referred to as inconsistency cycles. □

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†A graph theoretic proof by recursion is also possible, but we choose not to diverge in it in the context of this discussion.

The above definition reflects the fact that inconsistency cycles in $(\succ_1, \sim_1)$ are sources of violation of the properties of partial semi-orders by the eventual P3-closure of $(\succ_1, \sim_1)$. Strategies for dealing with inconsistency cycles in a partial relation $(\succ_1, \sim_1)$ resulting from the decision rule of section 4.2 are discussed in the next section.

4.4 Approaches for Correcting Inconsistencies

To place things in perspective, an initial partial binary preference-indifference relation $(\succ_1, \sim_1)$ is defined over a given set of options O using the mechanism specified in section 4.2. The properties of this relation being generally unrestrictive, the specific preference configuration obtained may not be consistent with the basic postulates of section 3.3; situations in the graph of $(\succ_1, \sim_1)$ that imply violations by the P3-closure of $(\succ_1, \sim_1)$ of the properties of partial semi-orders consist of certain types of directed cycles identified in section 4.3 and referred to as inconsistency cycles.

We are thus faced with an initial relation $(\succ_1, \sim_1)$ containing a number of such inconsistency cycles. This section describes three possible approaches, along with their underlying preferential assumptions and normative implications, for dealing with these inconsistencies so as to obtain a final relation $(\succ, \sim)$ which is consistent with our basic postulates. Again, consistency is defined in terms of the P3-closure of $(\succ, \sim)$, denoted by $(\succ^P, \sim^P)$, satisfying properties P1 to P4 of section 3.3.

The first of these corrective approaches, described in subsection 4.4.1, treats each inconsistency cycle as an indication of indifference among its elements. The second approach seeks the "weakest link" in each inconsistency cycle, as described in subsection 4.4.2. Finally subsection 4.4.3 describes the third approach which declares mutual incomparability of cycle elements and presents a summary comparative assessment of these approaches.

4.4.1 Approach 1: Indifference Among Cycle Elements

This approach assumes that two elements of O connected by an inconsistency cycle are preferentially indifferent to each other. Thus two elements are considered preferentially indifferent (by the "final" relation $\sim$) when-

ever they are preferentially indifferent by the initial $\sim_1$, or there exists an inconsistency cycle in $(\succ_1, \sim_1)$ connecting them. In order to express this formally, we introduce the following definition:

Definition 4.13: Let Γ_1 be a relation defined over O such that, for a given o_i, $o_j \in O$:

$[o_1 \ \Gamma_1 \ o_j] \Leftrightarrow$ {"there exists an inconsistency cycle in $G(\succ_1, \sim_1)$ connecting o_i and o_j"}. □

We thus state, that given o_i, $o_j \in O$,

$$[o_i \sim o_j] \text{ if and only if } \{[o_i \sim_1 o_j] \text{ or } [o_i \ \Gamma_1 \ o_j]\}.$$

While this approach bears similarity with the reduction procedure associated with ELECTRE (described in 4.3.1), it differs from it in that it does not require the final indifference relation to be transitive; thus $\sim$ is not necessarily an equivalence relation.†

In this approach, the final preference relation $\succ$ is obtained directly from $\succ_1$, with some node pairs initially connected by $\succ_1$ changing to indifference pairs in the final relation (when they are elements of an inconsistency cycle). Formally,

$$[o_i \succ o_j] \text{ if and only if } \{[o_i \succ_1 o_j] \text{ and not } [o_i \ \Gamma_1 o_j]\}.$$

The resulting final relation $(\succ, \sim)$ would thus not contain any inconsistency cycles, meaning that no violation of the properties of partial semi-orders are present in its P3-closure $(\succ^P, \sim^P)$.

This approach makes relatively strong preferential assumptions to resolve

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†The reader who is not familiar with the exact definitions of the preference theoretic terminology is again referred to Appendix B.

inconsistencies, and does not really address the question of why these inconsistencies arise. In other words, violations of the initial postulates guiding the selection among alternative options ought to be looked at more carefully as they might provide clues to mistakes or anomalies in the result of the process up to that point.

4.4.2 Approach 2: "Breaking" Weakest Link

Alternatively, one could attempt to remove the inconsistency in a cycle by "breaking" its weakest (one or more) critical link(s) (in the sense that if removed, so is the inconsistency). Ideally, of course, all the elements conncected by an inconsistency cycle could be reappraised on a pairwise basis as well as globally by the decision-maker so as to identify that critical link. However, especially when a large number of options are under consideration,† it might be useful to devise operational rules, based on the principles underlying this decision-aiding approach, to detect such inconsistency links, with minimal risk of violating the decision-maker's true preference structure.

In the general case of any partial binary relation with inconsistency cycles, the approach requires some rule for identifying the critical link(s). Once identified, the end nodes of a critical link are declared incomparable, thus eliminating the source of inconsistency.

In this subsection, this approach is described for the particular case when the partial binary relation $(\succ_1, \sim_1)$ is obtained by the mechanism specified in section 4.2. In this case, the concordance and discordance thres-

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†Which is generally the case when the use of such a decision aid is warranted.

holds ($\bar{c}$ and $\bar{d}$ respectively) used to define the outranking relation R and to subsequently build ($\succ_1$, $\sim_1$) provide a powerful mechanism for this task. The identification decision rule is discussed hereafter, following a description of its underlying logic.

From the definition of R (see subsections 4.2.2 and 4.2.3), the higher the value of $\bar{c}$ and the lower that of $\bar{d}$ (within the range [0, 1]), the "stronger" is the assertion of outranking and the lesser the risk associated with the decision to prefer one random prospect over another.† Subsequently, an outranking assertion which can be made for a given threshold level value pair ($\bar{c}_o$, $\bar{d}_o$) is less than or as risky as a corresponding assertion with ($\bar{c}_1$, $\bar{d}_1$) where $\bar{c}_o \geq \bar{c}_1$ and $\bar{d}_o \leq \bar{d}_1$ (with one of these inequalities strict for the "less than" to hold). This is best illustrated by an example.

Example 4.8: Consider o_i, o_j, o_k, $o_\ell \in O$, characterized by the following:

$$c_{ij} = 0.9 \quad \text{and} \quad d_{ij} = 0.2$$
$$c_{k\ell} = 0.8 \quad \text{and} \quad d_{k\ell} = 0.3,$$

where c_{ij} and d_{ij} are given by equations (4.1) and (4.2) respectively. The decision [o_i R o_j] is considered less risky than and thus to take precedence over [o_k R o_ℓ] in a given decision problem.§ □

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†For instance, if $\bar{c} = 1$ (maximum value), we can accept $o_i R o_j$ if and only if o_i outperforms o_j on every possible state of nature, i.e., if $x_{ik} \geq x_{jk}$, $\forall S_k \in S$, which tolerates no risk whatsoever that o_i might not be preferred to o_j.

Note also that the concept of riskiness of a decision was discussed amply in conjunction with the basic considerations in Section 2.2. Roy (1977, 1977a) also elaborates on the risk rationale of outranking relations in decision aids.

§As a matter of fact, we used this same rationale to build ($\succ_1$,$\sim_1$), in Section 4.2.3, when both c_{ij} and c_{ji} were greater than or equal to $\bar{c}$ and d_{ij} and d_{ji} were both less than or equal to $\bar{d}$.

In the above example, if $\bar{c}$ and $\bar{d}$ are respectively set at 0.7 and 0.4 (based on overall problem considerations), then both $[o_i \; R \; o_j]$ and $[o_k \; R \; o_\ell]$ would be true. If these two assertions do not lead to inconsistences, then both are accepted. However, if inconsistencies occur, then we consider $[o_i \; R \; o_j]$ to take precedence as being more credible than $[o_k \; R \; o_\ell]$, since it can be reached with a higher $\bar{c}$ and $\bar{d}$.

The operational rules for removing inconsistencies in a given cycle derive from the above logic. Thus the weakest link in an inconsistency cycle is that which can be broken for the lowest $\bar{c}$ and highest $\bar{d}$. Mathematically, given a directed inconsistency cycle $C = \{o_1, o_2, \ldots, o_\ell\}$, with $o_\ell = o_1$, we seek a pair $(o_k, o_{k+1}) \in C$ such that:

$$\left[c_{k,k+1} = \min_{i \mid o_i \in C} (c_{i,i+1})\right] \text{ and } \left[d_{k,k+1} = \max_{i \mid o_i \in C} (d_{i,i+1})\right] .$$

If such a pair (o_k, o_{k+1}) exists in C, then o_k, o_{k+1} are declared incomparable by the final relation $(\succ, \sim)$, thus removing the inconsistency. If more than one pair of options in C satisfy the above conditions, then all such relations are declared not to hold.

In the event that no such pair of consecutive elements exists in C (and there is no compelling reason for one to exist), the problem is akin to a bi-criterion evaluation problem,[†] where we are minimizing one of the criteria ($\bar{c}$) and maximizing the other ($\bar{d}$), and no solution exists that achieves both objectives simultaneously.[§] In this case, we first identify

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[†]The decision space of course consists of the set of all pairs of consecutive elements (o_k, o_{i+1}) in a given inconsistency cycle C.

[§]We will not elaborate here on the characteristics of multi-criteria decision problems and their solution techniques. For a review, see Mahmassani (1980b), Keeney and Raiffa (1976, chapter 3), Cohon (1979) among others.

the efficient set of option pairs along a given inconsistency cycle. The efficient set is defined in the usual manner as comprising all pairs $(o_i, o_{i+1}) \in C$ for which $\{\nexists(o_j, o_{j+1}) \in C \mid c_{j,j+1} \leq c_{i,i+1}$ and $d_{j,j+1} \geq d_{i,i+1}\}$. The link to be removed is then selected from the efficient set in one of two alternative ways:

1. on a situational basis, whereby the decision maker is presented with all the efficient pairs in a given inconsistency cycle and asked to directly select the "weakest" relation among them.† The risk of misrepresenting the decision maker's preferences is thus minimized as he is given the opportunity to reconcile his own inconsistencies in the specific situation where they occur (and not in an abstract a priori way). However, whenever a large number of inconsistency cycles have to be resolved and the decision maker's (as well as the analyst's) time is severely constrained, this approach quickly becomes impractical, thus the motivation for the second rule hereafter.

2. On the basis of the concordance (or probability of dominance) measure only, whereby we remove the link (o_i, o_{i+1}) with the lowest $c_{i,i+1}$ in the efficient set associated with (inconsistency) cycle C. A strong case could be made for this rule in conjunction with the binary relation $(\succ_1, \sim_1)$ of section 4.2. The underlying rationale is that for a given level of acceptable regret (defined as the difference in potential payoffs for a given state of nature between o_i and o_{i+1}), the higher the associated $c_{i,i+1}$, the "stronger" the decision and the less risky it is. Thus, since we have accepted the $\bar{d}$ at which the inconsistency cycle arise, it seems plausible to feel more comfortable with a relation offering a greater probability

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†In that sense, this selection approach would be analogous to that typically followed in the so called "generating" techniques for multi-objective optimization problems (Cohon, 1979; Mahmassani, 1980b).

of outperforming (or no occurrence of regret) rather than a lower one.

The analyst could of course devise other "weakest link" identification strategies if appropriate to a particular problem situation. In any event, once identified, that link is removed by declaring its end nodes incomparable by the final relation $(\succ, \sim)$.

A numerical example is presented hereafter to illustrate the link removal approach to resolving inconsistencies.

Example 4.9: Consider a problem characterized by seven possible states of nature $S = \{S_1, S_2, S_3, \ldots, S_7\}$ with the associated probability of occurrence vector $P = \{0.1, 0.1, 0.1, 0.1, 0.2, 0.2, 0.2\}$. We have the subset of options $O = \{o_1, o_2, o_3, o_4\}$, with the conditional impact matrix shown in Table 4.1 and the values of the associated c_{ij}'s and d_{ij}'s shown in Table 4.1 as well.

If we set $\bar{c} = 0.6$ and $\bar{d} = 0.7$, we obtain the following initial binary relation $(\succ_1, \sim_1)$:

$$[o_1 \succ_1 o_2];\quad [o_4 \succ_1 o_1];\quad [o_4 \succ_1 o_2];\ [o_3 \succ_1 o_4];\ \text{and}\ [o_2 \sim_1 o_3].$$

The graph representation of this relation, depicted in Figure 4.11.a readily reveals the existence of two inconsistency cycles: $\{o_4, o_2, o_3, o_4\}$ and $\{o_1, o_2, o_3, o_4, o_1\}$. In trying to remove the lowest link from the latter cycle, (o_3, o_4) is easily detected as having the lowest concordance value and highest discordance value (among those pairs included in the relation $[\succ_1, \sim_1]$). Removing link (o_3, o_4) from the graph of Figure 4.11.a, resolves the inconsistency in both of the above cycles, as shown in Figure 4.11.b which represents the final relation $(\succ, \sim)$ on O. Note that $[o_3\ I\ o_4]$,

Table 4.1: Impact Matrix of Example 4.9

States	S_1	S_2	S_3	S_4	S_5	S_6	S_7
Probas	0.1	0.1	0.1	0.1	0.2	0.2	0.2
Options	150	100	130	170	110	160	180
o_1	150	100	130	170	110	160	180
o_2	130	125	130	170	125	110	175
o_3	135	120	150	190	125	110	155
o_4	130	120	140	180	110	130	190

Risk Measures

$c_{12} = 0.7$ $\quad d_{12} = 0.3$

$c_{21} = 0.5$ $\quad d_{21} = 1.0$

$c_{13} = 0.5$ $\quad d_{13} = 0.4$

$c_{31} = 0.5$ $\quad d_{31} = 1.0$

$c_{14} = 0.5$ $\quad d_{14} = 0.4$

$c_{41} = 0.7$ $\quad d_{41} = 0.6$

$c_{23} = 0.7$ $\quad d_{23} = 0.4$

$c_{32} = 0.7$ $\quad d_{32} = 0.4$

$c_{24} = 0.4$ $\quad d_{24} = 0.4$

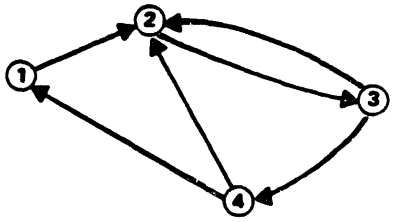
$c_{42} = 0.7$ $\quad d_{42} = 0.3$

$c_{34} = 0.6$ $\quad d_{34} = 0.7$

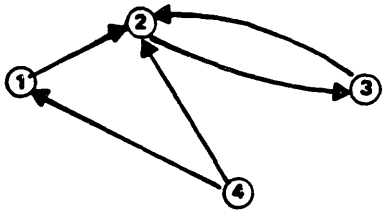
$c_{43} = 0.5$ $\quad d_{43} = 0.3$

Note: $d^{max} = 50$ in this problem

Figure 4.11: Graph Representation of Problem of Example 4.9



a. Inconsistency Cycles



b. After Removing (o_3, o_4)

indicating their incomparability in ($\succ$, $\sim$). Moreover, in this particular example, $\{o_3, o_4\}$ is the kernel; thus, this decision aid has succeeded in eliminating o_1 and o_2 and selecting o_4 and o_3 for further comparison. □

The third approach for resolving inconsistency cycles is presented in the next subsection.

4.4.3 Approach 3: Mutual Incomparability Among Cycle Elements

This approach makes more extensive use of the non-completeness property, and is the most conservative of the three approaches (in terms of risking misrepresentation of preferences). Simply stated, it consists of declaring all elements connected by an inconsistency cycle as mutually incomparable. Formally,

$$[o_i \; \Gamma_1 \; o_j] \Rightarrow [o_i \; I \; o_j]$$

This approach is similar to one proposed by Whitt (1979) in conjunction with non-transitive stochastic dominance. The rationale for our suggesting it in this context is quite obvious: in as much as inconsistency cycles indicate possible trouble, the least risky assertion concerning a given pair of options is that they are incomparable, indicating the absence of sufficient basis for a more positive statement. It is of course expected that closer scrutiny of those options will take place at a later stage of the decision aiding procedure, if no other basis exists for the elimination of any of them. Therefore, this strategy is indeed a very cautious one, and is "passive" preferentially in that it does not require strong preferential assumptions. However, the price the analyst pays for this "unobtrusive" approach is a potentially drastic reduction in the decidability of the overall problem.

In summary, while the first approach gives a positive interpretation of inconsistency cycles by equating their presence to the prevalence of mutual preferential indifference among options connected by such cycles, the second and third approaches treat them as possible causes of preference misrepresentation or other errors in the course of the conduct of the decision-aiding process. Therefore, rather than taking this risk in a "mechanical" fashion, the options in question are declared incomparable according to the available preferential information. The difference between the second and third approaches lies in how specific they are in terms of "diagnosing" the problem and detecting it: while the third one "refuses" to accept any of the comparisons of the inconsistency connected options, the second one takes further steps in pointing out to a specific pair of options that is at the root of the inconsistency cycle.

It should be noted that one need not be confined to the use of a single approach, as it is possible to use any two or all three of them in the same problem. For instance, all elements in an inconsistency cycle could first be declared incomparable (i.e. approach 3), followed by rebuilding of the relations using the relational precedence concepts of approach 2. Or, certain cycles could be construed as indicating indifference (if for example, all the relations connecting pairs along the cycle are all efficient with respect to $\bar{c}$ and $\bar{d}$), while other cycles are reduced by approach 2. The range of possibilities is quite wide here thus providing flexibility to adjust to specific decision contexts.

Ultimately, we obtain a partial relation $(\succ, \sim)$, over the option set O, which is free of inconsistency cycles. Its P3-closure, $(\succ^P, \sim^P)$, therefore follows all four properties of section 3.3. A reduced subset $K \subseteq O$ of superior options (containing the most preferred one) can then be obtained

by solving for the kernel of $(\succ^P, \sim^P)$. As discussed in section 4.1, a kernel can be shown to exist when the relation is consistent with our basic postulates, as neither type 3.a or 4.a cycles (which are allowed and possible here) prevent kernels from existing (as could some other types of cycles, as discussed in subsection 4.1.2, and in Hansen et al. [1975]).† It is however possible to solve for the kernel of $(\succ^P, \sim^P)$ directly from $(\succ, \sim)$ (as obtained after removing inconsistency cycles), without having to first establish $(\succ^P, \sim^P)$ (by implementing definition 4.7 of the P3-closure of a relation). Indeed, we prove hereafter that a necessary and sufficient condition for an element of $o_i \in O$ to be in the kernel of $(\succ^P, \sim^P)$ is that there exists at least one element $o_j \in O$ which is strictly preferred to it by the relation $\succ$.

LEMMA: Let $(\succ, \sim)$ be a partial relation defined over O, such that the graph of $(\succ, \sim)$ does not contain inconsistency cycles (cf. definition 4.12). If $(\succ^P, \sim^P)$ is the P3-closure of $(\succ, \sim)$ (cf. definition 4.7), and K the kernel of $(\succ^P, \sim^P)$ (cf. definition 4.1), then

$$\{o_i \in K\} \text{ if and only if } \{\nexists\, o_j \in O \mid o_j \succ o_i\}, \quad \forall\, o_i \in O.$$

PROOF: To establish the necessary condition, consider $o_i \in K$ and assume that $\{\exists o_j \in O \mid o_j \succ o_i\}$. If $o_j \in K$, and noting that by definition 4.7 (of the P3-closure of a relation), $[o_j \succ o_i] \Rightarrow [o_j \succ^P o_i]$, then there is a contradiction to condition (ii) in definition 4.1 (of K). On the other hand,

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†It is of course possible to solve instead for the kernel of $(\succ, \sim)$, which is possible larger than that of its transitive closure $(\succ^P, \sim^P)$. Such a kernel can be shown to exist for $(\succ, \sim)$ as well since the only cycle types it allows are type 3.a and 4.a cycles. Of course, one could always solve for the quasi-kernel, if a kernel does not exist, in the event that $(\succ, \sim)$ has not been made to be consistent with the properties of partial semi-orders.

if $o_j \notin K$, then $\{\exists\, o_\ell \in K \mid o_\ell \succ^P o_j\}$, by condition (i) in definition 4.1. However, by P3-closure, $\{[o_j \succ^P o_i]$ and $[o_\ell \succ^P o_j]\} \Rightarrow [o_\ell \succ^P o_i]$, which also contradicts condition (ii) of definition 4.1. Having shown that $\{o_i \in K\} \Rightarrow \{\nexists\, o_j \in O \mid o_j \succ o_i\}$ establishes the necessary condition in the lemma.

The sufficient condition can be established by shown that if $\{\nexists\, o_j \in O \mid o_j \succ o_i\}$, then $o_i \in K$. Assume that $\{\nexists\, o_j \in O \mid o_j \succ o_i\}$; if $o_i \notin K$, then $\{\exists\, o_\ell \in K \mid o_\ell \succ^P o_i\}$, by condition (i) of definition 4.1. Noting that, from definition 4.7, a necessary condition for $[o_\ell \succ^P o_i]$ is that $\{\exists\, o_j \in O \mid o_j \succ o_i\}$, it is not possible that $o_i \notin K$. This establishes the sufficient condition, thus completing the proof. □

The above lemma has two very important algorithmic implications in terms of solving for the kernel of $(\succ^P, \sim^P)$. The first is that it is not necessary to implement the P3-closure of $(\succ, \sim)$[†] in order to solve for its kernel. The second consists of the above straightforward rule for identifying the elements of this kernel, namely to eliminate all options $o_i \in O$ for which $\exists\, o_j \in O \mid o_j \succ o_i$.

The next section summarizes this chapter, and presents some concluding remarks concerning the conformance of the methodological development herein with the basic considerations of Chapter 2.

[†]Provided of course that $(\succ, \sim)$ is free of inconsistency cycles, such as after applying the corrective approaches described in this section.

4.5 Summary and Concluding Remarks

This chapter has presented a methodology for conducting the pairwise comparisons phase of the overall decision aiding framework.† While the structure of the methodology and some of the supporting theoretical results are applicable to all problem subclasses formulated in section 1.4, the details of its major components were pursued for problem subclass 1, characteristic of frequently encountered "alternative scenarios" situations.

Section 4.1 provided an overview of the general structure and logic of the methodology. It consists of building a partial binary preference-indifference relation ($\succ$, $\sim$) over the set of options O, insuring its consistency with the starting postulates (of section 3.3), and then using that relation to obtain a reduced subset K of superior options. Solution concepts for the reduced subset K and their properties were discussed in subsection 4.1.2. The remainder of the chapter addressed the procedure for building the binary relation ($\succ$, $\sim$).

This procedure starts by building an initial partial binary relation ($\succ_1$, $\sim_1$), developed here for problem subclass 1, and reflecting the basic considerations of section 2.2. After restating the problem and elaborating on its characteristics in subsection 4.2.1, an outranking relation R was developed in subsection 4.2.2, using a multidimensional representation of the risk implications of a binary choice situation to convey information to and seek input from the decision maker. Rules for using the above relation and its associated risk measures to build the above mentioned initial relation ($\succ_1$, $\sim_1$) were specified in subsection 4.2.3. The properties of

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†This framework was presented in section 2.3.

$(\succ_1, \sim_1)$, thoroughly discussed in subsection 4.2.4, were then shown to be quite unrestrictive and not necessarily in conformance with the starting postulates.

While not claimed as an absolute model of preferences for random prospects, partial semiorders, as discussed in section 3.3, could be considered as possessing relatively rational properties for the final relation $(\succ, \sim)$ to follow, within the context of operationally forming that relation.† For that reason, sections 4.3 and 4.4 developed rules and operations for transforming the initial unrestrictive $(\succ_1, \sim_1)$ to a final relation $(\succ, \sim)$ consistent with those properties. After briefly presenting other possible assumptions for proceeding beyond $(\succ_1, \sim_1)$ in subsection 4.3.1, the notion of P3-closure of a relation was defined in subsection 4.3.2, and obstacles in the way of its implementation, which constitute inconsistencies with the properties of partial semiorders, were identified. Accordingly, different types of such inconsistencies, which take the form of cycles in the graph of the binary relation $(\succ_1, \sim_1)$ were described and rules for their detection specified, in subsection 4.3.3.

Finally, in section 4.4, three approaches for correcting these inconsistencies were described, resulting in a relation $(\succ, \sim)$ satisfying the desired properties, which could then be used to obtain a reduced subset of options (in this case, a kernel, as discussed in section 4.1.2). These three approaches are applicable in the general case of resolving inconsistencies (with the desired properties) in a partial binary relation; however, since the details of the second approach are dependent on the

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†See also Chapter 2 for further discussion on this and related points, in conjunction with the basic underlying considerations.

the specific mechanism for building the initial binary relation, we pursued their development for the specific relation $(\succ_1, \sim_1)$ described in section 4.2.

From the overall perspective of the thesis, this chapter has proposed an alternative to existing decision models for problem subclass 1, reviewed in Chapter 3 and found inadequate with respect to the basic considerations presented in Chapter 2. This methodology distinguishes itself from those models in a number of features, reflecting the above-mentioned basic considerations. In particular, it does not embody an overriding normative rule, nor does it assume an a priori complete preference system on behalf of the decision maker. Instead, it recognizes the importance of an intuitively understandable decision rule, that can be explained and communicated, yet captures the essential implications of the choice situation in a given problem context. The regret-based risk measures and the associated evaluation rationale, in our opinion, possesses those features, as discussed in section 4.2, and reflect a number of specific behavioral considerations presented in Chapter 2.[†]

Moreover, consistently with proposition 2.2.6, only satisficing information, known to be relatively easy to supply, is required from the decision maker.[§]

However, its usefulness is contingent upon the willingness of the decision maker to accept and trust its logic, as explained in section 4.2. It is however possible to use other risk measures in conjunction with the same evaluation rationale, thus defining a family of possible decision rules

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[†]In particular, the points in conjunction with proposition 2.2.7 were explicitly considered, as discussed in section 4.2.

[§]Of course, the fit of the pairwise comparison methodology of this chapter within the overall decision aiding frameowrk (of section 2.3) reflects proposition 2.2.6 even further.

which increase the flexibility of the methodology in being fine-tuned to specific decision situations. Such alternative risk measures are discussed in the next chapter.

Furthermore, a significant feature of this methodology is due to proposition 2.2.8, namely the recognition of incomparability and threshold phenomena in an operational decision aid for choice among transportation options under uncertainty. As such, the basic postulates articulated in section 3.3 whereby partial semiorders were proposed as an alternative model of preferences among uncertain complex options (instead of the usually completely formed order), and the development of a procedure consistent with these postulates, constitute a departure from previous and existing decision making techniques.

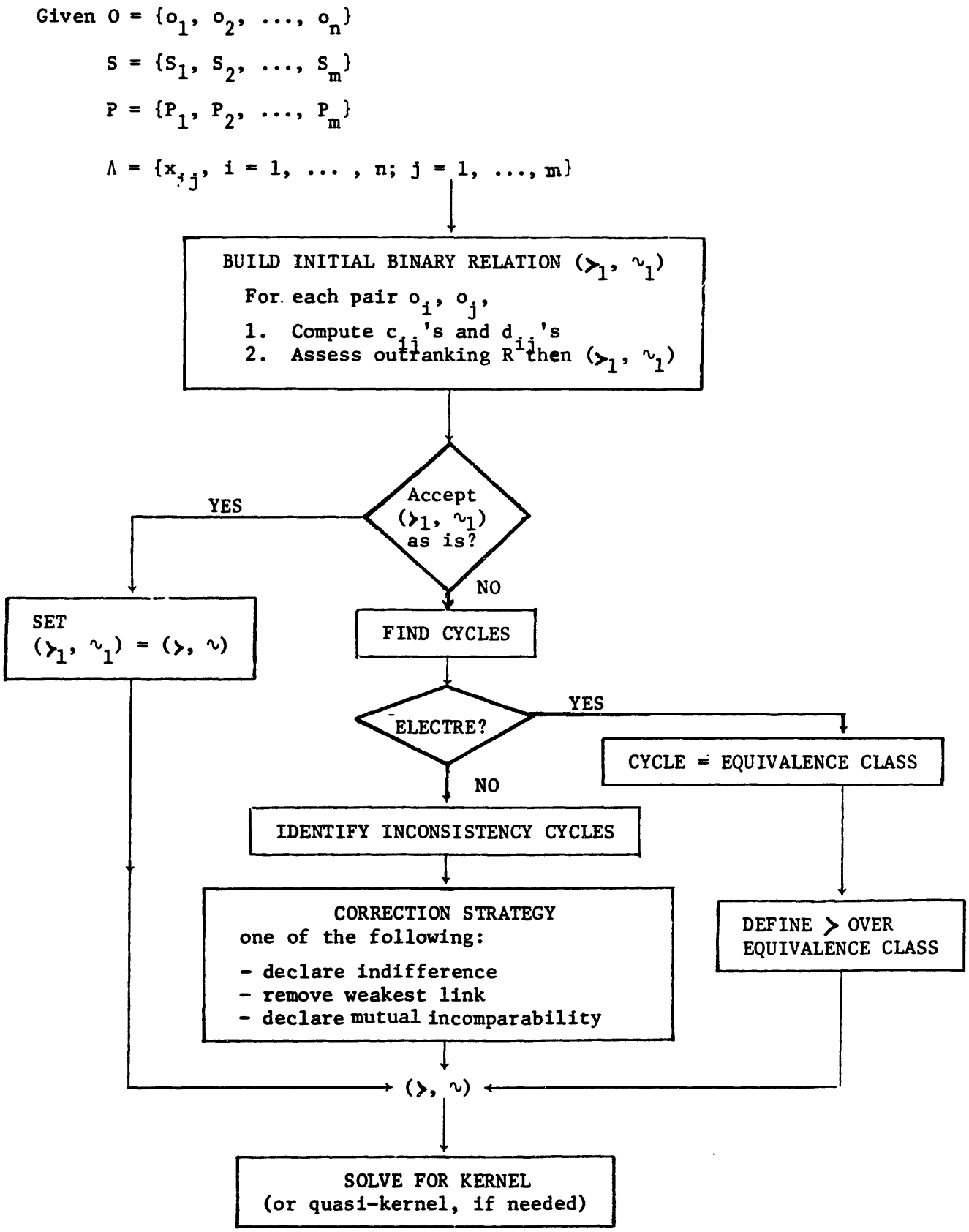
However, as we noted in section 4.1, other assumptions concerning preferences are possible which would require different methodological steps. In particular, we identified 2 such other assumptions and associated procedures, namely:

1. accept initial $(\succ_1, \sim_1)$ "as is"; and
2. declare cycles as equivalence classes, as in the ELECTRE method.

These assumptions are presented as possible options, alongside our basic postulates, in the highly schematic summary of the methodological structure depicted in Figure 4.12.

In the next chapter, extensions and variations of this methodology are investigated.

Figure 4.12: Overview of Pairwise Comparisons Methodology



CHAPTER 5

VARIATIONS AND EXTENSIONS

This chapter presents a number of variations and extensions of the methodology developed in Chapter 4. Section 5.1 describes alternative functional forms for the risk measures used in section 4.2 to build the outranking relation R and the subsequent initial preference-indifference relation ($\succ_1$, $\sim_1$). Problem situations where these alternative definitions would be appropriate are identified. These definitions are independent of the rest of the methodology of Chapter 4, and could be directly substituted to their respective counterparts in section 4.2 to define an initial relation ($\succ_1$, $\sim_1$).

Section 5.2 presents an alternative evaluation rationale featuring the notion of "target" or "achievement level" (for the impact resulting from the implementation of an option). The use of this feature to construct an "initial" binary preference indifference relation between options is described. While this relation can be used within the same methodological structure presented in Chapter 4, some associated details concerning the corrective approaches for inconsistency cycles (of section 4.4) are specified. Furthermore, we briefly discuss the use of a target oriented approach for individual option elimination (which precedes the pairwise comparisons phase in the decision aiding framework of section 2.3).

Both types of variations in sections 5.1 and 5.2 respectively are aimed at increasing the flexibility of the methodology , thus allowing the analyst to fine tune its details to the specific decision problem situation and decision-maker being counseled. In essence, a family of

decision rules is developed herein for use within the methodological structure of the previous chapter, for problem subclass 1.

Section 5.3 generalizes the methodology of Chapter 4 as well as its above extensions to the case where options consist of continuous probability distributions, i.e., problem subclass 2. Essentially, the task primarily consists of reformulating the risk measures in the continuous case, resulting in some loss of intuitive appeal (compared to the discrete case).

Section 5.4 then presents an extension to problem subclass 3 where options are explicitly evaluated with respect to their uncertain impact along a multiplicity of criteria. In this case, when more than one criterion forms the basis of the pairwise comparisons, the methodological structure described in chapter 4 is directly applicable. The principal problem specific element needed is the mechanism for building the initial preference-indifference relation over the set of options. The extent of our development in this section is limited to illustrating how the basic mechanism introduced in section 4.2, and extended throughout the previous sections in this chapter, may be applicable to the multi-criteria case.

The contents of this chapter are subsequently summarized in section 5.5.

5.1 Alternative Functional Forms for Risk Measures

This section describes alternative functional forms of the risk measures introduced in section 4.2 as the basic mechanism for establishing the initial binary preference-indifference relation $(\succ_1, \sim_1)$. Alternative definitions for both risk dimensions (of section 4.2) are presented and their implications discussed. Subsection 5.1.1 considers alternative forms for the concordance (or regret avoidance) measure (Eq. 4.1), and proves additional properties concerning transitivity of the resulting relation and cycle formation. Subsection 5.2.2 discusses alternative measures of regret magnitude, or discordance (equation 5.2), highlighting the applicability of each through numerical illustrations.

5.1.1 The Concordance Measure

Using the notation introduced in Chapter 4, the concordance measure c_{ij} was formulated in equation 4.1 as the probability that option o_i performs at least as well as o_j (where o_i, $o_j \in O$), thus making it an indicator of degree of weak dominance of o_i over o_j. We consider here two cases where the index is defined so as to reflect only strict dominance of one option over another. In the first case, the marginal probability of one option strictly outperforming the other is considered, whereas in the second the conditional probability of outperforming (given that payoff from either option would not have been identical) is presented.

a. Strict Concordance

Let us denote this version by c^+_{ij}. For o_i, $o_j \in O$, c^+_{ij} is defined as:

$$c^+_{ij} = \sum_{k|S_k \in B_{ij}} P_k \qquad (5.1)$$

where the sum is taken only over those states of nature where o_i strictly outperforms o_j, i.e., the elements of B_{ij}, defined in section 4.2 as the dominance set. Formally, $B_{ij} = \{S_k \in C_{ij} \subseteq S \mid x_{ik} > x_{jk}\}$.† Of course, c^+_{ij} is in the interval [0,1]. Moreover, it is very easy to see that $c^+_{ij} + c^+_{ji} \leq 1$, with the difference $(1 - c^+_{ij} - c^+_{ji})$ indicating the marginal probability of o_i and o_j yielding an indentical outcome.

The use of c^+_{ij} (instead of c_{ij}) to define an outranking relation could be justified in situations where a more clear cut dominance is judged necessary before outranking can be accepted (though this aspect could be reflected to a large extent with a higher $\bar{c}$ in the case of c_{ij}). Rather than advocate the desirability of either index over the other, we simply point out to the possibility of its use, and prove an important property of the transitive behavior of the resulting $(\succ_1, \sim_1)$. which does not hold with the previous c_{ij}. It is thus assumed hereafter that c^+_{ij} is used in exactly the same fashion as was c_{ij}, in conjunction with d_{ij}, to define the outranking relation R, as described in subsection 4.2.2,and subsequently the "first pass" relation $(\succ_1, \sim_1)$ described in subsection 4.2.3 (as shown in Figure 4.3, which is a prerequisite for a clear understanding of this discussion). Note that $\bar{c}$ still denotes the threshold level of the concordance measure, though it applies to c^+_{ij}. In this case, i.e., given o_i, $o_j \in O$. we declare $[o_i \; R \; o_j]$ if and only if $c^+_{ij} \geq \bar{c}$ and $d_{ij} \leq \bar{d}$.§

The principal result that we want to establish in conjunction with this alternative definition of $(\succ_1, \sim_1)$ concerns its transitive behavior, and in particular the relationship between the threshold value $\bar{c}$ and the occur-

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†It is essential to note that in the definition of c_{ij} in 4.2, the sum was taken over C_{ij}, the concordance set with its elements $S_k \in S$ such that $x_{ik} \geq x_{jk}$.

§Note that Jacquet-Lagreze (1974) uses a parallel to c^+_{ij}, along with other indices, to define, in a different manner, an outranking relation for multicriteria decision problems under certainty.

rence of violations to transitivity, as alluded to in subsection 4.4.2. In that subsection, obstacles in the way of building the transitive closure of $\succ_1$ were identified. For instance, given o_i, o_j, $o_k \in O$, with $[o_i \succ_1 o_j]$ and $[o_j \succ_1 o_k]$, a serious inconsistency exists if either $[o_k \succ_1 o_i]$ or $[o_k \sim_1 o_i]$ is true. We show hereafter that either case can only occur if $\bar{c}$ is set at or below certain levels.

LEMMA: If $[o_i \succ_1 o_j]$ and $[o_j \succ_1 o_k]$, a necessary condition for either $[o_k \sim_1 o_i]$ or $[o_k \succ_1 o_i]$ is that the threshold (strict) concordance measure, $\bar{c}$, be less than or equal to $\frac{2}{3}$, $\forall o_i, o_j, o_k \in O$.

PROOF: First note that a necessary condition for either violation is that $[o_k \; R \; o_i]$. For this latter to be true, it is necessary that $c^+_{ki} \geq \bar{c}$ and $d_{ki} \leq \bar{d}$. We are interested in the relative performance of o_i with respect to o_k for each state of nature $S_\ell \in S$, given that $[o_i \succ_1 o_j]$ and $[o_j \succ_1 o_k]$. We have that since $[o_i \succ_1 o_j]$, then $[o_i \; R \; o_j]$, and thus $c^+_{ij} \geq \bar{c}$, implying that the probability of occurrence of a state of nature on which o_j strictly outperforms o_i is at most equal to $(1 - \bar{c})$ i.e., $\sum_{\ell \mid S_\ell \in B_{ij}} P_\ell \leq (1 - \bar{c})$.[†] Similarly, since $[o_j \succ_1 o_k]$ the elements of B_{kj} are also such that $\sum_{\ell \mid S_\ell \in B_{kj}} P_\ell \leq (1 - \bar{c})$.

For any state of nature $S_\ell \in S$ where $x_{k\ell} > x_{i\ell}$, and is thus an element of B_{ki}, we can have that either $S_\ell \in \{B_{ij} \cup E_{ij}\}$ or $S_\ell \in D_{ij}$, since, by definition, $B_{ij} \cap E_{ij} \cap D_{ij} = \{\phi\}$ and $B_{ij} \cup E_{ij} \cup D_{ij} = S$. If the former is true,

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[†]In the case $E_{ij} = \{\phi\}$, then $\sum_{\ell \mid S_\ell \in B_{ji}} P_\ell = (1 - \bar{c})$.

i.e. $S_\ell \in \{(B_{ij} \cup E_{ij}) \cap B_{ki}\}$, then $(S_\ell \in \{B_{ij} \cup E_{ij}\})$ and $(S_\ell \in B_{ki}) \Rightarrow$ $x_{k\ell} > x_{i\ell} \geq x_{j\ell} \Rightarrow x_{k\ell} > x_{j\ell}$, meaning that $S_\ell \in B_{kj}$ as well, therefore, $\{(B_{ij} \cup E_{ij}) \cap B_{ki}\} \subseteq B_{kj}$, which allows us to write that

$$\sum_{\ell | S_\ell \in \{(B_{ij} \cup E_{ij}) \cap B_{ki}\}} P_\ell \leq \sum_{\ell | S_\ell \in B_{kj}} P_\ell \quad .$$

Since we have established above that $\sum_{\ell | S_\ell \in B_{kj}} P_\ell \leq (1 - \bar{c})$, then $\sum_{\ell | S_\ell \in \{(B_{ij} \cup E_{ij}) \cap B_{ki}\}} P_\ell \leq (1 - \bar{c})$. If, on the other hand, $S_\ell \in \{D_{ij} \cap B_{ki}\}$, then noting that $D_{ij} = B_{ji}$ and $\sum_{\ell | S_\ell \in B_{ji}} P_\ell \leq (1 - \bar{c})$, it is obvious that $\sum_{\ell | S_\ell \in \{D_{ij} \cap B_{ki}\}} P_\ell \leq (1 - \bar{c})$. We are, of course, interested in $c^+_{ki} = \sum_{\ell | S_\ell \in B_{ki}} P_\ell$. We have that:

$$\sum_{\ell | S_\ell \in B_{ki}} P_\ell = \sum_{\ell | S_\ell \in \{(B_{ij} \cup E_{ij}) \cap B_{ki}\}} P_\ell + \sum_{\ell | S_\ell \in \{D_{ij} \cap B_{ki}\}} P_\ell < 1 - \bar{c} + (1 - \bar{c})$$

which yields that

$$c^+_{ki} \leq 2 - 2\bar{c} \tag{5.2}$$

We noted earlier that a necessary condition for either of the two above-mentioned violations is that $c^+_{ki} \geq \bar{c}$. However, since $c^+_{ki} \leq 2 - 2\bar{c}$, the necessary condition as to the value of $\bar{c}$ is that:

$$2 - 2\bar{c} \geq \bar{c}$$

which yields $\bar{c} \leq \frac{2}{3}$. □

The implication of this result is of course that if (the decision maker supplied) $\bar{c} > \frac{2}{3}$, then the above described violation to transitivity will not occur, suggesting that a conservative (risk minimizing) attitude (reflected in high $\bar{c}$) will not lead to the above-mentioned violations of transitivity.

However, another type of obstacle to the transitive closure of $\succ_1$ might occur, namely cycle formation, as described in subsection 4.2.4, whereby we might obtain $o_1 \succ_1 o_2 \cdots \succ_1 o_m \succ_1 o_1$. Therefore, while no obstacles to transitivity will occur for $\bar{c} > \frac{2}{3}$ for any three elements o_i, o_j, $o_k \in O$ where $[o_i \succ_1 o_j]$ and $]o_j \succ_1 o_k]$, there is no guarantee that, for some $o_m \in O$ such that $[o_k \succ_1 o_m]$, we do not obtain either $[o_m \succ_1 o_i]$ or $[o_m \sim_1 o_i]$, even if $\bar{c} > \frac{2}{3}$.

We show hereafter that it is still possible to find values of $\bar{c}$ for which no cycles connecting a given finite number of elements will emerge. However, our results hereafter indicate that, starting at o_i, the further down we go along a path emanating from o_i, the larger the value of $\bar{c}$ needed to prevent a "reverse" arc to o_i which would consecrate the formation of a cycle. Let us first examine the situation with four elements o_i, o_j, o_k, $o_m \in O$ such that $[o_i \succ_1 o_j]$, $[o_j \succ_1 o_k]$ and $[o_k \succ_1 o_m]$. We are interested in the relative performance of o_m with respect to o_i, so as to compute an upper bound for c^+_{mi} which will allow us to assess whether it can lead to $[o_m \; R \; o_i]$ and, if so, for what values of $\bar{c}$.

LEMMA: If $[o_i \succ_1 o_j]$ and $[o_j \succ_1 o_k]$ and $[o_k \succ_1 o_m]$ (with $[o_i \; I \; o_k]$ and $[o_j \; I \; o_m]$[†]), a necessary condition for either $[o_m \succ_1 o_i]$ or $[o_m \sim_1 o_i]$ is

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[†]Note that $[o_i \; I \; o_k]$ means that o_i and o_k are incomparable by $(\succ_1, \sim_1)$.

that the threshold (strict) concordance measure, $\bar{c}$, be less than or equal to $\frac{3}{4}$, $\forall\, o_i, o_j, o_k, o_m \in O$.

PROOF: Since $[o_i \succ_1 o_j]$ and $[o_j \succ_1 o_k]$, we have established above (in equation 5.2) that $\sum_{\ell | S_\ell \in B_{ki}} P_\ell \leq 2 - 2\bar{c}$. Moreover, because $[o_k \succ_1 o_m]$, we have that $\sum_{\ell | S_\ell \in B_{mk}} P_\ell \leq 1 - \bar{c}$. We are interested in the value of the sum of the P_ℓ's for $P_\ell \in B_{mi} \subseteq S$. For any state of nature $S_\ell \in B_{mi}$, we have that either $S_\ell \in \{B_{ik} \cup E_{ik}\}$ or $S_\ell \in D_{ik}$. If the former is true, i.e., $S_\ell \in \{(B_{ik} \cup E_{ik}) \cap B_{mi}\}$ then $S_\ell \in \{B_{ik} \cup E_{ik}\}$ and $S_\ell \in B_{mi} \Rightarrow x_{m\ell} > x_{i\ell} \geq x_{k\ell} \Rightarrow x_{m\ell} > x_{k\ell}$, meaning that $S_\ell \in B_{mk}$ as well; therefore $\{(B_{ik} \cup E_{ik}) \cap B_{mi}\} \subseteq B_{mk}$, implying that $\sum_{\ell | S_\ell \in \{(B_{ik} \cup E_{ik}) \cap B_{mi}\}} P_\ell \leq \sum_{\ell | S_\ell \in B_{mk}} P_\ell \leq 1 - \bar{c}$. If, on the other hand, $S_\ell \in \{D_{ik} \cap B_{mi}\}$ then, again noting that $D_{ik} = B_{ki}$, and $\sum_{\ell | S_\ell \in B_{ki}} P_\ell \leq 2 - 2\bar{c}$, we obtain that $\sum_{\ell | S_\ell \in \{D_{ik} \cap B_{mi}\}} P_\ell \leq 2 - 2\bar{c}$.

Using the definition of c^+_{mi} (i.e., equation 5.1), we have:

$$c^+_{mi} = \sum_{\ell | S_\ell \in B_{mi}} P_\ell = \sum_{\ell | S_\ell \in \{D_{ik} \cap B_{mi}\}} P_\ell + \sum_{\ell | S_\ell \in \{B_{ik} \cup E_{ik}\} \cap B_{mi}} P_\ell \leq 1 - \bar{c} + 2 - 2\bar{c}.$$

which yields that:

$$c^+_{mi} \leq 3 - 3\bar{c} \tag{5.3}$$

Since a necessary condition for either $[o_m \sim_1 o_i]$ or $[o_m \succ_1 o_i]$ is that $c^+_{mi} \geq \bar{c}$, we can compute a minimum value for $\bar{c}$ if this condition is to be satisfied, by setting

$$3 - 3\bar{c} \geq \bar{c}$$

which yields $\bar{c} \leq \frac{3}{4}$. □

Therefore, only if $\bar{c} \leq \frac{3}{4}$ can we have that $[o_m \succ_1 o_i]$ <u>or</u> $[o_m \sim_1 o_i]$ when $[o_i \succ_1 o_j]$, $[o_j \succ_1 o_k]$ <u>and</u> $[o_k \succ_1 o_m]$ (and $[o_i \; I \; o_m]$, and $[o_i \; I \; o_k]$). We notice that to absolutely prevent the formation of 4 node cycles, we need to set $\bar{c}$ at a higher level than needed to prevent 3 node cycles ($\frac{3}{4}$ vs. $\frac{2}{3}$ respectively). We can now generalize the results obtained so far to cycles connecting any number of elements. First of all however, note that all the results derived so far in section 5.1 concerning the transitivity of $\succ_1$ actually apply to the transitivity of the underlying outranking relation R.† Therefore, this discussion concerning cycle formation applies to all types of of cycles (identified in subsection 4.3.3) and not just strict preference cycles (i.e., type 1 cycles). For this reason, to be more complete in presenting the following general necessary conditions for cycle formation, we use R instead of $\succ_1$.

Consider a situation were $[o_1 \; R \; o_2]$, $[o_2 \; R \; o_3]$, ... , $[o_{m-1} \; R \; o_m]$, $[o_m \; R \; o_{m+1}]$, for $m \geq 2$ with no two elements o_i, o_j with $j \neq i-1$ and $j \neq i+1$ otherwise connected (except by incomparability, of course). In other words, we have a directed path from o_1 to o_{m+1}, with (m+1) nodes and m arcs, such that no two non-consecutive nodes along it are connected by R. We are interested in finding a sufficient condition on the value of $\bar{c}$ as a function of m for $[o_{m+1} \; R \; o_1]$ not to be true, thus preventing an (m+1)-node cycle from forming. We prove hereafter that:

<u>LEMMA</u>: For any $o_1, o_2, o_3, \ldots, o_m, o_{m+1} \in O$, $m \geq 2$, such that $[o_1 \; R \; o_2]$, $[o_2 \; R \; o_3]$, ..., $[o_{m-1} \; R \; o_m]$, $[o_m \; R \; o_{m+1}]$ <u>and</u> $\{[o_i \; I \; o_j], \forall i, j \in [1, m+1] \mid j \neq i-1 \text{ } \underline{\text{and}} \text{ } j \neq i+1\}$, a necessary condition for $[o_{m+1} \; R \; o_1]$ is that $\bar{c} \leq \frac{m}{m+1}$.

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†Indeed, the only property of $o_i \succ_1 o_j$ needed in the above proofs is that $c^+_{ij} \geq \bar{c}$, which is a necessary condition of $o_i \; R \; o_j$ as well.

PROOF: We first prove the following result, for all $m \geq 2$:

$$c^{+}_{m+1,\,1} \leq m - m\bar{c} \tag{5.4}$$

We proceed by recursion. We can readily see that equation 5.4 is valid for the three cases examined so far, i.e., for m = 2, m = 3 and m = 4. Assume that it is valid for m = i+1; we need to show that it is valid for m = i+2 to complete the proof.

We thus have $[o_1 \; R \; o_2]$, ..., $[o_i \; R \; o_{i+1}]$, $[o_{i+1} \; R \; o_{i+2}]$. We have, from (5.4) that $c^{+}_{i+1,1} = \sum_{\ell | S_\ell \in B_{i+1,1}} P_\ell \leq i - i\bar{c}$ and also that $\sum_{\ell | S_\ell \in B_{i+2,\,i+1}} P_\ell \leq 1 - \bar{c}$, since $[o_{i+1} \; R \; o_{i+2}]$. Again, we are interested in the elements of $B_{i+2,1} \subseteq S$. For any $S_\ell \in B_{i+2,1}$ we have that either $S_\ell \in D_{1,\,i+1}$ or $S_\ell \in \{B_{1,\,i+1} \cup E_{1,\,i+1}\}$. If $S_\ell \in \{B_{1,\,i+1} \cup E_{1,\,i+1}\}$ and $S_\ell \in \{B_{i+2,\,1}\}$, then $x_{i+2,\ell} > x_{1,\ell} \geq x_{i+1,\ell} \Rightarrow x_{i+2,\ell} > x_{i+1,\ell}$, implying that $S_\ell \in \{B_{i+2,\,i+1}\}$ as well, which gives us as before, that $\sum_{\ell | S_\ell \in \{(B_{1,i+1} \cup E_{1,i+1}) \cap B_{i+2,1}\}} P_\ell \leq 1 - \bar{c}$. If on the other hand $S_\ell \in \{D_{1,i+1} \cap B_{i+2,1}\}$ then, since $D_{1,i+1} = B_{i+1,\,1}$, then $\sum_{\ell | S_\ell \in \{D_{1,i+1} \cap B_{i+2,1}\}} P_\ell \leq c^{+}_{i+1,1} \leq i - i\bar{c}$ by equation (5.4). We can thus write:

$$c^{+}_{i+2,1} = \sum_{\ell | S_\ell \in B_{i+2,1}} P_\ell = \sum_{\ell | S_\ell \in \{(B_{i,i+1} \cup E_{1,i+1}) \cap B_{i+2,1}\}} P_\ell + \sum_{\ell | S_\ell \in \{B_{i+2,1} \cap D_{1,i+1}\}} P_\ell \leq 1 - \bar{c} + 1 - i\bar{c} = (i+1) - (i+1)\bar{c}$$

Having established that $c^{+}_{i+2,1} \leq (i+1) - (i+1)\bar{c}$ proves the result stated in equation (5.4) above.

Since a necessary condition for either $[o_{m+1} \sim_1 o_i]$ or $[o_{m+1} \succ_1 o_i]$ is that $c^+_{m+1,1} \geq \bar{c}$, we can compute a minimum value for $\bar{c}$ if this condition is to be satisfied, by setting (using equation 5.4)

$$m - m\bar{c} \geq \bar{c}$$

which yields

$$\bar{c} \leq \frac{m}{m+1} \quad , \tag{5.5}$$

thus completing the proof. □

At the limit, i.e., to absolutely prevent the formation of cycles of any length ($m \rightarrow +\infty$), the threshold value $\bar{c}$ would have to be set at:

$$\bar{c} > \lim_{m \rightarrow +\infty} \frac{m}{m+1} = 1$$

which means of course that only absolute dominance (on every possible $S_\ell \in S$) will guarantee that cycles are not formed in R. Table 5.1 shows the minimum value of $\bar{c}$ needed to prevent the formation of cycles of different lengths (defined as the number of elements connected by the cycle, i.e., number of nodes), using equation 5.5. It is interesting to note that progressively higher values of $\bar{c}$ are needed to prevent longer cycles.

To conlude this part of c^+_{ij}, the strict concordance measure, it should be noted that the results derived have assumed a "worst case" analysis (in that the maximum possible performance contradicting transitivity was assumed) therefore, the minimum values $\bar{c}$ derived to eradicate intransitivities are only necessary conditions for the existence of cycles, but are by no means sufficient. It would indeed be almost pathological to find real world problems exhibiting such "worst case" characteristics as the one assumed to derive the above conditions. However, we are not advocating the use of

Table 5.1: Minimum Values of $\bar{c}$ Needed to Prevent Cycles of Length n

Cycle Length n	Minimum $\bar{c}$
2	0.5
3	0.667
4	0.75
5	0.8
6	0.833
7	0.857
8	0.875
9	0.889
10	0.9
15	0.933
20	0.95
30	0.967
100	0.99

c^{+}_{ij}, but merely pointing out to the possibility of defining a stricter criterion than c_{ij} (which includes equivalence along with strict dominance), and more importantly to some of its interesting properties which are not equally valid for c_{ij}.

Though different decision rules can be used with c^{+}_{ij} and c_{ij} than the ones in subsection 4.2.2 and 4.2.3, which we have assumed to apply to this discussion, the latter rules could be inadequate when dealing with situations where considerable equivalence in conditional payoffs is present. The next section presents a more effective alternative concordance index from the viewpoint of treating states on which different options yield identical payoffs.

b. Normalized Strict Concordance Measure

The rationale for this specification is that only states on which two given options do not have equal or otherwise identical payoffs should be considered in comparing these two options with the purpose of establishing preference. In other words, if, for $S_\ell \in S$, $x_{i\ell}=x_{j\ell}$, then S_ℓ is not a decisive factor in the comparison. We thus define the normalized or conditional strict concordance measure, c'_{ij} as:

$$c'_{ij} = \sum_{\ell \mid S_\ell \in B_{ij}} \frac{P_\ell}{\sum_{\ell \mid S_\ell \in \{B_{ij} \cup D_{ij}\}} P_\ell} \tag{5.6}$$

The term in the denominator is the probability that the payoffs from o_i and o_j are not equal, and $\left(\frac{P_\ell}{\sum_{\ell \mid S_\ell \in \{B_{ij} \cup D_{ij}\}} P_\ell}\right)$ is the conditional probability of S_ℓ

given that a "decisive" state for o_i and o_j occurs (i.e., one where payoffs are not equal). Thus c'_{ij} can be interpreted as the conditional probability that the outcome obtained from o_i is better than that of o_j given that a decisive state occurs. Of course, the higher c'_{ij} the more desirable o_i is with respect to o_j.

We illustrate this measure with the following numerical example.

Example 5.1: Assuming six equiprobable states of nature, $S = \{S_1, S_2, S_3, S_4, S_5, S_6\}$, thus $P_j = \frac{1}{6}$, $\forall\ S_j \in S$, consider two options o_i and o_j with the following impact matrix

$$\begin{bmatrix} 100 & 150 & 120 & 160 & 200 & 170 \\ 100 & 110 & 180 & 190 & 200 & 170 \end{bmatrix}$$

In this example, $E_{ij} = E_{ji} = \{S_1, S_5, S_6\}$, and only $B_{ij} \cup D_{ij} = \{S_2, S_3, S_4\}$ are the decisive states, with a marginal probability of occurrence equal to $\frac{1}{2}$. Since $B_{ij} = \{S_2\}$, $c'_{ij} = \frac{1}{6} / \frac{1}{2} = \frac{1}{3}$, and with $B_{ji} = D_{ij} = \{S_3, S_4\}$, $c'_{ji} = \left[\frac{1}{6} + \frac{1}{6}\right] \cdot 2 = \frac{2}{3}$. □

The three alternative concordance measures considered so far (equations 4.1, 5.1 and 5.6, respectively) are equal in magnitude only if $E_{ij} = \{\phi\}$. However, generally, $c'_{ij} \geq c^{+}_{ij}$, and c'_{ij} can be obtained from c_{ij} as follows:

$$c'_{ij} = \frac{c_{ij} - p(E_{ij})}{1 - p(E_{ij})}$$

where $p(E_{ij})$ is the probability of occurrence of an element of E_{ij}, i.e., $p(E_{ij}) = \sum_{\ell \mid S_\ell \in E_{ij}} P_\ell$.

When c'_{ij} is substituted for c_{ij} in section 4.2, the resulting outranking relation and initial ($\succ_1$, $\sim_1$) possess the same general properties as with the initial specification c_{ij}. A possible difference could be that if $\bar{c}$ is set at a value greater than 0.5 (in conjunction with c'_{ij}) then it is no longer possible to have $[o_j \; R \; o_i]$ if $[o_i \; R \; o_j]$, $\forall o_i, o_j \in O$, nor to obtain indifference by $\sim_1$.[†] Thus c'_{ij} allows the decision maker or his analyst to exercise a stricter basis for accepting the preference of one options over another.[§]

In conclusion, in as much as one can control the value of $\bar{c}$, a stricter basis for preference could be imposed using a high value of $\bar{c}$ in conjunction with the initially specified c_{ij}. However, in problem situations where options results in equal payoffs under a number of states of nature, one of the strict measures above might be more appropriate.

The concordance measure is, however, only one dimension entering the comparison of options. The other, namely the discordance measure, reflecting the maximum loss of regret resulting from a "wrong" decision, is quite as important, and, as discussed hereafter, can be very effective in adapting the methodology to problems exhibiting different general characteristics.

5.1.2 The Maximum Regret or Discordance Measure

In Section 4.2, the discordance d_{ij} between o_i and o_j was defined in equation 4.2 as:

[†]The reason is that $c'_{ij} + c'_{ji} = 1$, $\forall o_i, o_j \in O$, therefore, if $\bar{c} > 0.5$, then $[o_j \; R \; o_i]$ would be impossible if $[o_i \; R \; o_j]$ since it would imply that $c'_{ij} + c'_{ji} \geq \bar{c} + \bar{c} > 1$.

[§]Note that the possibilities of using such an index, along with others to define an outranking relation are quite numerous; see, for example, Jacquet-Lagreze (1974) for a relation based on a similar strict index used in conjunction with an index of equivalence defined over E_{ij} (for multicriteria problems under certainty).

$$d_{ij} = \underset{\{k \mid S_k \varepsilon D_{ij}\}}{\text{Max}} \frac{(x_{jk} - x_{ik})}{d^{max}} \tag{5.7}$$

where d^{max} is the maximum difference in payoff between any two options in O for a given state of nature. As such, d_{ij} is the maximum loss that might occur if o_i is selected over o_j, appropriately normalized over all options pairs in O. We present here four alternative specifications for d_{ij}, which we denote as d^{ℓ}_{ij}, d^{c}_{ij}, d^{m}_{ij} and d^{r}_{ij} respectively, and briefly discuss problem situations where each might be more appropriate or desirable for defining the initial relation $(\succ_1, \sim_1)$. The basic form of all four specifications is the same, namely, it is the ratio of a quantity $\mathcal{L}_{ij}$ reflecting maximum regret due to o_i with respect of o_j, to $\mathcal{L}^{max}$, the maximum value of $\mathcal{L}_{ij}$ for all pairs; this form always yields an index in the range [0, 1]. We now give the mathematical definition for the alternative measures.

a. Likelihood Compensated Discordance Index: d^{ℓ}_{ij}

This index is specified as follows:

$$d^{\ell}_{ij} = \underset{\{k \mid S_k \varepsilon D_{ij}\}}{\text{Max}} \left[\frac{(x_{ji} - x_{ik})\ P_k}{[\Delta.P]^{max}} \right] \tag{5.8}$$

where $[\Delta.P]^{max}$ is defined as $\underset{i,j \mid o_i, o_j \varepsilon O}{\max} \left\{ \underset{k \mid S_k \varepsilon D_{ij}}{\max} [(x_{jk} - x_{ik})P_k] \right\}$, or the largest probability weighted difference on any given state between any two elements of O; of course $[\Delta.\ P]^{max}$ is the same for all pairs of options o_k, o_j ε O. The numerator is what we refer to as a likelihood compensated loss which might result from the implementation of o_i instead of o_j, and is simply the difference between the two outcomes for a given state multiplied by the probability of occurrence of that state.

The rationale for likelihood compensation is relatively intuitive in that the maximum loss for a state in D_{ij} might occur for a state of nature which has very low probability of occurrence. Thus, a more appropriate reflection of worst case performance in building R is to multiply loss for each state $S_k \in D_{ij}$ by its probability of occurrence P_k, thus providing a measure which reflects the magnitude of a loss as well as its likelihood relative to other possible losses,† as illustrated in the following example.

Example 5.2: Consider a situation with $S = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ and associated $P = \{0.25, 0.15, 0.03, 0.15, 0.22, 0.2\}$; we also have $o_1, o_2 \in O$ with the following impact matrix:

$$\begin{matrix} o_1 \longrightarrow \\ o_2 \longrightarrow \end{matrix} \begin{bmatrix} 100 & 100 & 170 & 95 & 130 \\ 100 & 120 & 90 & 155 & 90 \end{bmatrix}$$

Using equation 5.7 in computing d_{21}, we would have had the difference occurring at S_3 in the numerator, i.e., 170 - 90 = 80. However, when we realize that $P_3 = 0.03$, it might seem more appropriate to be concerned by the difference in S_6, equal to (130 - 90) = 40, but with $P_6 = 0.2$. Thus, in d^{ℓ}_{21}, we would have Max {(80 x 0.03), (40 x 0.2)} = Max {2.4, 8} = 8, in the numerator, as the basis for a preferential statement between o_1 and o_2. The denominators in both cases are, of course, a function of the rest of the options in O, and though different for each of the measures respectively, they do not affect the general conceptual issue highlighted in this example. □

It should be noted, of course, that such likelihood compensation, as in d^{ℓ}_{ij} is not necessarily desirable in all problem situations and, furthermore, might be quite misleading and dangerous in others. For instance, when a

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†See Guiguou (1971) for a similar logic in the context of multicriteria problems under certainty.

very high loss of a catastrophic nature is possible, though it has a very small probability of occurrence, it would be desirable to be aware of such a loss and have it appropriately captured in a preferential comparison, without it being unduly scaled down by multiplication by the very small probability of occurrence. Such problems usually occur in the context of public safety hazards, where very small probabilities of catastrophic failure exist, such as nuclear power plant siting and operation. Such situations might also arise in transportation problems, where some options might result in significant irreversible impact. Thus, when "catastrophic" losses might be involved, the use of d^{ℓ}_{ij} instead of d_{ij} is discouraged as it might mask some essential information. However, when high non-catastrophic losses could occur with very low probabilities, then d^{ℓ}_{ij} might be more appropriate.

b. Conditional Weighted Discordance Index: d^{c}_{ij}

The alternative definition is best motivated through the following example:

Example 5.3: Consider a decision problem with $S = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ and associated probability vector $P = \{0.2, 0.2, 0.1, 0.2, 0.1, 0.2\}$, and two options o_1 and o_2 characterized by the impact matrix:

$$\begin{array}{c} o_1 \\ o_2 \end{array} \begin{bmatrix} 100 & 150 & 150 & 100 & 100 & 125 \\ 100 & 150 & 100 & 150 & 150 & 100 \end{bmatrix}$$

It is easy to verify that $c_{12} = c_{21} = 0.7$. Each option is strictly outperformed by the other on two states of nature respectively, with $D_{12} = \{S_4, S_5\}$ and $D_{21} = \{S_3, S_6\}$; using the initial definition of d_{ij} (equation 5.7), we obtain that $d_{12} = d_{21} = 50/d^{max.}$ Moreover, simple inspection reveals

to us that the payoff from o_2 under S_4 and S_5, i.e., when o_2 outperforms o_1, exceeds o_1 by 50 for both S_4 and S_5. However, payoff from o_1 for S_3 and S_6, where o_1 would outperform o_2, exceeds o_2 by 50 in the case of S_3 (with $P_3 = 0.1$), and only by 25 in the case of S_6 (with $P_2 = 0.2$). Because it yields a lower probability of lower loss (everything else being equal), o_2 should be chosen over o_1. However, using equation 5.7 for the discordance measure in conjunction with any of the concordance measures described so far, would have yielded that $[o_1 \sim_1 o_2]$, indicating preferential indifference where there seems to be a clear case for unidirectional preference. □

We now define the conditional weighted discordance index d^c_{ij}, which would have given the procedure the discriminatory power needed to detect cases such as the one above:

$$d^c_{ij} = \frac{\sum_{k|S_k \epsilon D_{ij}} (x_{jk} - x_{ik}) \cdot P_{k|D_{ij}}}{[\bar{\Delta}_c]^{max}} \tag{5.9}$$

where $P_{k|D_{ij}}$, denotes the conditional probability of occurrence of S_k given the occurrence of an element of D_{ij}, thus

$$P_{k|D_{ij}} = \frac{P_k}{\sum_{\ell|\ell \epsilon D_{ij}} P_\ell} \quad ; \tag{5.10}$$

and $$[\bar{\Delta}_c]^{max} = \max_{i,j|o_i,o_j \epsilon O} \left\{ \sum_{k|S_k \epsilon D_{ij}} (x_{jk} - x_{ik}) \cdot P_{k|D_{ij}} \right\} .$$

The numerator of equation 5.9 is the conditional expected loss from choosing o_i over o_j given that the realized state (denoted by S*) is one where o_j outperforms o_i (i.e., $S^* \varepsilon D_{ij}$). The denominator is defined for normalizing purposes as described earlier in subsection 5.1.2.

In example 5.3 above, using equation 5.9 yields

$$d_{12}^c = (50 \times \frac{0.1}{0.3} + 50 \times \frac{0.2}{0.3}) \; / \; [\bar{\Delta}_c]^{max} = 50/[\bar{\Delta}_c]^{max} \text{ and}$$

$$d_{21}^c = (50 \times \frac{0.1}{0.3} + 25 \times \frac{0.2}{0.3}) \; / \; [\bar{\Delta}_c]^{max} = 33.3/[\bar{\Delta}_c]^{max} \quad ,$$

thus reflecting the better performance of o_2 vis a vis o_1 (everything else being equal as in this problem).

However, as in any linear additive rule, d_{ij}^c could conceal important differences between states. Therefore, in situations where very large variations in relative loss exist between states for a given pair of options, d_{ij}^c is not recommended, especially when the elements of D_{ij} have strongly dissimilar relative probabilities. In general, the initial d_{ij} is more conservative than d_{ij}^c as it gives the absolute worst case preemptive power, and as such, prevents the "dilution" of possibly unacceptable losses by other less drastic ones. Again, the salient issue here is that of the catastrophic loss, low probability situation which d_{ij} handles quite well but the other indices might not detect. On the other hand, many problems have less drastic relative loss profiles, and as such, might be more adequately represented by measures which consider more than just the worst possible loss (such as d_{ij}^c), or that compensate for likelihood of occurrence in the definition of highest loss (such as d_{ij}^{ℓ}). The next alternative measure presented here attempts to reflect both concerns respectively represented by d_{ij}^c and d_{ij}^{ℓ}; we refer to it as the marginal weighted discordance index: d_{ij}^m.

c. Marginal Weighted Discordance Index: d^m_{ij}

This measure is specified as follows:

$$d^m_{ij} = \frac{\sum_{k|S_k \varepsilon D_{ij}} (x_{ji} - x_{ik}) \cdot P_k}{[\bar{\Delta}]^{max}} \tag{5.11}$$

$$\text{where } [\bar{\Delta}]^{max} = \max_{i,j|o_i,o_j \varepsilon O} \left\{ \sum_{k|S_k \varepsilon D_{ij}} (x_{jk} - x_{ik})\ P_k \right\} \tag{5.12}$$

The main difference between this index and the previous one (d^c_{ij}) is that the probabilities used in (5.11) and (5.12) are the marginal probabilities rather than the conditional $P_{k|D_{ij}}$ used in the definition of d^c_{ij} Thus the numerator in (5.11) is the expected loss or regret of o_i with respect to o_j.† It is obtained from the numerator of (5.9) by multiplying the latter by $\sum_{k|S_k D_{ij}} P_k$, or the probability that $S^* \varepsilon D_{ij}$. In essence then, d^m_{ij} is a likelihood compensated version of d_{ij}, in the same sense that d^ℓ_{ij} is a likelihood compensated version of d_{ij}. The difference between d^m_{ij} and d^ℓ_{ij} is however similar to that between d^c_{ij} and d_{ij} in that d^c_{ij} considers all the sates in D_{ij} in the definition of the discordance index, whereas d_{ij} considers the worst one only.

Again, the same caution expressed earlier in connection with the use of d^c_{ij} in problem situations where catastrophic-like losses might exist applies here. Furthermore, d^m_{ij}, when used with the concordance index c_{ij} to define the preference-indifference relation ($\succ_1$, $\sim_1$), has some redundancy in the information it conveys. In particular, given c_{ij}, the probability that $S^* \varepsilon D_{ij}$ is simply $1 - c_{ij}$, which is also used in the defini-

†Note that we do not mean net loss here, nor do we implicitly assume that gains are negative losses.

tion of d^m_{ij}. Therefore, rather than give an independent complementary item of information on the choice situation (such as the maximum possible loss), d^m_{ij} reflects again the probability of o_j outperforming o_i. As such, d^c_{ij} might be a more effective complement to c_{ij} in defining R than d^m_{ij} would.

The next, and last, measure we discuss, introduces a dimension not present in any of the above-mentioned four indices, namely the willingness to accept a larger absolute potential loss for larger payoffs than for lesser ones.

d. Relative Discordance Index: d^r_{ij}

We motivate the definition of this measure by the following example:

Example 5.4: Consider a decision situation with $S = \{S_1, S_2, S_3, S_4\}$ and associated probability vector $P = \{0.3, 0.2, 0.2\}$, and $o_1, o_2 \in O$, characterized by the following impact matrix:

$$\begin{matrix} o_1 \\ o_2 \end{matrix} \begin{bmatrix} 1400 & 200 & 150 & 2000 \\ 1400 & 200 & 50 & 2100 \end{bmatrix}$$

It can be seen that o_1 and o_2 perform better than the other respectively on states S_3 and S_4, which have $P_3 = P_4 = 0.2$. While d_{ij} and all the other indices discussed so far would have yielded equal values of the maximum regret measure, since $(x_{13} - x_{23}) = (x_{24} - x_{14}) = 100$, it seems that having lost 100 when we receive 50 (if o_2 is selected over o, and S_3 occurs) is more "serious" than losing 100 where we receive 2000 (if o_2 is selected and S_4 occurs). In the first case, the loss is equal to 200% of the realized payoff, whereas in the second, it is only 5%. □

This suggests the following alternative specification of the regret measure:

$$d^{r}_{ij} = \max_{k|S_k \epsilon D_{ij}} \left[\frac{(x_{jk} - x_{ik}) / x_{ik}}{[\Delta_r]^{max}} \right] \tag{5.13}$$

$$\text{where } [\Delta_r]^{max} = \max_{i,j|o_i,\, o_j \epsilon O} \left\{ \max_{k|S_k \epsilon D_{ij}} \left(\frac{x_{jk} - x_{ik}}{x_{jk}} \right) \right\} \tag{5.14}$$

This specification is especially appropriate in decision problems where the range of payoffs over S of the different options is quite wide (i.e., order[s] of magnitude), whereby the ratio of relative losses is significantly different from that of absolute ones.†

As noted earlier in this section, all indices discussed above are interchangeable as far as their role in the mechanism for building the initial preference-indifference relation ($\succ_1$, $\sim_1$), described in Chapter 4, and thus do not change the general properties of the resulting relation. The differences lie in their respective preferential implications, and the different degrees to which they lead to a more faithful representation of the decision maker's preferences (by conveying relevant information) in any given decision situation.

The last point to be made concerning the discordance measures is that they are by no means mutually exclusive, as one could opt for the use of more than one index, such as the use of d^{c}_{ij} in conjunction with d_{ij}, whereby the latter guarantees that no catastrophic losses are being diluted by d^{c}_{ij},

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†Note that the use of d^{r}_{ij} effectively corresponds to the existence of a concave monotonically increasing utility function over X, the impact domain, since the marginal utility of X-units is increasing at a rate which decreases with higher values in X.

which would otherwise prevail as a basis of relative discordance between options.

5.2 A Target Oriented Approach

The motivation behind this section is the realization of the importance of the notion of "target" or "aspiration level" in many problem situations in the evaluation of transport options, and thus the desirability of including such features in the decision aiding approach pursued in this thesis. A modified evaluation rationale is thus presented herein, which extends the "concordance" and "discordance" risk measures to include this feature. The principal difference with the rationale previously introduced (in section 4.2) is that each option is characterized by its performance with respect to a common "benchmark" consisting of a target value for the desired impact or payoff from the implemented option.

In addition to incorporating this feature in constructing an initial binary preference-indifference relation ($\succ_1$, $\sim_1$) over the set O, that could be used within the methodological structure of Chapter 4, this section presents a target oriented elimination rule for use in the first stage of the decision aiding framework (of section 2.3). While we have not elaborated on the elimination stage for this subclass of problems,† because it is relatively straightforward, and exception is made for the target oriented rule presented herein, which can be viewed as a particularly useful extension of the concepts of section 4.2.

Subsection 5.2.1 elaborates further on the notion of target and its interpretation in this context, and describes the use of this feature for individual option elimination. Subsection 5.2.2 presents a target oriented approach for pairwise comparisons. The properties of the resulting binary preference indifference relation are then investigated in subsection 5.2.3, and their methodological implications briefly discussed in subsection 5.2.4. Subsection 5.2.5 offers some concluding remarks on this approach.

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†Of course, the discussion of elimination presented in Appendix A is applicable here as well.

5.2.1 Target Oriented Elimination Rule

The notion of "target" or "critical level " is germaine to theories of decision making and is featured in a number of well-known approaches. From the earlier concepts of satisficing and bounded rationality (Simon, 1955) in the decision sciences, to the more recent "process" oriented decision making theories and models in the context of multi-criteria decision making (Benayoun et al., 1971, Zeleny, 1975, 1977a), including the well-known goal programming techniques (Ignizio, 1976, Sfeir-Younis and Bromley, 1977), the concept of "target" is fundamental. Note however, that the understanding of "target" for this approach is not one of "minimum acceptable level" (as in satisficing) nor one of (usually unattainable) "ideal point" (Zeleny, 1977a), but somewhere in between. In other words, the desired target should be achievable but with uncertainty as to its realization, yet it should not be a "bare" minimum requirement that loses most of its discriminating power.

For the class of decision problems under uncertainty addressed here, the desired option is one that offers a high probability of achieving that (high) target, yet does not yield an unacceptable loss (or regret) otherwise. This trade off is operationalized hereafter.

Let x_t denote the target set in a given problem situation. In an analogous manner to the outranking relation R defined in subsection 4.2.2, we define the concordance set $C_{it} \subseteq X$ as the subset of states of nature where the impact from option o_i meets the target x_t; its complement in S, the discordance set D_{it} consists of course of those states where the impact from o_i falls short of the desired target. Mathematically,

$$C_{it} = \{S_k \in S \mid x_{ik} \geq x_t\}$$
$$D_{it} = \{S_k \in S \mid x_{ik} < x_t\}$$

A target achievement, or concordance, measure is thus computed as:

$$c_{it} = \sum_{k \mid S_k \in C_{it}} P_k \tag{5.15}$$

This measure is simply the probability that the payoff from o_i is not below x_t.[†] Alternative specifications for the concordance index, described in subsection 5.1.1, could also be used.

It is obvious that larger values of c_{it} are more desirable than lower ones. However, as in the development of the outranking relation R, another dimension needs to be captured, namely the possible loss or underachievement should the outcome S* be a state of D_{it} instead of C_{it}. A discordance measure d_{it} is thus defined as follows, as in equation (4.2).

$$d_{it} = \max_{k \mid S_k \in D_{it}} \frac{(x_t - x_{ik})}{d_t^{max}} \tag{5.16}$$

where d_t^{max} is the maximum difference between the performance of any option in O (on any state $S_k \in S$) and t, i.e., $d_t^{max} = \max_{j \mid o_j \in O} \left\{ \max_{k \mid S_k \in D_{jt}} (x_t - x_{jk}) \right\}$.

This index reflects the maximum possible underachievement of option o_i with respect to the target value, i.e., the potential regret. It is thus clear that lower values of d_{it} are more desirable than higher ones.

[†]This measure can be recognized as the "aspiration criterion" defined by Charnes and Cooper (1963) and used by Geoffrion (1967) in conjunction with stochastic programming. It is also a special case of Stone's family of risk measures described in section 3.1.

The decision rule for the elimination of an option becomes transparent when the target is viewed as a hypothetical option o_t yielding x_t under any state of nature; the risk measures thus defined can be seen to be those comparing all $o_i \in O$ to o_t by the outranking relation R defined in subsection 4.2.2. In this case, however, only one-directional outranking is of interest, i.e., whether or not o_i outranks o_t.

Denoting the subset of options reduced using the target x_t by O^t, the elimination rule can be formally stated as:

$o_i \in O^t$ if and only if $c_{it} \geq \bar{c}$ and $d_{it} \leq \bar{d}$

thus $o_i \notin O^t$ if and only if $c_{it} < \bar{c}$ or $d_{it} > \bar{d}$.

As before, $\bar{c}$ and $\bar{d}$ are user-supplied thresholds levels.† Such threshold values are often used in practice, along with the intuitive logic implied in the above decision rule, as illustrated hereafter.

Example 5.5: Consider a situation where a design standard is set requiring a proposed facility to carry vehicles at a 45 mph speed. Alternative proposals thus generated will be evaluated according to the extent that they meet this standard. However, deviations from this desired service level will usually be tolerated, whereby a proposed option will be accepted if it achieves this level 90% of the time. In this example, $x_t = 45$ and $\bar{c} = 20$, while the underachievement magnitude is not taken into account (which is equivalent to setting $\bar{d} = 1.0$). □

In summary, this elimination rule is a mere formalization of frequently used and very intuitive concepts. The interesting aspects are that:

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† Use of the same notation as in subsection 4.2.2 by no means implies that the thresholds mentioned in this case should have the same numerical value as those used in 4.2.2 to define the outranking relation R.

1) it is a two-dimensional representation of the risk of an individual option, unlike the unidimensional measures discussed in section 3.1; and 2) it follows the same logic used in the initial outranking relation of section 4.2.

5.2.2 Pairwise Comparisons Relative to Target

In this subsection, we investigate how performance vis-a-vis a target can provide a basis for defining a binary preference-indifference relation between the elements of O, for the pairwise comparisons stage of the decision-aiding framework. This is thus an alternative mechanism for defining the initial relation $(\succ_1, \sim_1)$ introduced in Chapter 4. The same notation is used here, in that $(\succ_1, \sim_1)$ does not refer to the specific relation obtained with the comparison mechanism of section 4.2.

To further place matters in perspective, the same methodological structure of Chapter 4 for the pairwise comparisons stage is adopted here. Therefore, once the initial $(\succ_1, \sim_1)$ is formed according to the new target oriented mechanism, the same issues faced earlier are present. As it turns out, the relation defined in this subsection is not necessarily consistent with the properties of partial semiorders (postulated in section 3.3), as shown in subsection 5.2.3. Thus subsection 5.2.4 relies on the results of the previous section to suggest the relation specific details of a correction strategy for removing these inconsistencies from $(\succ_1, \sim_1)$ and thus convert it to the final relation $(\succ, \sim)$, as discussed is section 4.4.

In what follows, the target-oriented mechanism for building $(\succ_1, \sim_1)$ is presented. Each $o_i \in O$ is characterized by the two risk dimensions defined in the previous section, namely c_{it} and d_{it}, as per equations 5.15 and 5.16 respectively. The problem of comparing two options,

o_i, o_j ε O can thus be viewed as a bi-criterion evaluation problem, where it is desired to jointly maximize c_{it} and minimize d_{it} over all o_i ε O. As is usually the case with such problems,† the first thing to do is to eliminate dominated (or inferior) options. In order to keep the presentation free from non-essential elements, we restrict our defintion of $(\succ_1, \sim_1)$ to the non-dominated (or efficient) subset of options.§ Furthermore, in keeping with the notation used throughout Chapter 4, we still denote by O the set of options over which $(\succ_1, \sim_1)$ is defined, even though it is assumed that dominated options have already been eliminated.

While it is theoretically feasible to assess a joint value function over "concordance" and "discordance"& and use it as a basis to induce a complete order over the elements of the efficient subset, it is not as appropriate approach for this problem, given the nature of the concordance and discordance criteria.¢ Instead, the interpretation of these criteria as risk measures, coupled with the following assumption on the preferences over the payoff X-domain, suggests a possible definition of $(\succ_1, \sim_1)$ using the same logic as in section 4.2.

As an "operational" assumption on the preferences in X-domain, payoff

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†See earlier discussion in section 4.4.

§It is implicitly assumed that this subset is sizeable and requires further reduction (prior to the Global Comparison stage of the decision aiding framework of section 2.3).

&See Keeney and Raiffa (1976), Chapter 3, for a value function approach to trade-offs under certainty, as well as a brief survey of other possible approachs; see also Dyer and Sarin (1979).

¢Those criteria are only meaningful within a specific unique problem context and do not have a consistent value in units that the decision maker is familiar with or conducts transactions with.

values exceeding the target level are considered preferentially equivalent. Formally, if $x_{ik} \geq x_t$ and $x_{jk} \geq x_t$, $\forall S_k \in S$ and $\forall o_i, o_j \in O$, then x_{ik} and x_{jk} are preferentially equivalent. An equivalence class is thus effectively defined over the range of X containing all values greater than or equal to x_t. This assumption is critically discussed in subsection 5.2.5, as it may be quite inappropriate in some situations. It is however used here to interpret the target-based concordance measures (c_{it} and c_{jt}, relative to individual options o_i and o_j respectively) into a pairwise concordance measure, reflecting the probability of outperformance, as per section 4.2. It is easy to show that this assumption allows us to state that if $c_{it} \geq c_{jt}$, then $c_{ij} \geq c_{ji}$ (where c_{ij} and c_{ji} are as defined in section 4.2).

Furthermore, a measure of pairwise discordance (reflecting relative potential regret) could be defined by considering the difference between the respective maximum possible shortfall from the target, characterizing each option.

Subsequently, the initial relation $(\succ_1, \sim_1)$ is specified as follows: given $o_i, o_j \in O$;[†]

a. $[o_i \succ_1 o_j]$ if and only if

   1. $\{(c_{it} - c_{jt}) > \delta_c$ and $0 \leq (d_{it} - d_{jt}) < \delta_d^1\}$

      or

   2. $\{0 \leq (c_{jt} - c_{it}) \leq \delta_c$ and $(d_{jt} - d_{it}) > \delta_d^2\}$;

b. $[o_i \sim_1 o_j]$ if and only if

   1. $\{0 \leq c_{it} - c_{jt} \leq \delta_c$ and $0 \leq (d_{it} - d_{jt}) \leq \delta_d^1\}$

      or

   2. $\{0 \leq (c_{jt} - c_{it}) \leq \delta_c$ and $0 \leq (d_{jt} - d_{it}) \leq \delta_d^1\}$,

---

[†]The reader is again reminded that all options in O are assumed to be non-dominated. Therefore, it is not possible to have o_i and o_j such that $(c_{it} \geq c_{jt})$ and $(d_{it} \leq d_{jt})$.

where δ_d^1, δ_d^2 and δ_c are user-supplied threshold levels.

δ_d^1 is a "rejection" threshold which, if exceeded by the difference between d_{it} and d_{jt} provides a sufficient basis for rejecting the preference that would have otherwise prevailed based on the concordance measure alone.

δ_d^2 is an "acceptance" threshold which, if exceeded by the difference $(d_{it} - d_{jt})$, provides a strong enough basis for accepting the preference of o_j over o_i, subject to the difference between c_{it} and c_{jt} not exceeding the concordance rejection threshold δ_c.†

In general, δ_d^1, δ_d^2, $\delta_c \in [0,1]$ and $\delta_d^1 \leq \delta_d^2$. Of course, o_i and o_j are declared incomparable whenever we have <u>not</u> $([o_i \succ_1 o_j]$ <u>or</u> $[o_j \succ_1 o_i]$ <u>or</u> $[o_i \sim_1 o_j])$. The decision rule underlying $(\succ_1, \sim_1)$ is summarized in Figure 5.1. The properties of this relation are discussed in greater detail hereafter.

5.2.3 Properties of Target-Oriented $(\succ_1, \sim_1)$

It is clear that $(\succ_1, \sim_1)$ as defined above is a partial relation, since incomparability is allowed. The indifference relation $\sim_1$ is by definition reflexive and symmetric; it is however not necessarily transitive, as a simple counter example can show, and is as such consistent with proposition 2.2.8.

On the other hand, the preference relation $\succ_1$ is irreflexive and antisymmetric (since $[o_i \succ_1 o_j] \Rightarrow$ <u>not</u> $[o_j \succ_1 o_i]$). It is however not an order because it is not necessarily transitive. Intransitivities occur primarily as a result of the threshold concepts used in defining this

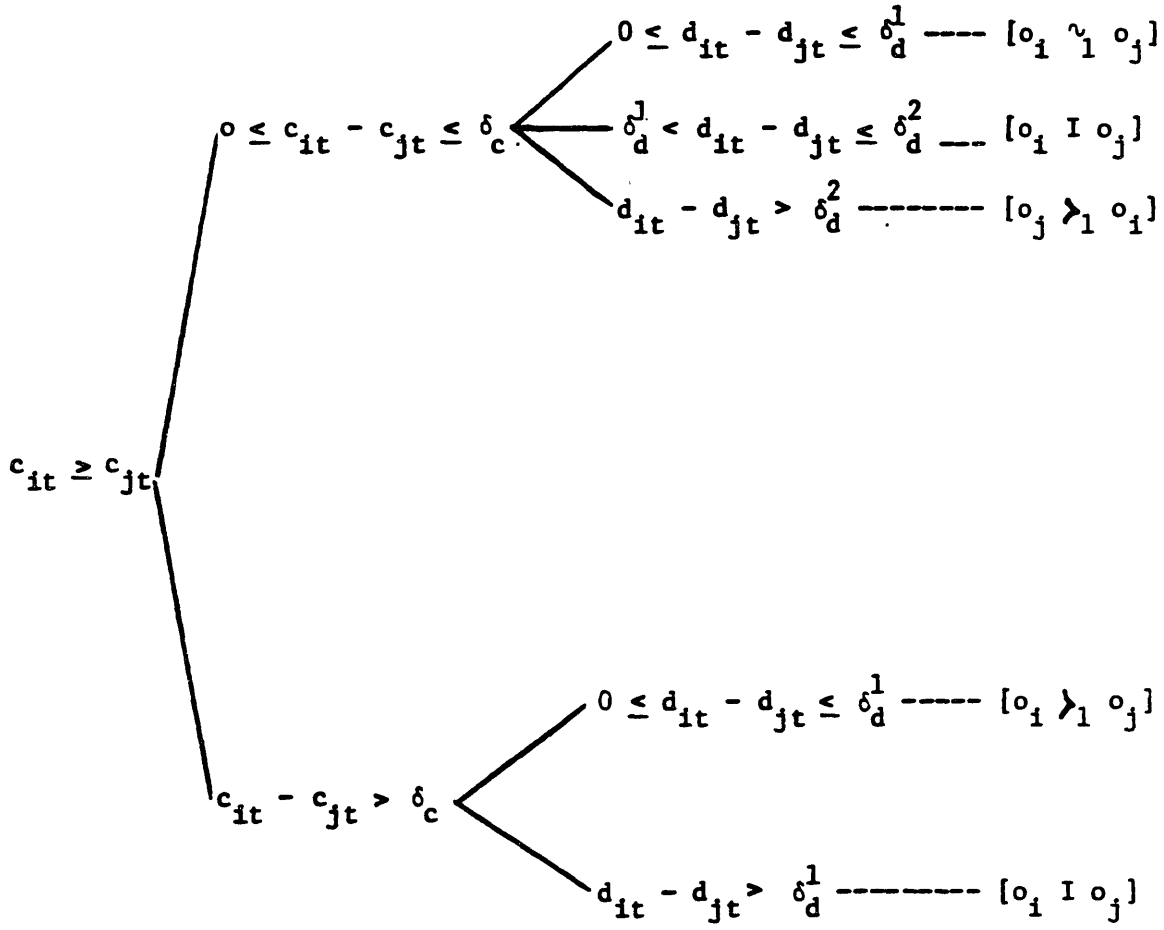
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†A family of such thresholds is discussed by Roy (1977) in the context of multiple criteria decision aids.

Figure 5.1: Decision Rule for Target Oriented $(\succ_1, \sim_1)$

Given:

$$\{o_i, o_j \in O \mid \text{not } (c_{it} \geq c_{jt} \text{ and } d_{it} \leq d_{jt})\}$$



Legend:

$\succ_1$: Strict Preference

$\sim_1$: Indifference

I: Incomparability

relation.† However, it is possible to prove an important result concerning the type of intransitivities encountered. In particular, keeping in mind the discussion in subsection 4.2.4, we show that no obstacles arise in the face of the transitive closure of $\succ_1$. In particular, we prove that only incomparability can exist when preference implied by transitivity does not hold.§

LEMMA: If $\{[o_i \succ_1 o_j]$ and $[o_j \succ_1 o_k]\}$, then either $\{[o_i \succ_1 o_k]$ or $[o_i \; I \; o_k]\}$.

PROOF: From Figure 5.1,

$$[o_i \succ_1 o_j] \Rightarrow \begin{cases} 0 \leq c_{jt} - c_{it} \leq \delta_c \text{ and } d_{jt} - d_{it} > \delta_c^2 & \text{(Condition I.1)} \\ \text{OR} \\ c_{it} - c_{jt} > \delta_c \text{ and } 0 \leq d_{it} - d_{jt} \leq \delta_d^1 & \text{(Condition I.2)} \end{cases}$$

$$[o_j \succ_1 o_k] \Rightarrow \begin{cases} 0 \leq c_{kt} - c_{jt} \leq \delta_c \text{ and } d_{kt} - d_{jt} > \delta_d^2 & \text{(Condition II.1)} \\ \text{OR} \\ c_{jt} - c_{kt} > \delta_c \text{ and } 0 \leq d_{jt} - d_{kt} \leq \delta_d^1 & \text{(Condition II.2)} \end{cases}$$

We are interested in the relationship between (c_{it} and c_{kt}) on one hand and (d_{it} and d_{kt}) on the other.

If conditions I.1 and II.1 hold, then the inequalities or the concordance index yield $0 \leq c_k - c_i \leq 2\delta_c$, while the inequalities on the discordance measure yield

$$d_{kt} - d_{it} > 2\delta_d^2 > \delta_d^2 \Rightarrow \{\text{not } [o_k \succ_1 o_i] \text{ and not } [o_k \sim_1 o_i]\}.$$

---

†As such, intransitivities constitute the potential price that the analyst pays in order to obtain an intuitively appealing and flexible decision aid.

§As discussed in subsection 4.2.4, incomparability is not considered an inconsistency in the same way as indifference or especially preference in the "opposite" direction.

Furthermore, depending on the value of $(c_{kt} - c_{it})$, then either $[o_i \succ_1 o_k]$ (for $0 \leq (c_{kt} - c_{it}) \leq \delta_c$) or $[o_i \; I \; o_k]$ (for $\delta_c < (c_{kt} - c_{it}) \leq 2\delta_c$).

If conditions I.2 and II.1 hold, we obtain that ($c_{kt} < c_{it}$ and $d_{it} < d_{kt}$), which means that o_i dominates o_k, which is impossible by the very definition of the domain of $(\succ_1, \sim_1)$. Thus conditions I.2 and II.1 are mutually exclusive.

If I.1 and II.2 hold, then ($c_{it} > c_{kt}$ and $d_{it} < d_{kt}$), thus proving that I.1 and II.2 are mutually exclusive, by the same argument as above.

Lastly, if I.2 and II.2 hold, then $c_{it} - c_{kt} > 2\delta_c$ ($\Longrightarrow$not $[o_k \succ_1 o_i]$) and $0 \leq d_{it} - d_{kt} \leq 2\delta_d^1$; again, depending on the value of the latter quantity, we can obtain that either $[o_i \succ_1 o_k]$ (for $d_{it} - d_{kt} \leq \delta_d^1$) or $[o_i \; I \; o_k]$ (for $\delta_d^1 < d_{it} - d_{kt} \leq 2\delta_d^1$). The proof is thus complete. □

This result has significant bearing on the strategy to be followed for detecting and correcting inconsistencies with any desired properties of the resulting preference-indifference relation, as discussed in subsection 5.2.4. This proof can be extended to show that in general no type 1 cycles can occur in the above relation.†

This result does not however rule out the occurrence of other types of cycles, especially since the indifference relation $\sim_1$ is not transitive. Recalling properties P3 and P4 of partial semi-orders (defined in section 3.3), it can be shown that violations of both properties can indeed occur, leading to the formation of cycles in $G(\succ_1, \sim_1)$. We support this contention hereafter by presenting a case where violations of P3 could occur.

Example 5.6:

Consider o_i, o_j, o_k, $o_\ell \in O$, such that $[o_i \succ_1 o_j]$, $[o_j \sim_1 o_k]$ and $[o_k \succ_1 o_\ell]$. P3 of course stipulates that $[o_i \succ_1 o_\ell]$ should hold as well.

†Types of cycles are defined in section 4.3; type 1 cycles are preference cycles without indifference pairs.

Of particular concern are situations where $[o_i \sim_1 o_\ell]$ or $[o_\ell \succ_1 o_i]$ instead of $[o_i \succ_1 o_\ell]$.† We show that if three sufficient conditions for $[o_i \succ_1 o_j]$, $[o_j \sim_1 o_k]$ and $[o_k \succ_1 o_\ell]$ respectively hold simultaneously, then it is possible that $[o_i \sim_1 o_\ell]$. These three conditions are:

C1. $c_{it} - c_{jt} > \delta_c$ and $0 \leq d_{it} - d_{jt} \leq \delta_d^1$ $(\Rightarrow [o_i \succ_1 o_j])$

C2. $0 \leq c_{kt} - c_{jt} \leq \delta_c$ and $0 \leq d_{kt} - d_{jt} \leq \delta_d^1$ $(\Rightarrow [o_j \sim_1 o_k])$

C3. $0 \leq c_{\ell t} - c_{kt} \leq \delta_c$ and $d_{\ell t} - d_{kt} > \delta_d^2$ $(\Rightarrow [o_j \succ_1 o_k])$

From C1 and C2 above, we obtain

C4. $0 \leq c_{kt} < c_{it}$ and $d_{it} - d_{kt} \leq \delta_d^1$,

which, in turn, coupled with C3 yields:

C5. $0 \leq c_{\ell t} - c_{it} < \delta_c$ and $d_{\ell t} - d_{it} > \delta_d^2 - \delta_d^1$

Since we have that $\delta_d^2 \geq \delta_d^1$, then $d_{\ell t} - d_{it} > \delta_d^2 - \delta_d^1 \geq 0$, thus $d_{\ell t} > d_{it}$. Since $0 \leq c_{\ell t} - c_{it} < \delta_c$, the value of the difference $(d_{\ell t} - d_{it})$ determines the relation between o_ℓ and o_i.

If $d_{\ell t} - d_{it} \leq \delta_d^1$, which is feasible, then $[o_i \sim_1 o_\ell]$, which is a violation of property P3, and can thus lead to an inconsistency cycle in G $(\succ_1, \sim_1)$.

Consider for instance the following numerical illustration:

$c_{it} = 0.72$	$d_{it} = 0.5$
$c_{it} = 0.69$	$d_{jt} = 0.47$
$c_{kt} = 0.71$	$d_{kt} = 0.48$
$c_{\ell t} = 0.73$	$d_{\ell t} = 0.53$

Assume further that for this problem, $\delta_c = 0.03$, $\delta_d^1 = 0.03$ and $\delta_d^2 = 0.04$. It is easy to see that the values above satisfy conditions

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†The same definition and rationale of inconsistencies as in Chapter 4 apply here. Thus $[o_i \; I \; o_\ell]$ is not considered an obstacle in the face of the P3 closure of a relation.

C1, C2 and C3. Furthermore, we have that $c_{\ell t} - c_{it} = 0.01 < \delta_c$ and $d_{\ell t} - d_{it} = 0.03 \leq \delta_d^1$, implying that $[o_i \nu_1 o_\ell]$, thus defining a type 2 cycle, which was found in section 4.3 to be inconsistent with the properties of partial semi-orders. □

This subsection has discussed the properties of the target-oriented binary relation $(\succ_1, \nu_1)$ defined in the previous subsection, and concluded that inconsistency cycles, as defined in section 4.3, with the exclusion of type 1 cycles, could occur in the graph of $(\succ_1, \nu_1)$.

5.2.4 Implications for Correcting Inconsistencies

The problem from this point on is essentially the same as the one addressed in section 4.3, i.e., an initial partial relation $(\succ_1, \nu_1)$ has been established with properties that may or may not be acceptable to the decision-maker and/or his counsel. According to the framework of Figure 4.12, we face the methodological choice of how to proceed beyond establishing $(\succ_1, \nu_1)$ over O. The same discussion of sections 4.3 and 4.4 applies here as well. The only element which can benefit from further specification concerns correction strategies aimed at removing inconsistencies from $G(\succ_1, \nu_1)$ so as to be compatible with the properties of partial semi-orders.

The same three strategies described in section 4.4 are thus applicable here, namely:

1. assuming all cycles to be equivalence classes
2. identifying "weakest link" in each cycle and declaring its end nodes incomparable
3. declaring all elements connected by a cycle as mutually incomparable.

Only the second strategy requires procedures whihc are specific to the particular relation under consideration. In particular the definition

of the "weakest link" in a cycle in G ($\succ_1$, $\sim_1$) needs to be specified. Recalling the definition given in section 4.4.2, the "weakest" or critical link in a cycle is the one with the highest risk of misrepresenting the decision-maker's preferences. An interpretation of this definition is thus sought in the context of ($\succ_1$, $\sim_1$).

In the decision rule of ($\succ_1$, $\sim_1$), the assertion of $[o_i \sim_1 o_j]$ can be recognized as being prone to preference misrepresentation, since it posits preferential indifference in situations where incomparability might have been more appropriate. Moreover, since no type 1 cycles (i.e., without indifference pairs) can occur in G ($\succ_1$, $\sim_1$), a strong case can be made to consider option pairs connected by $\sim_1$ the likely candidates for "weakest link."

Furthermore, among those likely candidates in a given cycle, the magnitude of the difference in the target achievement measure (i.e., concordance index) between the two connected options provides the basis for further identifying the link(s) to be removed. Thus, given o_i, o_{i+1} o_j, o_{j+1}, all part of an inconsistency cycle in G($\succ_1$, $\sim_1$), with $[o_i \sim_1 o_{i+1}]$ and $[o_j \sim_1 o_{j+1}]$, the indifference pair (o_i, o_{i+1}) is considered "weaker" than (o_j, o_{j+1}) if and only if $|c_{it} - c_{i+1,t}| > |c_{jt} - c_{j+1,t}|$.[†] The rationale is that the larger that difference, the less reason there is for the two options to be indifferent preferentially, for a given value of δ_d^1.[§]

In summary, given an inconsistency cycle C = $\{o_1, o_2, \ldots o_\ell\}$, with

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[†]Of course, since $[o_1 \sim_1 o_{i+1}]$ and $[o_j \sim_1 o_{j+1}]$, we have that $|c_{it} - c_{i+1,t}| \leq \delta_c$ and $|c_{jt} - c_{j+1,t}| \leq \delta_c$, according to the rules of Figure 5.1.

[§]It is assumed in this operational rule that the dichotomy of threshold levels (δ_d^1 and δ_d^2) for the discordance measure allows a more faithful representation of preferences than the single threshold δ_c.

$o_\ell = o_1$, in G ($>_1$, $\sim_1$), we search for the option pair(s) $(o_i, o_{i+1}) \in C \times C$ such that $[o_i \sim_1 o_{i+1}]$ and $|c_{it} - c_{i+1,t}| = \max_{o_j \in C} |[o_j \sim_1 o_{j+1}]| c_{jt} - c_{j+1,t}|$, where the maximum operator is taken over all indifference pairs in C. The pair(s) of options thus obtained is then declared incomparable, thus removing the inconsistency with the properties of partial semi-orders, and forming the final relation $(>,\sim)$.

5.2.5 Concluding Comments on Target-Oriented Approach

A number of comments are in order concerning the above target-oriented binary preference-indifference relation. In particular, questions as to why such an approach is at all needed (that is in addition to or as alternate for the more soundly anchored and carefully motivated binary relation developed in section 4.2), and how it is different from that relation of section 4.2.

Three significant yet interrelated points speak against the target-oriented approach vis a vis the previous one:

a. It is critically dependent on the target level; changing that target (x_t) yields a different relation altogether, with possibility of preference reversal at higher target levels.

b. The assumptions made in subsection 5.2.3 concerning preferences over the payoff domain, whereby all values exceeding the target form one equivalence class, may be wholly inappropriate in some situations.

c. It uses a limited portion of the information on relative performance of each option on each state $S_k \in S$, and does not directly compare options on each state, beyond whether or not they exceed the target level x_t. In that sense, it transmits less information to the decision-maker and increases the risk of making an inappropriate selection.

The relation of section 4.2 does not possess these limitations, and is indeed generally more appropriate for pairwise comparisons than the target-oriented approach. However, there are situations where the latter would nevertheless be useful, namely:

a. when the above-mentioned assumption of preferential indifference for all values in X greater than x_t is for all practical purposes, valid;

b. when only limited information is available;

c. in the case where the options $o_i \in O$ consist of (probabilistically) independent density functions defined over a range of X. The extension of the techniques developed in this section to that case are relatively straightforward, as shown in section 5.3 hereafter.

Note that the above comments apply only to the binary preference-indifference relation presented in this section, and not to the sequential elimination rule described in subsection 5.2.1. For the latter case, the target-oriented approach is quite appropriate and remains a powerful and intuitively appealing tool for the elimination stage of the decision aiding framework, and as such could be used to reduce the set of options prior to using any binary preference-indifference relation (in particular, that of Chapter 4).

5.3 Generalization to the Case of Continuous Distributions

This section generalizes the applicability of the pairwise comparisons methodology developed in Chapter 4, along with its major variations presented earlier in this chapter, to problem subclass 2 (defined in section 1.4). In this subclass, each option is characterized by a (known) continuous probability density function over the impact domain. The methodological structure of Chapter 4 is applicable to this subclass as well; thus the principal task in this section is the specification of a binary preference indifference relation which can then be integrated within that structure. In particular, the contribution of this section is to adapt the mechanism of section 4.2, articulated in the context of the more apparent problem subclass 1, where uncertainty is represented through discrete states of nature, to the less intuitive continuous case of subclass 2. This is primarily achieved by appropriately respecifying the risk measures used within that mechanism. Though straghtforward mathematically, some conceptual issues have to be resolved in the process, in particular with the measure of potential regret, i.e., discordance.

The next subsection presents the formal definition of problem subclass 2, followed in subsection 5.3.2 by a specification of the risk measures for the relation described in section 4.2. A straightforward generalization for the "target oriented" approach of the previous section is presented in subsection 5.3.3, which concludes this section.

5.3.1 Problem Formulation

The problem consists of selecting a most preferred option o*, where the decision space consists of a finite set of n options $O = \{o_1, o_2, \ldots, o_m\}$.

The impact x_i^* that would result from the implementation of o_i is not known with certainty and is therefore represented as a random variable following a known (objectively or subjectively assessed) probability density function f_i, defined over the impact space X (usually $\mathbb{R}$, the set of real numbers).

The random variables x_i^*, for $i = 1, \ldots, n$, are not, in the general case, independently distributed.† The joint (or compound) probability density functions of all pairs x_i^*, x_j^*, for all o_i, $o_j \in O$, are thus assumed to be known; $f_{i,j}$ denotes the bi-variate probability density function of the pair $[x_i^*, x_j^*]$, defined over $X \times X$

Additional notation used in this section includes:

F_i = cumulative distribution function of x_i^*

$F_{i,j}$ = joint cumulative distribution function of (x_i^*, x_j^*)

$F_{i|x_j^*=x}$ = conditional cumulative distribution function of x_i^* given that the random variable x_j^* has taken the actual value x.

$f_{i|x_j^*=x}$ = conditional pdf of x_i^* given $x_j^* = x$.

A fundamental difference exists between this formulation and the discrete states problem (subclass 1) in so far as building the preference-indifference relation is concerned. In the latter, the experimental outcome consists of a state of nature which determines not only the payoff from the selected option, but also the forgone payoff from the unselected ones. This results in unambiguous conditional preference between any two options given the state of nature, thus giving regret-based risk measures a clearer and more direct significance than in the general case. This

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†This point can be seen very clearly in the special case of the "discrete states" formulation (1), where the impact of a given option is conditional upon the realization of an exogenous state of nature.

point is apparent when interpreting and respecifying the risk measures underlying the decision rule of section 4.2, but it is less pronounced in the target oriented approach, as seen hereafter.

5.3.2 Adapting the Pairwise Regret Based Risk Measures

In the introduction, respecification of the two risk dimensions of section 4.2 was identified as the principal task needed for the generalization of the methodology of Chapter 4 to this problem.

Concordance Measure

Given two options o_i, $o_j \in O$, the concordance measure c_{ij} was defined in Equation 4.1 as the probability of no regret resulting from the implementation of o_i (relative to o_j). As such, it is quite straightforward to adapt in this case, especially since it was already introducted in section 3.2 for the general case in conjunction with the critique of the independence axiom.

The measure c_{ij} is thus the probability that the value taken by the random variable x_i^* exceeds, or is equal to x_j^*; mathematically,

$$c_{ij} = p(x_i^* \geq x_j^*) = \int_X F_{j|x_i^*=x}(x) \cdot f_i(x)dx \qquad (5.17)$$

where the terms are as defined above.

In general, the conditional cumulative distribution function of x_j^* given x_i^*, $F_{j|x_i^*}$, is such that $F_{j|x_i^*} = \frac{F_{i,j}}{F_i}$. Of course, when the two random variables x_i^* and x_j^* are independent, then $F_{j|x_i^*} = F_j$, and equation (5.17) becomes

$$c_{ij} = \int_X F_j(x) \cdot f_i(x)dx \qquad (5.18)$$

It is of course possible to similarly express the alternative functional forms presented in section 5.1. For instance, the strict concordance measure c^{+}_{ij} defined in equation 5.1, simply becomes:

$$c^{+}_{ij} = p(x^{*}_{i} > x^{*}_{j}) = \int_{X} [1 - F_{i|x^{*}_{j}=x}(x)] . f_{j}(x) dx$$

$$= 1 - \int_{X} F_{i|x^{*}_{j}=x}(x) . f_{j}(x) dx \qquad (5.19)$$

The normalized strict concordance measure, c'_{ij}, defined in equation (5.), also becomes

$$c'_{ij} = p(x^{*}_{i} > x^{*}_{j} \mid x^{*}_{i} \neq x^{*}_{j}) = \frac{[p(x^{*}_{i} > x^{*}_{j})]}{[1 - p(x^{*}_{i} = x^{*}_{j})]}$$

$$= \frac{1 - \int_{x} F_{1|x^{*}_{j}=x}(x) \cdot c_{j}(x) dx}{1 - \int_{x} f_{i,j}(x,x) dx} \qquad (5.20)$$

The interpretation of these measures in the general case is essentially the same as in the discrete states case and will not be duplicated here. However, this is not the case for the potential regret, or discordance measure as shown hereafter.

Discordance Measure

The initial specification of that measure in section 4.2 (equation 4.2) expressed the maximum potential regret that might result from the preference of a given option over another. In the discrete states formulation, that maximum occurred under a particular state of nature. Such a definition

is however not meaningful in the continuous case, especially if one of the random variables is not bounded. Even if both random variables are within bounded ranges, it is not clear that a maximum possible regret (defined as the difference between the upper bound of one variable and the lower bound of another) is an appropriate measure for this case, as the probability of such a worst case occurence would typically be barely significant.†

Rather than consider the worst performance at a point (which was very meaningful in the discrete states case), it might be more appropriate to consider a definition of regret across a continuum, as in the conditional weighted discordance index, d^c_{ij}, defined in subsection 5.1.2 (equation 5.9). This measure reflects the conditional expected regret from selecting o_i given that regret occurs, i.e., that $x^*_j > x^*_i$.

Keeping in mind the general ratio form of the discordance measures, described in subsection 5.1.2, we specify the numerator Δ^c_{ij} of d^c_{ij} .

Letting z^*_{ij} denote the random variable equal to$(x^*_j - x^*_i)$, we are interested in the conditional mathematical expectation of z^*_{ij} given that $z^*_{ij} > 0$, i.e., that $x^*_j > x^*_i$. Denoting the pdf of z^*_{ij} by $f_{z^*_{ij}}$, we have that:

$$f_{z^*_{ij}} = \int_X f_{i,j}(x,\ x+y)dx$$

and the conditional pdf of z^*_{ij} given that $z^*_{ij} > 0$ is:

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†Cases where such a worst case are of interest would typically be formulated as a strategic planning problem with discrete states of nature, where one of the states would exhibit this worst case behavior.

$$f_{z^*_{ij} | z_{ij}>0} = \frac{\int_X f_{i,j}(x,\ x+y)dx}{P(x^*_j > x^*_i)}$$

$$= \frac{\int_X f_{ij}(x,\ x+y)dx}{1 - \int_X F_{j|x^*_i}(x) \cdot f_i(x) \cdot dx} \qquad (5.21)$$

By inspection, the denominator of equation 5.21 is equal to $1 - c_{ij}$ (see equation 5.17), which yields:

$$f_{z^*_{ij} | z^*_{ij}>0} = \frac{1}{1 - c_{ij}} \int_X f_{i,j}(x,\ x+y)dx \qquad (5.22)$$

The expression for Δ^c_{ij} is therefore:

$$\Delta^c_{ij} = \frac{1}{1 - c_{ij}} \int_0^{+\infty} y \int_X f_{i,j}(x, x+y)dxdy \qquad (5.23)$$

Following the general form of subsection 5.1.2, we denote the denominator of d^c_{ij} by $[\bar{\Delta}_c]^{max}$, whereby $[\bar{\Delta}_c]^{max}$ is the maximum value of Δ^c_{ij} across all pairs (o_i, o_j) in O, i.e.,

$$[\bar{\Delta}_c]^{max} = \max_{i,j | o_i, o_j \in O} \{\Delta^c_{ij}\} \ .$$

Thus, for problem subclass 2, d^c_{ij} can be specified as:

$$d^c_{ij} = \frac{1}{[1 - c_{ij}]\ [\bar{\Delta}_c]^{max}} \int_0^{\infty} y \int_X f_{i,j}\ (x,\ x+y)dxdy \qquad (5.24)$$

It should be noted that the above risk measures can be substituted for their counterparts in section 4.2 to build an initial binary preference indifference relation which can then be integrated within the methodological structure of Chapter 4.

5.3.3 Adapting the Target Oriented Approach

The extension of the target oriented approach to the general case is relatively more straightforward than that of the relation specified in section 4.2. In the target oriented approach, options are characterized individually, vis-a-vis the target, regardless of other options in the choice set. However, while the target achievement (concordance) measure is directly generalizable, the discordance measure raises the same issues faced in the preceding subsection.

Given an option $o_i \in O$, the target achievement measure c_{it}, defined in equation 5.15 for the discrete states case, can be respecified as follows:

$$c_{it} = p(x_i^* \geq x_t) = \int_{x_t}^{+\infty} f_i(x)dx \qquad (5.25)$$

where x_t is the target level, as defined in the previous section.

The definition of the maximum shortfall from the target, or discordance, is however, not as simple. Note that the same comment made earlier concerning bounded random variables applies here. However, in this case, the use of the relative difference between the target and the lower bound on x_i^* may not be totally inappropriate, especially if the bounds are realistic for the problem under consideration. However, for the general case, a definition parallel to the one described in the previous subsection is presented.

Given option $o_i \in O$, the target-oriented conditional weighted discordance index, denoted by d_{it}^c, reflects the conditional expected shortfall from the target level x_t given that $x_i^* < x_t$.

It is easy to show that the numerator Δ_{it}^c of d_{it}^c is then given by:

$$\Delta_{it}^c = \frac{1}{1 - c_{it}} \cdot \int_0^{+\infty} y \cdot f_i(x_t - y) \cdot dy \qquad (5.26)$$

The denominator $[\bar{\Delta}^c]^{max}$ is again defined as $\max_{i|o_i \, O} \{\Delta_{it}^c\}$, yielding the following expression for d_{it}^c :

$$d_{it}^c = \frac{1}{[1 - c_{it}]\,[\bar{\Delta}^c]^{max}} \int_0^{+\infty} y \cdot f_i(x_t - y) \cdot dy \qquad (5.27)$$

5.4 Extension to Multicriteria Decision Problems Under Uncertainty

This section addresses problem subclass 3, where options are explicitly evaluated with respect to their uncertain impact along a multiplicity of criteria. Most of the concepts, results and procedures developed in the previous chapters are applicable to this subclass as well, including the underlying basic considerations and the overall decision aiding framework (described in Chapter 2). Furthermore, in that framework, the details of the elimination stage were developed and described in Appendix A in the context of this subclass of problems. As stated in section 1.4, it may be possible in many problem situations to satisfactorily deal with the multiplicity of criteria through elimination, leaving only one as a basis for pairwise comparisons, which is the problem directly treated in the previous chapter.

However, even when more than one criterion forms the basis of the pairwise comparisons, the methodological structure described in chapter 4 is directly applicable. The principal problem-specific element needed is the mechanism for building the initial preference-indifference relation over the set of options. It is naturally possible to develop and use various such relations, which may nonetheless be consistent with our set of basic considerations. The extent of our development in this chapter is however limited to illustrating how the basic mechanism introduced in section 4.2, and extended throughout the previous sections, may be applicable to the MC case.

Having placed this section in its proper perspective, the mathematical problem formulation is presented in subsection 5.4.1, primarily to reintroduce the necessary notation. The pairwise comparisons methodology is then pursued, with section 5.4.2 defining an initial preference-indifference relation which integrates the relation of section 4.2 within a lexicographic

model. The properties of the resulting relation are then discussed in section 5.4.3, followed by their methodological implications, particularly with regards to strategies for correcting inconsistencies. Concluding remarks and a summary of this section are presented in subsection 5.4.4.

5.4.1 Multicriteria Discrete States Problem

In this section, the mathematical problem formulation is presented, followed by a brief discussion of major solution approaches.

As stated in section 1.4, the multicriteria discrete states problem is defined as follows. Given a Pareto-optimal or non-dominated set of n options $O = \{o_1, o_2, \ldots, o_n\}$, the problem is to select a most preferred option $o^* \varepsilon O$. Uncertainty is represented through a set of m mutually exclusive, collectively exhaustive states of nature $S = \{S_1, S_2, \ldots, S_m\}$, with known probabilities of occurrence $P = \{P_1, P_2, \ldots, P_m\}$, where $P_j \varepsilon P$ denotes the probability of occurrence of state $S_j \varepsilon S$. It is again assumed that the P_j's are independent of which option $o^* \varepsilon O$ is selected.

The impact of each option is expressed by a vector of impacts along p criteria $C = \{C_1, C_2, \ldots, C_p\}$, conditional upon the realization of a state of nature. Let $X_{ij} = \{x_{ij}^1, x_{ij}^2, \ldots, x_{ij}^p\}$ denote the conditional impact of o_i given the realization of $S_j \varepsilon S$, where x_{ij}^k is the impact of o_i along criterion C_k should S_j occur. A given option $o_j \varepsilon O$ is thus characterized by the m-tuple $\bar{X}_i = \{X_{i1}, X_{i2}, \ldots, X_{im}\}$.

The practical relevance of this formulation was discussed in Chapter 1.

The principal formal approach for dealing with this and other multicriteria evaluation problems under uncertainty is multiattribute utility theory (MAUT), which satisfies the Expected Utility model discussed in Chapter 3. Since MAUT relies on VN-M's axioms, the critical discussions

of EUT in Chapter 3 is applicable here as well. In addition to those axioms, MAUT makes further restrictive assumptions on the decision-maker's preferences, in the form of different independence axioms, in order to obtain operationally tractable functional forms. A review of this vast body of literature is outside the scope of this chapter; for further details, see Fishburn and Keeney (1974), Keeney and Raiffa (1976), Fishburn (1977), Farquhar (1977) or Yilmaz (1978), among others. A critical review of MAUT relevant to the premise of this thesis can be found in Schoemaker (1980).

Also based on EUT are extensions of stochastic dominance (described in subsection 3.1.3) to the multivariate case, for which only few operationally useful results exist (Fishburn, 1977; Fishburn and Vickson, 1978). A [illegible]unification of stochastic dominance relations, set in an interactive format, was recently described by White and Sage (1981). However, relatively little else outside the framework of MAUT is available for MC evaluation problems under uncertainty.

The principal difference between this formulation and problem subclass 1 is of course that it is no longer possible to establish a complete conditional order over the set O under each of the possible states of nature. This is a result of the multiplicity of criteria along which this conditional impact is expressed; of course, by collapsing these criteria into a single index, the problem is recast as a single criterion problem to which the ad hoc approaches discussed in subsection 4.2.1 are applicable. We proceed hereafter to show the applicability of the decision aiding framework of section 2.3 and the subsequent methodological development articulated in the context of the single criterion case to the MC problem.

5.4.2 Pairwise Comparisons: Defining An Initial Binary Relation

As stated in the introduction to this section, the same methodological structure for pairwise comparisons developed in Chapter 4 is applicable here.† This subsection addresses the first component in that structure, which consists of building an initial binary preference-indifference relation over the options set O. In particular, its purpose is to show how the measures and associated evaluation rationale developed for the single criterion case, in section 4.2, can be used in the presence of multiple criteria.

It should first be noted that we assume properties of the preference-indifference relation over the options set O to be consistent with those of partial semiorders, as postulated in section 3.3, and for the same reasons advanced there. This relation therefore allows incomparability of options and does not require transitivity of indifference.

Further assumptions specific to the relation developed herein are next presented, along with the rationale underlying the binary decision rule, followed by the mathematical description of its elements.

Comparison Rationale

The preference-indifference relation is predicated upon the existence of a hierarchy over the set C of criteria. When comparing two options o_i and o_j, it is assumed that o_i cannot be preferred to o_j unless it is either better than or indifferent to o_j on the "most important" criterion. We thus assume the comparison to proceed lexicographically, whereby a given

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†The reader is referred to subsection 4.4.1 for an overview of the methodological structure; see also Figure 4.12.

criterion comes into play only if indifference exists on the more important criteria.†

The mechanism for comparing o_i and o_j thus starts with the most important criterion. If preference of one option over another can be asserted with respect to that criterion, then lower ranked criteria are not considered. However, if indifference exists on the first criterion, then comparison with respect to the second criterion becomes the basis of option-wise overall comparison. Noting that the impact of an option along any criterion is not known with certainty, the special feature of this model (vis-a-vis the standard lexicographic decision model [Fishburn, 1979]) is to allow incomparability along a particular criterion, which is translated into option-wise overall incomparability.

The choice situation faced in comparing two options along a particular criterion can be seen to be analogous to the single criterion discrete states problem (i.e., subclass 1) addressed in Chapter 4 and in the previous sections of this chapter. The measures and decision rules introduced in section 4.2 (and extensions in Chapter 5) are thus applicable to it, as described hereafter.

Mathematical Specification

The problem is to define an initial partial binary preference-indifference relation, denoted by $(\succ_1, \sim_1)$, over the set O. As before, $\succ_1$, denotes strict preference, whereas $\sim_1$, denotes preferential indifference.

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†The lexicographic idea in decision models is described in a number of references, including Fishburn's (1974) excellent review article offering both pragmatic and mathematical perspectives on the subject. Early advocates include Georgescu-Roegen (1954, 1968), Encarnación (1964).

Incomparability of options by $(\succ_1, \sim_1)$ is denoted by I. Therefore, given $o_i, o_j \in O$, we describe here the mechanism used to establish which one of the following holds: 1) $[o_i \succ_1 o_j]$; 2) $[o_j \succ_1 o_i]$; 3) $[o_i \sim_1 o_j]$, or 4) $[o_i \; I \; o_j]$.

Let r denote the lexicographic rank of a particular criterion, with r = 1 for the most important criterion and r = p for the least important. The preference-indifference relation over O, based on the r-th ranked criterion only, is denoted by $(\succ_1, \sim_1)$, where, for a given $o_i, o_j \in O$:

- $[o_i \succ'_r o_j]$ or $[o_j \succ'_r o_i]$ indicates preference based on a comparison between the respective impact of o_i and o_j along the r-th ranked criterion, given that this impact is conditional upon the realization of one of a set of mutually exclusive and collectively exhaustive states of nature S. In other words, it is a comparison between the respective marginal probabilty mass functions of the impact of o_i and o_j along the r-th ranked criterion.†

In addition, $[o_i \; I'_r \; o_j]$ indicates incomparability of o_i and o_j by the relation $(\succ'_r, \sim'_r)$.

As we said earlier, $(\succ'_r, \sim'_r)$, for r = 1, ..., p, can be constructed using the procedure of section 4.2, as shown in figure 4.3. The details associated with the redefinition of the risk measures in that section are relatively straightforward and are left to the end of this subsection, following the remainder of the description of the mechanism for using the r-th criterion relations $(\succ'_r, \sim'_r)$ to establish the option-wise

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†if C_k is the r-th ranked criterion, then the marginal probability mass function of o_i's impact along the r-th ranked criterion consists of a probability P_ℓ of obtaining $x^k_{i\ell}$, for $\ell = 1, \ldots, m$. Note however that the impacts of o_i and o_j are jointly conditioned upon the realization of a state of nature $S^* \in S$.

overall relation ($_1$, $\sim_1$).[†]

We first give a methematical definition of the preference-indifference relation ($_1$, $\sim_1$).

Definition 5.1: The partial binary preference-indifference relation ($\succ_1$, $\sim_1$) is defined over the options set O (for the problem stated in subsection 5.4.1)[§] as follows:

1) $o_i \succ_1 o_j$ if and only if

$$\{\exists k \in [1,\ldots,p] \mid ([o_i \sim'_r o_j] \text{ for all } r=1,\ldots,k-1) \text{ and } [o_i \succ'_k o_j])\}$$

2) $o_i \sim_1 o_j$ if and only if

$$\{[o_i \sim'_r o_j] \text{ for all } r=1,\ldots,p\} .$$

3) o_i I o_j otherwise. □

In other words, in order to establish ($\succ_1$, $\sim_1$) between two options o_i, $o_j \in O$, we start with the highest ranked criterion, i.e., r = 1. By the 1st ranked criterion relation ($\succ'_1$, $\sim'_1$), one of the following holds:

1) $[o_i \succ'_1 o_j]$; 2) $[o_j \succ'_1 o_i]$; 3) $[o_i \sim'_1 o_j]$ or 4) $[o_i \; I'_1 \; o_j]$.

If $[o_i \succ'_1 o_j]$ is true, then $[o_i \succ_1 o_j]$, by the preference relation $\succ_1$ defined above over the set of options. Similarly, if $[o_j \succ'_1 o_i]$, then $[o_j \succ_1 o_i]$.

If $[o_i \; I'_1 \; o_j]$, then $[o_i \; I \; o_j]$, i.e., option-wise (or overall) incomparability is declared between the two options, regardless of the other criteria.

If $[o_i \sim_1 o_j]$, then the next highest ranked criterion is considered; i.e., we set r = 2.

[†] Note that throughout this section the symbols denoting the criterion-specific relations are primed, whereas the overall relations taking the multiplicity of criteria into account are not.

[§] Note that each option $o_i \in O$ in this problem consists of a multivariate probability mass function.

Therefore, the second-ranked criterion relation ($\succ_2'$, $\sim_2'$) is used to compare o_i and o_j only if $[o_i \sim_1' o_j]$. Clearly then, given that $[o_i \sim_1' o_j]$,

- if $[o_i \succ_2' o_j]$, then $[o_i \succ_1 o_j]$ and vice-versa if $[o_j \succ_2' o_i]$
- if $[o_i \; I_2' \; o_j]$, then $[o_i \; I \; o_j]$
- if $[o_i \sim_2' o_j]$, then the next level is considered, i.e., we set $r = 3$, and so on ... until one of the following is obtained: $[o_i \succ_1 o_j]$, $[o_j \succ_1 o_i]$, $[o_i \sim_1 o_j]$, or $[o_i \; I \; o_j]$.

One further comment concerns elicitation of the rank order over the set of criteria. Two approaches come to mind:

1) a priori, before establishing ($\succ_1$, $\sim_1$) over O

2) progressively, whereby the decision maker first selects the most important and all pairs are compared by the relation ($\succ_1'$, $\sim_1'$). If there are any option pairs $(o_i, o_j) \in O \times O$ such that $[o_i \sim_1' o_j]$, then the second most important criterion is selected and all eligible pairs are compared by ($\succ_2'$, $\sim_2'$).

Similarly, the next highest criterion is identified and all eligible pairs compared, and so on, until either:

- all p criteria have been identified (i.e., we reach $r = p$)
- $r < p$ but no pairs (o_i, o_j) in $O \times O$ are such that $[o_i \sim_r' o_j]$.

Comments on Establishing R-th Ranked Criterion Relation ($\succ_r'$, $\sim_r'$)

It was mentioned earlier that the mechanism of section 4.2 is applicable here to obtain ($\succ_r'$, $\sim_r'$). The applicability is quite clear; let C_k be the r-th ranked criterion. If C_k is the only criterion of interest in the comparison, then the discrete states formulation (problem subclass 1) is obtained.

We can thus compute the risk measures c_{ij}^k and d_{ij}^k, based on the

respective impact of o_i and o_j along C_k, using Equations 4.1 and 4.2. The rules summarized in Figure 4.3 are then directly applicable. The only preferential information needed from the decision-maker for a particular single criterion relation consists of the two threshold values $\bar{c}^k$ and $\bar{d}^k$,† as defined in section 4.2.§

Having described the mechanism for comparing any two options in O, the properties of the resulting initial binary relation are discussed in the next subsection.

5.4.3 Properties of Initial Relation $(\succ_1, \sim_1)$ and Correction Strategies

This subsection addresses the properties of the initial option-wise preference-indifference relation developed in the previous section, with particular emphasis on its consistency with the properties of partial semiorders. To the extent that such consistency does not hold, and it is desired to remove sources of inconsistencies, our methodological structure provides for general corrective strategies (cf. section 4.4), with one requiring further relation specific definitions, as discussed in subsection 6.4.2.

Properties

The properties of the option-wise relation $(\succ_1, \sim_1)$ integrating the single criterion relation of section 4.2 within a lexicographic model, clearly depends on the properties of the criterion-specific relations $(\succ'_r, \sim'_r)$, $r = 1, \ldots, p$.

In particular, in section 4.2, it was determined that the properties of the criterion based relations are generally unrestrictive (cf. subsec-

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†Note that we have superscripted these measures by "k" so as to indicate that they apply to impacts along C_k.

§Of course, different thresholds apply for each criterion.

tion 4.2.4), and not necessarily consistent with partial semiorders. Such inconsistencies in $(\succ_r', \sim_r')$ for r = 1, ..., p, can obviously carry over to the overall relation $(\succ_1, \sim_1)$, along with other properties, as briefly discussed hereafter.

Completeness: $(\succ_1, \sim_1)$ is of course not complete as it allows incomparability of options. It should be noted in this context that incomparability effectively limits the applicability of the lexicographic assumption only to those comparisons where a preferential statement is clear cut (cf. discussion in section 4.2 and 2.2 on incomparability). Conceivably, compensatory action between criteria can then be exercised among options incomparable by the lexicographic relation.†

Reflexivity and Symmetry: Following standard usage, the preference relation $\succ_1$ is irreflexive and asymmetric, while the indifference relation $\sim_1$ is reflexive and symmetric.

Transitivity: Because the criterion-based relations $(\succ_r', \sim_r')$, r = 1, ..., p are not necessarily transitive (neither for preference $\succ_r'$ nor for indifference $\sim_r'$). the option-wise overall $(\succ_1, \sim_1)$ can easily be shown to inherit this property.

Properties P3 and P4: Counterexamples are presented in subsection 4.2.4 to show that the criterion-based relation $(\succ_r', \sim_r')$ does not necessarily follow properties P3 and P4 of semiorders (defined in section 3.3). It is again trivial to establish that those properties do not hold for $(\succ_1, \sim_1)$.

Therefore, in general, $(\succ_1, \sim_1)$ is not necessarily consistent with the

†In that sense, the underlying rationale has features of Luce's (1978) lexicographic trade-off structures.

properties of partial semi-orders. As discussed extensively in section 4.3, a particular relation ($\succ_1$, $\sim_1$) defined over a specific set of options may or may not violate those properties. However, when they do occur, such violations are manifested through the presence of cycles in the graph $G(\succ_1, \sim_1)$ associated with that relation. Types of cycles and whether they are inconsistency cycles were described in section 4.3, and that discussion is of course applicable to this case as well.

Furthermore, approaches for correcting inconsistencies, described in section 4.4, are applicable here as well. However, the second corrective strategy ("breaking weakest link") requires interpreting the "weakest link" in the context of the relation ($\succ_1$, $\sim_1$) specified in subsection 5.4.2.[†]

Correction Strategies

An interesting feature associated with this relation is that a particular link in G ($\succ_1$, $\sim_1$) can be characterized by the lexicographic rank r at which it was established. For example, preference between o_i and o_j (ε 0), denoted $o_i \succ_1 o_j$, is established at rank 3, when

$$[o_i \sim_1' o_j] \text{ and } [o_i \sim_2' o_j] \text{ and } [o_i \succ_3' o_j].$$

A special case is of course option-wise indifference, which can only be established at rank p (where p is the number of criteria.)[§]

Subsequently, a given cycle in $G(\succ_1, \sim_1)$ may consist of links that are either: 1) established at the same lexicographic rank, or 2) established at different ranks. In the first case, the problem of identifying the

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[†]In the same way that it was interpreted in section 4.4 for the relation of section 4.2, and in subsection 5.2.4, for the target-oriented relation of subsection 5.2.2.

[§]By definition, $[o_i \sim_1 o_j] \Leftrightarrow \{[o_i \sim_r' o_j], r=1, \ldots, p\}$ (cf. subsection 5.4.2).

weakest link becomes identical to the one addressed in subsection 4.4.2, in conjunction with the (univariate) discrete states case, and the same approach applies here. In the second case however, when the different links forming the inconsistency cycles are established at different ranks, the lexicographic feature of the model suggests a different interpretation.

In particular, links established at a "lower level" criterion (i.e., higher values of r) are considered prime candidates for "weakest link".† The implication is thus that, in a given cycle with links established at differing ranks, the end nodes of the link established last (i.e., at the lowest rank) are declared incomparable, thus removing the inconsistency. In the event that multiple pairs in the cycle qualify under the above rule, then either: 1) all are declared incomparable, or 2) the same rules described in subsection 4.4.2 can be used to select among those multiple pairs.

Of course, the other two correction strategies (of subsection 4.4.1 and 4.4.3) are applicable here as well, as they are general and require no relation-specific definitions.

5.4.4 Concluding Remarks on Section 5.4

This section has described the sequential elimination structure and associated decision rules for the multicriteria discrete states problem In addition, it illustrated the use of the pairwise comparisons methodology developed in the previous chapters to the MC case. In particular, it described an example initial binary preference-indifference relation (over the options set) that integrates the single criterion technique (of section 4.2) within a lexicographic comparison structure. Its properties were

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†The rationale is a direct result of our definition of $(\succ_1, \sim_1)$ in subsection 5.4.2, whereby lower rank preference indicates indifference up to that rank, meaning that other option pairs along the same cycle exhibit more clear cut dominance on the higher ranked criteria.

discussed and implications for building the final binary preference-indifference relation suggested.

In the remainder of this section, we elaborate on the lexicographic assumption for the initial binary relation developed in this chapter. It is clear that this assumption is quite strong and may be untenable in some instances (see Fishburn, 1974; 1977 for a discussion). On the other hand, it is not necessarily inappropriate in all situations for our class of problems, especially when one considers the following:

a. In many instances in urban transportation evaluation (especially in the context of developing countries), economic criteria (usually an aggregate measure of users benefits) are still quite prominent, whereas other environmental and social criteria are primarily handled on a satisficing basis.

b. The sequential elimination phase can ensure the consideration of these important factors by eliminating those unsatisfactory options that do not meet some minimum levels.

c. The pairwise comparisons stage would then recognize the precedence of economic criteria, and, if two options are preferentially indifferent with respect to the highest ranked criterion, then the less important criteria come into play again.

d. By allowing incomparability, the criterion-specific relations as well as the option-wise overall preference-indifference relation recognize the restrictiveness of a strictly lexicographic relation, and allow for a full consideration of trade-offs, over a reduced number of superior options, in the global comparison stage of the decision aiding framework.

However, our methodology does not propose anything by way of explicitly capturing preferences for trade-offs among criteria. Further-

more, the lexicographic feature is not advocated as a normative principle to follow. We merely presented an example binary preference-indifference relation, which draws upon the basic methodological development of the previous chapters, and incorparates a feature which is often observed in practice and as such may be useful in the pairwise comparison stage of the decision aiding framework.

It should be noted that generalization of the contents of this section to the MC equivalent of problem subclass 2 is feasible. In particular, for the pairwise comparisons stage, the only difference is at the level of the criterion-specific preference-indifference relations, wheregy the continuous version of the comparison measures (of section 4.2), as described in subsection 5.3.2, can be utilized. Beyond this minor modification, the lexicographic aspect of the option-wise overall relation remains unchanged.[§]

[†]In this formulation, uncertainty as to the impact of each option $o_i \in O$ is represented by p-variate probability density (or mass) functions of $(x_i^1, x_i^2, \ldots, x_i^p)$, where x_i^k is the random variable denoting the impact of o_i on C_k.

[§]Note however that this formulation is not representative of actula applications due to its very definition and steep impact information requirements.

5.5 Summary of Chapter 5

This chapter has consolidated the contribution of Chapter 4 by extending the methodology in four directions:

1) Specification of alternative functional forms for the risk dimensions used to build the initial preference indifference relation, in section 5.1;

2) Integration, in section 5.2, of the well-known notion of target within the methodology, as an elimination rule and as an alternative mechanism for defining the initial relation.

3) Generalization of the methodology, in section 5.3, to problem subclass 2 where options are characterized by continuous probability density functions over the impact domain.

4) Extension, in section 5.4, to problem subcalss 3, where the options' impacts are expressed along a multiplicity of criteria and uncertainty is represented via discrete states of nature.

In conjunction with the first point, an important result was established concerning the occurrence of intransitivities in conjunction with a strict concordance measure. In addition, problem situations where each of the alternative specifications is appropriate were identified, and numerical illustrations provided.

The second point extended the rationale of the decision rule of section 4.2 to problem situations where options are evaluated with respect to their performance vis-a-vis a target level for the desired impacts. The properties of the so-defined binary relation were examined and some of their implications for devising corrective strategies (as per section 4.1) discussed.

A family of decision rules compatible with the methodological struc-

ture presented in Chapter 4 have thus been developed in the first two sections of this chapter. More important than the specific rules and measures described is the fact that they underscore: 1) the importance of adapting the methodology to the specifics of a given problem situation, and 2) the flexibility of the decision aiding methodology developed in this thesis in that regard.

The third extension presented in this chapter concerned the more general problem formulation where options are represented as continuous probability density functions. The applicability of the methodology to this subclass of problems was shown to be possible, subject to respecifying the underlying risk measures. It became apparent, however, that some loss of intuitive appeal occurs as the discordance measure is redefined.

The fourth and last extension addressed the case where options are explicitly evaluated with respect to their uncertain impact along a multiplicity of criteria. The same methodological structure for pairwise comparisons is applicable here as well, and the problem specific element needed is the mechanism for building the initial preference-indifference relation over the options set. An example binary preference-indifference relation was defined, primarily to illustrate how the measures and associated evaluation rationale developed for the single criterion case (in section 4.2) may be used in the presence of multiple criteria.

Further comments are presented in the next chapter summarizing the thesis and its principal contributions, and suggesting directions for further research.

CHAPTER 6

CONCLUSIONS

In this concluding chapter, a summary of the thesis, highlighting its principal contribution, is presented in section 6.1. In addition, section 6.2 reviews the major concerns facing the application of the methodology by an analyst interacting with a decision maker in an environment similar to the one depicted in section 1.1. These concerns include the various assumptions and possibilities open within the structure of the methodology, and aspects associated with the interaction process. Finally, directions for further research are outlined in section 6.3.

6.1 Summary

This thesis was motivated by the need to explicitly consider uncertainty in transportation planning decision models. While the field is currently deficient in that respect, the restrictive assumptions and/or high input requirements of existing formal methodologies make these generally inappropriate for the realities of urban transportation decision problems and their complex environment. The objective pursued in this thesis was the development of a decision aid, consisting of a family of decision rules embedded within a coherent methodological structure, for a representative class of problems (of the urban transport context), involving selection among a large finite set of options whose impact is not known with certainty. This class of problems was considered to arise within a prototypical decision environment consisting of an agency where an analyst is engaged in aiding a single decision maker (or "group which speaks with one voice") in the decision process.

A set of basic considerations underlying the methodological development of the decision aid was derived from the characteristics of the decision environment as well as behavioral aspects of individual decision making and judgement, including relevant results from the behavioral science literature. In addition to further defining the role of decision aiding, for our class of problems, this set of considerations specified the special features that the decision aiding methodology should possess, and as such constitutes an important output of this research. Based on those considerations, an overall framework for decision aiding was developed, consisting of three sequential activities: elimination, pairwise comparisons and global comparison. The second of those was the primary focus of the analytical development in this thesis.

This set of considerations provided the basis upon which existing normative decision models for choice under uncertainty were assessed, with special emphasis on expected utility theory. The axiomatic foundation of the latter theory was critically reviewed, especially the independence axiom whose plausibility was questioned using a new argument relying on the implications of that axiom on the risk dimensions underlying a choice situation. Less restrictive postulates, reflecting our basic considerations, were proposed for the methodology developed in this thesis. In particular, the preference-indifference relation over the set of options was postualted to follow the properties of partial semi-orders, thus allowing incomparability of options as well as intransitivity of the indifference relation. This is a behaviorally justified departure from the usual assumption of the existence of a complete order in decision models, and as such is an integral part of the contribution of this research. It should be noted, however, that partial semi-orders were not claimed as an absolute normative representation of the preference-indifference relation, but as a model possessing relatively rational properties for that relation within the context of operationally forming that relation.

The main thrust of the methodological development in this thesis consists of the pairwise comparisons procedure (within the overall decision aiding framework), including its structure, as well as its specific elements. The methodology revolves around the construction of a binary preference-indifference relation over the set of options, that is consistent with our starting postualtes, and then using that relation to obtain a reduced subset of superior options. The first component of the methodology consists of a mechanism for comparing any two given options (yet allowing for incomparability), thus defining an initial partial binary preference-indif-

ference relation over the options set. Such a mechanism was introduced in Chapter 4 for a widely occurring problem subclass where the respective payoff of each element of a finite options set is conditional on the realization of one of a set of mutually exclusive and collectively exhaustive states of nature (with known probabilities of occurrence). This mechanism, consisting of regret-based risk measures (summarizing the implications of a binary choice situation), and an associated decision rule, requires preferential input from the decision maker in the form of a pair of threshold levels. The properties of the resulting initial preference-indifference relation were then shown to be quite unrestrictive and not necessarily in conformance with the starting postulates (i.e., properties of partial semi-orders).

The detection of such inconsistencies within a given problem context constitutes the second component of the pairwise comparisons methodology. This thesis thus identified inconsistencies with the properties of partial semi-orders as cycles in the graph theoretic representation of a given partial preference-indifference relation, and presented rules for their detection.

The third component of the methodology then consists of approaches for correcting those inconsistencies in the initial preference-indifference relation. Three such approaches were described, and the implications of each one examined. The remaining component of the methodology is to obtain the reduced subset of superior options (in this case the kernel) using the resulting "final" preference-indifference relation.

Variations on some of the specific elements of the core methodology were then presented. In particular, alternative functional forms of the risk measures sued to build the initial preference-indifference relation. In addition, the commonly used notion of target (or aspiration level) was integrated within the methodlogy, as an elimination rule as well as an alter-

native mechanism for defining the initial relation over the options set. A family of decision rules, comprising both types of variations, were thus defined for use, separately or jointly, within the above basic structure, thus increasing the flexibility of the methodology and allowing the analyst to fine tune its details to the specific decision problem situation. In addition to analyzing mathematical properties (and their implications) of the above variations, problem situations where each would be appropriate were identified and illustrated via numerical examples.

Generalization of the methodology to the case where uncertainty in the alternative options' impact is represented through probability density (or mass) functions was presented, requiring respecification of the underlying risk measures (for building the initial preference-indifference relation), all else remaining unchanged. In addition, extension to a problem subclass where options are explicitly evaluated with respect to their uncertain impact along multiple cirtieria was presented. For this subclass, the methodological structure is applicable as well, though its first component, i.e., the mechanism for building the initial preference-indifference relation over the options set, needs to be defined. An example preference-indifference relation was then presented, primarily to illustrate how the basic mechanism introduced earlier may be applicable, under some circumstances, to the multicriteria case.

Therefore, in addition to its stated overall premise and motivation, the principal contribution of this thesis consists of the development of a family of decision rules and associated methodological structure, for use within a decision aiding approach for a widely occurring (in the urban transportation environment) class of decision problems uncer undertainty. The structure

achieves consistency with specific postulates reflecting the set of basic considerations, which define the features of the methodology, and as such form an integral part of the contribution.

In addition to the above structure, the methodological contribution of this thesis includes:

- the principal binary preference-indifference relation developed (in section 4.2, with variations in section 5.1), based on a multidimensional representation of the risk implications of a binary choice situation; while the rationale is adapted from the ELECTRE technique (Roy, 1974; Nijkamp and Van Delft, 1977) for multicriteria decision problems under certainty, the adaptation to single criterion decision problems under uncertainty is new and particularly appropriate since the measures were derived from risk considerations of concern in the application context, as discussed in Chapter 4.

- the alternative assumptions as to the above measures and decision rules, including the target-oriented versions, aimed at increasing the operational flexibility of the approach.

- the "correction" strategies, in the third component of the above methodological structure, aimed at "removing" inconsistencies with the underlying postulates.

At a more theoretical or mathematical level, results that are worthy of note include:

- identification and detection rule for inconsistencies with partial semi-orders in the graph theoretic representation of a given partial preference-indifference relation;

- derivation of several proofs concerning the properties of the relations studied.

Other secondary contributions are present at various points in the thesis and are indicated in the chapters in which they occur. However, as emphasized throughout, it is the totality of the approach that constitutes the primary contribution of this thesis, rather than the sum of the "local" contributions.

6.2 Application Issues Facing Analyst

In this thesis, we have addressed the problem of an analyst, operating in the problem environment of section 1.1, in aiding a decision maker through the evaluation process. As stated in section 2.1, our basic premise has been that it is possible for the analyst to structure the decision aiding activity and use analytical tools (decision aids) within this structure. Basic considerations underlying the development of such decision aids were derived in Chapter 2, as well as an overall framework for decision aiding. Within this framework the focus of the subsequent methodological development was on the pairwise comparisons stage. However, consistently with the above mentioned basic considerations, which recognize the importance of context specifics, there is scope for interpretation and choice in the application of the methodology, allowing it to be fined tuned to the particular problem at hand.

The analyst thus faces a number of issues, the resolution of which is predicated upon a variety of factors, including: 1) the specific options under consideration and their associated impact matrix, and 2) characteristics of the decision maker, as well as many other circumstantial factors. A full characterization of all possible situations, and mapping of these situations into specific rigid procedural steps to be followed in a cookbook fashion by the analyst is certainly outside the scope of this thesis' contribution; furthermore, it is questionable that this degree of specificity is at all feasible for decision aiding tools in the complex urban transportation environment (as discussed in Chapters 1 and 2).

It is nonetheless possible to offer more or less specific "guidelines" or suggestions to assist the analyst in performing his role. For instance, most of the basic considerations of section 2.2 are also applicable to the

general activity of decision aiding (notably propositions 2.2.3 to 2.2.8). In addition, the identification of the principal choices available (to the analyst) and the factors affecting their resolution can be of considerable assistance. In the remainder of this section, we address a number of importatn concerns arising in conjunction with the application of the pairwise comparisons methodology developed in Chapters 4 and 5. These concerns are grouped in two categories: 1) "decision points" or assumptions relative to the selection of a principal element within the methodology, and 2) those related to the interaction between the decision maker and the analyst in his use of the methodology.

Subsections 6.2.1 and 6.2.2 respectively address each of the above categories.

6.2.1 Principal Methodological Choices

Two major methodological choices are of particular concern in this subsection. The first is that of the specific risk measures and associated decision rule underlying the mechanism for building the initial binary preference-indifference relation (i.e., in the first step of the methodological structure of section 4.1). The second is that of a correction approach or mix thereof, (in the third step of the methodological structure) for removing inconsistency cycles from the graph theoretic representation of the initial preference-indifference relation. These are discussed in turn hereafter.

Selection of Risk Measures

Within the family of measures introduced in section 4.2 and extended in section 5.1 and 5.2, the choice as to which ones to use in building the initial preference-indifference relation can be viewed as a two-stage one: the first stage involves determination of whether or not to use the target-

oriented measures (section 5.2), whereas the second stage determines the functional forms for the concordance and discordance measures, conditional upon the first decision.

As discussed in section 5.2, the (first stage) decision relative to the use of the target-oriented approach (to build the initial binary preference-indifference relation[†]) hinges upon the validity of an assumption on preferences over the payoff domain X, whereby all outcomes that achieve or exceed the target level x_t are considered to be preferentially equivalent. This assumption might, for all practical purposes, be acceptable whenever a target value x_t can be identified in the problem context; for instance, the options under consideration may have been designed to meet such a target level (in the form of a service standard, for example). Another situation where the use of the target would be quite appropriate is when impact information, for each option, is available only relative to a particular level, due to limitations of input data or impact prediction capabilities (or a variety of other reasons, including time constraints).

However, outside of the above two situations, the measures introduced in section 4.2 (or their variants in section 5.1) would be more appropriate as explained in section 5.2.

The second stage decision concerns the specific functional forms of the concordance and discordance measures respectively.[§] While the basic form was introduced in section 4.2, consistently with proposition 2.2.7 (as amply discussed in section 4.2 and 4.5), different variants were presented

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[†]Of course, this discussion applies to the sue of the target oriented approach for building the binary relation, and not to its use as the basis of an elimination rule.

[§]Note that this choice is open within the target-oriented approach as well. Though the presentation in section 5.2 did not explicitly show the target oriented parallels to the variants of section 5.1, these parallels are quite straightforward, and will thus be assumed available to the analyst.

in section 5.1. The different variants (and basic form) are listed in Figure 6.1 along with the associated notation to facilitate the reader's ability to follow this discussion.

This discussion in section 5.1 addressed the properties of those measures, and indicated problem situations (accompanied by simple numerical illustrations) where each would be appropriate. We will not duplicate that discussion here, but briefly highlight the principal concerns underlying each alternative form, and factors affecting their relative adequacy for a particular decision situation. Those factors can be grouped in the two types mentioned in this section's introduction; 1) those related to the options and 2) those related to the decision maker's preferences, and more importantly, the interaction between the two, as seen hereafter.

Concerning the choice of a functional form for the concordance measure, both "strict" measures (c^{+}_{ij} and c'_{ij}) are intended for situations where a stricter basis for preference is deemed necessary, relative to the "basic" form (c_{ij}). As discussed in section 5.1, they allow a sharper line for differentiating between preference and indifference. Such refinement in the representation of the decision maker's preferences may be needed when the options exhibit substantial "overlap", i.e., when the probability of receiving identical payoffs from two given options is relatively high.[†] Furthermore, when this overlap varies across options (given that it is sub-

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[†] This probability, in the discrete states formulation is given by $\sum_{k \mid S_k \epsilon E_{ij}} P_k$

Figure 6.1: Summary of Alternative Risk Measures

c_{ij} = basic form of concordance measure

c^{+}_{ij} = strict concordance measure

c'_{ij} = normalized strict concordance measure

d_{ij} = basic form of discordance measure

d^{ℓ}_{ij} = likelihood compensated discordance measure

d^{c}_{ij} = conditional weighted discordance measure

d^{m}_{ij} = marginal weighted discordance measure

d^{r}_{ij} = relative discordance measure

stantial overall), c'_{ij} or the normalized strict measure is particularly appropriate (cf. example 5.1, in section 5.1).†

The choice of discordance measure is also predicated upon the above two groups of factors. In particular, the factors related to the options' impact distributions consist primarily of the variation of impact within options, the variation across options, as well as the resulting regret or loss profile (i.e., distribution of the impact difference, for any given option pair). Perhaps the single most important factor is whether or not "catastrophic" losses are present. As discussed in section 5.2, such situations clearly require the use of the "basic" form (d_{ij}). When loss profiles are less drastic, yet exhibit substantial variation (within given option pairs), with some states having very low probability of occurrence, the likelihood compensated form (d^{ℓ}_{ij}) may be used§ (as illustrated in example 5.2). Furthermore, when more than just the "worst" state needs to be taken into account, the "weighted" variants (d^{c}_{ij} or d^{m}_{ij}) would be appropriate. In addition, as illustrated in example 5.4, when the range of payoffs of the different options is quite wide (i.e., order[s] or magnitude), the relative discordance measure (d^{r}_{ij}) may be appropriate, as it divides the absolute potential loss by the corresponding realized outcome.

While the above are mere qualitative considerations in the selection among alternative functional forms for risk measures, it is not possible

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†A particularly interesting case where either version of the "strict" measure may be appropriate is in the context of the multicriteria extension presented in section 5.4. Within the lexicographic structure, stricter preference may be required on the higher ranked criteria before accepting option-wise preference.

§Note that this discussion on the appropriate discordance measure is not applicable to problem subclass 2 (defined in section 1.4), since, as explained in section 5.3, the same range of choices (as in "discrete states" problems) is not available to the analyst.

at this point to specify more definitive, "hard" and content free procedures. Furthermore, the very nature of the general context of the decision problems of interest (as discussed in Chapters 1 and 2) precludes such specification, and dictates the need to allow the analyst (and decision maker) to exercise a fair amount of judgement in applying the decision aiding methodology. It is conceivable however that the accumulation of experience through repeated use of the methodology for a specific type of problem may produce more definite recommendations. It is however, felt that the basic forms of both concordance and discordance measures are generally the most appropriate and plausible for urban transportation applications. However, when some of the above mentioned special situations arise, the flexibility exists to accomodate them. Of course, the variants presented in section 5.1 and reviewed here have by no means exhausted all special situations nor all possible variants, as it is equally possible for the analyst to generate other variants if deemed necessary.

Selection of Correction Approaches

Three approaches were described in section 4.4 for affecting adjustments in an initial unrestrictive partial binary preference - indifference relation (containing inconsistency cycles which violate the properties of partial semi-orders) towards a final relation free of inconsistency cycles. These approaches were comparatively assessed in section 4.4 and the differences in the respective preferential implications highlighted. As a reminder, these approaches are:

1. preferential indifference among cycle elements,
2. breaking "weakest link", and
3. mutual incomparability among cycle elements.

As discussed in section 4.4, whereas the first makes a relatively strong preferential assumption (by interpreting the presence of a cycle as an indication of preferential indifference), which may be highly questionable in many situations, the second and third approaches utilize the non-completeness of the relation by declaring incomparability between two or more cycle elements. For instance, the second approach seeks in interpretation within a specific relation of the "weakest link" of a cycle, and subsequently "breaks" the cycle by declaring the endpoints of that link incomparable, as described in subsection 4.4.2. The third approach is more "passive" as it "refuses" to accept any of the inconsistently connected elements by declaring all cycle elements mutually incomparable.

It was further mentioned in section 4.4 that an analyst could utilize more than one approach in a particular problem. We elaborate hereafter on the analyst's possibilities, identify principal factors that s/he should consider, and accordingly suggest strategies which s/he may follow in certain situations.

The first decision of course concerns whether to adopt one correction approach throughout, and apply it to all cycles, or to employ different approaches for different cycles. This decision is to be taken by the analyst and decision maker being counseled; it is preferable to proceed situationally, by waiting to face a specific cycle configuration in the graph of the actual initial relation.

The factors that affect the analyst's choices at that point are:

1. number of inconsistency cycles, both in absolute and relative to the size of the problem;

2. size of each inconsistency cycle, i.e., the number of elements connected by it;

3. decision maker's preferences;

4. circumstantial characteristics of the interaction process including time constraints, decision maker's availability and others.

For instance, in the presence of only a few inconsistency cycles, within a problem with a relatively large number of options, the third approach would probably be recommended. It is clear that such an approach cannot misrepresent the decision maker's preferences, and can be executed without the decision maker's input. However, as explained in section 4.4, the price exacted for this value-neutrality could be a significant reduction in the decidability of the overall problem as in the case where a large number of options are connected by inconsistency cycles.

Similarly, a small-sized inconsistency cycle is better suited for resolution by the third approach than by any of the other two. Larger† cycles are more appropriately treated by the second approach, which only "destroys" a small number of links. Thus, for a problem with a large number of cycles, small cycles could be treated by the third approach, with larger ones to be resolved by the second approach.

The application of the second approach may further entail another decision for the analyst to take, as described in subsection 4.4.2. In particular, it was stated in that subsection that when the efficient set of option pairs (corresponding to end nodes of the links along a given cycle) is large, selection of the "weakest link" may proceed either situationally (by having the decision maker directly identify that link), or according to a set rule (specified in that subsection). It was clear that the situational approach is preferable from the standpoint of representing the decision

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†As an arbitrary rule of thumb, consider "large" cycles as those with 5 or more nodes.

maker's preferences, though it is more demanding on the decision maker's time. Therefore, in the presence of a large number of inconsistency cycles which cannot be resolved by the bi-criterion rule (cf. subsection 4.4.2), the analyst has to consider the constraints on the time available for the decision aiding activity in determining whether to seek the decision maker's input or to resort to the alternative assumption.

As stated earlier, the first approach is the most questionable theoretically. It may however be used if its underlying assumption conforms to the decision maker's preferences. There is, however, no way of determining in any conclusive fashion whether or not this is the case. Therefore, if informal interaction with the decision maker reveals the acceptablity of this assumption, then it may be used. It is however, recommended that its applicability be limited to small sized cycles.

In summary, a number of factors enter the determination of which correction approach to apply to a particular cycle. In general, it is preferable to limit the use of the first appraoch to smaller cycles, if at all. The third approach, on the other hand, is the least risky one, and is recommended for situations where its implementation would not lead to a drastic loss of decidability of the overall problem (such as in the presence of a small number of cycles in a problem with a large number of options). However, in cases where a substantial number of cycles are present, it is recommended that the second approach be used for large cycles, while the third one would resolve the small cycles (unless an unduly large number of those exist). In using the second approach, time constraints and the decision maker's availability are important factors in determining the identification rule (of the weakest link) to be followed.

It is clear that these are merely guidelines and suggestions that are meant to complement rather than supplement the analyst's judgement in conducting the decision aiding activity.

6.2.2 Interaction Between Analyst and Decision Maker

This subsection addresses a number of important concerns faced by the analyst with regard to the interaction with the decision maker in the use of the methodology developed in Chapters 4 and 5. In particular, the methodological development in Chapter 4 assumed the existence of decision maker supplied threshold values $\bar{c}$ and $\bar{d}$, as input to the mechanism for building the initial relation $(\succ_1, \sim_1)$. No further specification of the process by which those two values can be otained has been presented. As discussed in the introduction to this section, the nature and the mechanics of the interaction are better handled on a context specific basis, thus adapting to a number of factors, including:

- the specific set of options under consideration;
- characteristics of the decision maker, such as his familiarity with the decision problem at hand, his knowledge of the specific options at hand, his prior experience (if any) with the methodology, among others;
- time and other resource constraints;
- the specific problem under consideration; in particular, whether any of the preferential inputs of the evaluation are known a priori (in the form of legislated constraints, prior decision, etc.).

Again the final adaptation is the prerogative of the analyst. Figure 6.2 depicts a possible sequence of steps adapted from ELECTRE (Roy, 1971; Nijkamp and Van Delft, 1977) that can be followed by the analyst in conducting the pairwise comparisons stage of the decision aiding framework. This se-

quence reflects proposition 2.2.5 (progressive difficulty in preference elicitation), by letting the decision maker start at a high concordance threshold and low discordance threshold (reflecting a fairly "confident" statement on behalf of the decision maker[†]) and then progressively decreasing $\bar{c}$ and/or increasing $\bar{d}$ (reflecting less confident preferential statements). At every step (i.e., new threshold values pair), inconsistencies are identified and removed, and only the kernel is retained for the next step (cf. Chapter 4 for the methodological details.)

Another important associated concern is that of what information to display to the decision maker in the above process. Of course, proposition 2.2.7 is applicable here, in addition to having been already invoked in communicating the implications of a binary choice situation via the two intuitively understandable risk measures, which also serve as a medium for eliciting preferential information. Thus, the analyst should present the decision maker with all the concordance (c_{ij}'s) and discordance (d_{ij}'s) measures for all the comparisons that would be affected by the requested threshold values. However, in most realistic problems, the number of the (c_{ij}, d_{ij}) pairs under consideration is likely to be too large for useful (numerical) display, thus the need to summarize this information. Histograms (for c_{ij} and d_{ij} separately and also possibly jointly) may be particularly appropriate for this task.

Note in addition that the discordance information should be presented along with the value of d^{max} (see Equation 4.2)[§] in order to "anchor" the

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[†]Remember that $c_{ij} = 1$ and $d_{ij} = 0$ imply complete dominance of o_j by o_i

[§]Or, if a variant of this measure is used, the corresponding quarterly in the denominator should be displayed.

Figure 6.2: Possible Interaction Sequence

1. Analyst displays c_{ij}'s and d_{ij}'s to decision maker (DM)
2. DM supplies threshold values $(\bar{c}, \bar{d})$.
3. "Processing: with appropriate computational support, analyst
   - builds initial relation
   - identifies and corrects inconsistencies (may decide to seek DM input)
4. Analyst presents DM with reduced subset of options resulting from $(\bar{c}, \bar{d})$†
5. If DM is satisfied (i.e., reduced subset of sufficiently small), thus pairwise comparisons state is terminated.
   Otherwise, c_{ij}'s and d_{ij}'s, for reduced subset of options, are redisplayed.
6. If DM is not willing to further lower $\bar{c}$ and/or raise $\bar{d}$, then pairwise comparisons stage is terminated.
   Otherwise, DM lowers $\bar{c}$ and/or raises $\bar{d}$, and process goes back to step 3.

†In addition, the impact information may also be presented along with those options, as well as the list of eliminated options.

values of d_{ij} (which are all in the interval [0,1]). In some cases (depending on the decision maker), it may be preferable to display the discordance measures directly as differences in impact, prior to normalization. The chief reason is that it may be easier to elict the threshold value $\bar{d}$, reflecting the maximum tolerable regret, directly in the units in which the impact information is expressed.

Of course, the interaction pattern presented in this subsection is highly schematic. It does however give a flavor of how the methodology developed in this thesis might be used in the overall decision aiding process.

6.3 Directions for Further Research

There are three major directions along which further research related to this thesis can proceed:

- research on issues immediately related to the contents of this thesis.

- research towards major extensions of the methodology presented herein.

- research in pursuit of the same general aim guiding this thesis; this aim, stated in Chapter 1, consists of developing an integrated structured analytical apparatus for transportation evaluation and decision-making, that would be cognizant of the complexity of the decision problems, and operation in terms of being compatible with the realities of the decision environment.

Along each of the above directions, a number of issues and topics can benefit from further research, as discussed hereafter.

Immediate Issues

The principal issue arising in conjunction with this thesis is of a general nature as it concerns the evaluation of decision aids. In other words, it is legitimate to wonder whether a decision aid improves the "quality" of decision; however, the answer to that question is a dilemma, and remains as yet unresolved, as noted by Einhorn and Hogarth (1981). The latter also note the lack of validity of one-shot case studies as a basis of evaluation of a decision aid, especially when variations across decision-makers and their environments preclude any such generalization. Nevertheless, it is felt that applications to real problems will yield further insights into the usefulness of a decision aid, and may result

in better guidelines for its use.

At a more methodological level, specific points (related to the development presented in Chapter 4) which may benefit from further research include the possibility of formalizing the analyst's allocation of analysis resources to specific inconsistency cycles (in the graph of the initial partial binary preference-indifference relation) and subsequent matching of correction strategies. This may be accomplished by some form of prioritization scheme by which different inconsistency cycles would acquite different degrees of "urgency" (depending on their size, type and relation to other components of the graph).

Another specific research issue, primarily of a decision theoretic nature, is the characterization and identification of cycles that violate the properties of partial interval orders (a more general form of semi-orders proposed by Fishburn, 1973); the latter may be considered as an alternative assumption to partial semi-orders for representing the preference-indifference relation over the options set.

Major Extensions of the Methodology

In Chapter 5, we showed the applicability of the methodological structure to multicriteria decision problems under uncertainty, subject to the definition of an acceptable mechanism for building the initial binary preference-indifference relation. Such a mechanism was developed, also in Chapter 5, for a special case where a lexicographic assumption (based on a hierarchical rank ordering of the criteria) may be acceptable. The limitations of such an assumption were however recognized and discussed in that chapter. Thus the need remains for a more general mechanism for that extremely important formulation, which would nonetheless retain the

consideration is a major area for future development, which contributes towards the general aim guiding this thesis.

In a previous review (Mahmassani, 1980b), fundamental difficulties and "paradoxes" associated with aggregation of multiple actors preferences were highlighted. Furthermore, the dynamics of multiple actor decision-making processes, which involve bargaining and negotiation, clearly challenge the realism of formal normative decision models, especially in the urban transportation context where no explicit, systematic and mutually agreed upon rules govern the interaction process. However, in the decision environment depicted in section 1.1, it is possible to take institutional preferences into account in the decision aid via a number of mechanisms. For instance, a proposed approach with the following features seems worthy of further development:

a. analyst (using decision aid in interaction with single decision-maker) attempts to describe (or predict) the acceptability to all other actors of any option under consideration);

b. "acceptability" can be expressed in a number of ways. The following operational definition is proposed: "probability of accepting a particular option." This definition suggests the use of probabilistic discrete choice models to obtain that measure as a function of option attrbitues and a given actor's characteristics (along the lines of McFadden, 1977).

It is clear that the considerable experience accumulated in discrete choice modeling in the area of travel demand analysis renders the development of descriptive institutional preference models an accessible objective.

Another extremely important and continuing area of research, which

behavioral features of the basic considerations (cf. Chapter 2). In particular, first steps taken in this direction during the course of this research have suggested that such a relation should ideally allow for lexicography over subsets of criteria (instead of over individual criteria, as in the model presented in section 5.4). Furthermore, the rationale of Luce's (1978) lexicographic trade-off model, was considered particularly appealing for such a relation, whereby compensatory action (between different criteria) previals when preference by a lexicographic semi-order cannot be established. This last feature could be instilled in the extension described in section 5.4, by viewing the compensatory aspect as a means of reducing incomparability in the lexicographic model presented. However, the mechanism for capturing compensatory action in a manner that is consistent with our basic considerations was not specified, and as such remains a major area for further research.

Towards General Aim

In the description of the prototypical decision environment in section 1.1, it was stated that the final outcome of the decision process was predicated upon the interaction of a number of actors, with varying perceptions and (usually conflicting) objectives. It was further stated that the inter-actor interaction process had different methodological requirements than the intra-actor (i.e. single decision-maker interacting with analyst) process. However, despite its clear and acknowledged importance, taking multiple actors into account within systematic evaluation methodology has been relatively absent from the repertoire of transportation evaluation methods, as suggested by the review in section 1.3. Such explicit

is essential to the use of systematic decision aids, such as the one developed in this thesis, concerns problem formulation and the generation of the needed input information. Though the formulation(s) pursued in this thesis is realistic and representative of current modeling capabilities, and in some cases of current practice, our review in section 1.2 (of approaches for dealing with different types of uncertainty in transportation planning) revealed that considerable research opportunities exist in:

- recognizing sources and types of uncertainty in specific decision situations involving different transportation systems and problems. The categorization presented in section 1.2 is a first step in that direction.

- empirically analyzing those uncertainties, and analytically representing them in the prediction methodology.

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APPENDIX A

SEQUENTIAL ELIMINATION STRUCTURE AND DECISION RULES

This appendix specifies the details of the elimination stage of the decision-aiding framework presented in section 2.3. The rationable behind that stage was discussed in that section as well. It was further mentioned that it would proceed in an Elimination-by-Aspects, or EBA-like fashion, combining lexicographic features with conjunctive rules, based on proposition 2.2.6. Section A.2 describes the sequential elimination structure, followed in A.3 by the types of measures and associated elimination rules. The specification of these details is pursued in the context of problem subclass 3 (where impacts of options are expressed along multiple criteria and uncertainty represented via discrete states of nature)[†] which is first stated in A.1.

[†]The reasons for discussing elimination within this subclass were articulated in section 2.3. Note however that it is applicable to the other subclasses as well.

A.1. Problem Formulation

Given a Pareto-optimal or non-dominated set of n options $O = \{o_1, o_2, \ldots, o_n\}$, the problem is to select a most preferred option $o^* \in O$. Uncertainty is represented through a set of m mutally exclusive, collectively exhaustive states of nature $S = \{S_1, S_2, \ldots, S_m\}$ with known probabilities of occurrence $P = \{P_1, P_2, \ldots, P_m\}$, where $P_j \in P$ denotes the probability of occurrence of state $S_j \in S$. It is again assumed that the P_j's are independent of which option $o^* \in O$ is selected.

The impact of each option is expressed by a vector of impacts along p criteria $C = \{C_1, C_2, \ldots, C_p\}$, conditional upon the realization of a state of nature. Let $X_{ij} = \{x_{ij}^1, x_{ij}^2, \ldots, x_{ij}^p\}$ denote the conditional impact of o_i given the realization of $S_j \in S$, where x_{ij}^k is the impact of o_i along criterion C_k should S_j occur. A given option $o_j \in O$ is thus characterized by the m-tuple $\bar{X}_i = \{X_{i1}, X_{i2}, \ldots, X_{im}\}$.

The elimination stage of the decision aid seeks to reduce the size of the options set O by eliminating some of its elements from further consideration based on preferential information which operates at the individual option level, as described hereafter.

A.2 Sequential Elimination Structure

Tversky's (1972) EBA model described individual choice behavior as a sequential elimination activity, whereby an "aspect" (of the options being evaluated) is selected, and all options unsatisfactory on that aspect are eliminated.† This is followed by the selection of another aspect which itself becomes the basis of further elimination, etc. ... until all options but one are eliminated. To the extent that Tversky's model was concerned with describing observed choices (or revealed preferences), aspects selection was postulated as a stochastic phenomenon with the probability of selecting a given aspect proportional to its respective importance.

The use of elimination in our normative decision-aiding framework exhibits two important differences with respect to the above descriptive model:

1. the elimination phase is not generally intended to reach one single final solution, but merely to reduce the size of the set of options with a minimal amount of cognitive strain.§

2. aspects selection is not purely endogenous to the decision-maker, but affected by analyst interaction.

It follows from those points that the order in which aspects are considered is not really an issue (in terms of the outcome of the elimination phase), since an option that is not satisfactory on any of the considered aspects is ultimately eliminated.& Thus all aspects that the decision-

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†This model is considered by Fishburn (1974) as a special case of a more general class of lexicographic models.

§As such, it is patterned after mental operations and activities that the individual decision-maker is likely to feel comfortable with, especially at the beginning of the evaluation process, as empirical evidence cited in subsection 2.2.6. seems to indicate.

&Order is however a factor in Tversky's model when additional aspects are considered until a single option remains.

maker can form with confidence are considered. In that sense, the sequential elimination process ends by the inability to further form meaningful aspects, rather than by a rule based on the number of the remaining options. In that sense, the stopping rule is left up to the analyst in a particular problem setting rather than being a constraint of the methodology.†

The resulting general structure of the elimination phase is as follows:

a. An "aspect" (broadly defined) is formed as a result of analyst/decision-maker interaction.

b. Each option $o_i \in O$ is screened individually to determine whether or not it is satisfactory with respect to that aspect.

c. If o_i is not satisfactory on given aspect, then it is eliminated from further consideration. Otherwise, it remains eligible for further reduction.

d. After remaining options are determined, a "new" aspect is formed, also as a result of interaction with the decision-maker. Steps (b) to (d) are thus repeated until no further aspects can be formed.§

Note that we explicitly stipulated that "aspects" are broadly defined. Typically, they would consist of conjunctive rules (or set thereof) operating on deterministic or probabilistic measures computed from the m-tuples $\bar{X}_i$; $\forall o_i \in O$. This problem is addressed in the next subsection.

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†The question naturally arises as to what if all but one option are indeed eliminated before exhausting relevant aspects. The answer, pragmatically, is that the remaining option is thus the desired solution. However, in most problem situations where this decision aid is needed, such simplified strategies would not be sufficient to reach the desired solution.

§Again, we maintain the assumption that no unique single solution will emerge by this process.

A.3 Elimination Measures and Decision Rules

This subsection addresses the problem of forming aspects for use in the above sequential elimination structure. While many such aspects are straightforward, either because they correspond to current practice, or are extremely intuitive, the analyst can additionally suggest some less obvious measures judiciously selected for the problem context so as to capture some of its otherwise non-apparent interactions. This is particularly true for the multi-criteria (MC) problem, where the analyst may perceive "aspects" that the decision-maker should consider but may not have otherwise noticed in the initial impact information (i.e., the problem as defined in section A.1).†

As "aspect" is thus defined as a condition (or set thereof) that each option should meet in order not to be eliminated. The underlying decision rules therefore are conjunctive rules, which can operate on different types of information, including both deterministic and probabilistic information.

Deterministic Information

The simplest of course consists of a minimum acceptable score along any criterion, which may be state-specific or valid across all states of nature. Thus, if x^k_{min} denotes this minimum level for criterion $C_k \epsilon C$ the decision rule is simply:

if $x^k_{ij} > x^k_{min}$, $\forall S_j \epsilon S$, then reject o_i, for all $o_i \epsilon O$.

More generally, a vector $X_{min} = [x^1_{min}, x^2_{min}, \ldots, x^p_{min}]$ can be defined such that

if $X_{ij} < X_{min}$, $\forall S_j \epsilon S$, then reject o_i, for all $o_i \epsilon O$.

Within the above structure, such levels can be set either separately or jointly in any combination.

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†The reader is reminded of the description of the decision problem environment given in section 1.1, whereby the analyst and decision-maker are within the same agency and thus share an amount of context-specific knowledge.

Probabilistic Information

a. Marginal Probabilities

In this case, we are interested in controlling for the marginal probability of scoring below some critical level, x^k_{cr}, for some $C_k \epsilon C$. Whereas x^k_{min} is an absolute minimum level to be achieved with certainty, x^k_{cr} translates a situation where an inevitable finite probability exists of an undesirable (i.e., below x^k_{cr}) outcome. A typical example is controlling for the probability of "ruin" (defined with respect to x^k_{cr} on C_k) being below a given level. Mathematically,

if $p(x^k_{ij} < x^k_{cr}) < p^k_{max}$, then reject o_i, for all $o_i \epsilon O$,

where $p(x_{i.} < x^k_{cr}) = \sum_{j \mid x^k_{ij} < x^k_{cr}} P_j$

and p^k_{max} is the maximum acceptable probability of scoring below x^k_{cr}.

A useful way of framing that same rule is in terms of a target level, in the same sense as in subsection 5.2.1. In this case, we would be interested in the probability of exceeding that target being greater than a minimum value. While being mathematically equivalent, these two interpretations may have important operational distinctions. For instance, when dealing with a design service level that the contemplated facility should offer, framing in terms of the probability of achieving that target level is more appropriate (see section 5.2). Subsection 2.2.7 gives further explanation as to the importance of these distinctions.

It can be seen that the elimination rule presented in subsection 5.2.1. based on a two-dimensional target-oriented risk measure is an extension of the above rule, which is directly applicable here. The redefinition of the risk measures' functional form (specific in section 5.2.1.) is straightforward for the MC problem, and will thus not be presented here.

b. Joint Probabilities

More generally, one can control for the joint probability of scoring below a vector of critical levels defined over any subset of criteria, i.e., if $p(x^k_{i.} < x^k_{cr},\ k = 1, \ldots, p) > P^*_{min}$, then reject o_i, for all $o_i \varepsilon O$, where

$$p(x^k_{i.} < x^k_{cr},\ k = 1, \ldots, p) = \sum_{k=1}^{p} \sum_{j | x^k_{ij} < x^k_{cr}} P_j$$

and p^*_{min} is the minimum acceptable probability of obtaining a vector of scores below the critical vector.

Similarly, one could consider (both marginal and joint) probabilities of $x^k_{i.}$ being within pre-specified intervals† instead of simply exceeding a lower bound. The range of possible specifications is obviously quite wide.

Furthermore, it should be noted that the elimination phase is particularly well suited for handling non-quantitative critera, such as aesthetics, community disruption, etc. Aspects could then be defined based on categorical information (e.g., high, low, etc.).

To conclude this section, it should be noted that while the ideas underlying the elimination phase are to a large extent obvious, it is recognized as a plausible step which should be performed as part of the decision aiding framework. As discussed in Chapter 1, current practice in transportation planning includes many examples of the use of satisficing service standards, as in the case of statewide programming. We thus try to take advantage of an approach that decision-makers feel comfortable with to reduce the size of the problem.

A recent example from the Cairo, Egypt setting includes the development of a "political desirability index" and its use in conjunction with a

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†Odom et al. (1979) use this formulation in a multiobjective subset selection problem under uncertainty.

cut-off rule to eliminate options that are not institutionally acceptable (Mahmassani and Gakenheimer, 1981b).

APPENDIX B

MATHEMATICAL CONCEPTS AND DEFINITIONS USED IN DECISION THEORY

This appendix presents a number of fundamental mathematical concepts and definitions frequently used in decision theory, and at various instances throughout this thesis. The material presented herein draws heavily on Fishburn (1973a) and White (1976); those texts should be consulted for additional information and more advanced material.

Binary Relations

A binary relation R, defined over a set X consists of a particular property which exists between two elements x, y ϵ X. The notation [x R y] means as "x has the relation R to y." For example, if X is the set of legislators the relation "is more powerful than" is a binary relation, whereby [x R y] means that congressman x is more powerful than congressman y. The notation (x,y) ϵ R is sometimes used to denote [x R y]. Thus a binary relation R on a set X is a subset of ordered pairs in the product X x X.

If R is a binary relation on X, and x,y ϵ X, then exactly one of the following holds: 1) [x R y] and [y R x]; 2) [x R y] and not [y R x]; 3) not [x R y] and [y R x] or 3) not [x R y] and not [y R x].

Properties of Binary Relations

The following are a few properties that binary relations may possess that are of interest in decision theory:

1. Reflexivity: R is reflexive when [x R x] is a true assertion.

2. Irreflexivity: R is irreflexive when [x R x] cannot be a true assertion.

3. Symmetry: R is symmetric when [x R y] $\Rightarrow$ [y R x].

4. Asymmetry: R is asymmetric when [x R y] $\Rightarrow$ not [y R x].

5. Antisymmetry: R is antisymmetric when {[x R y] and [y R x]} $\Rightarrow$ x = y (the "equal" sign indicates that x and y are identical).

6. Transitivity: R is transitive when {[x R y] and [y R z]} $\Rightarrow$ x R y .

7. Completeness: R is complete when [x R y] or [y R x], $\forall$ x,y $\in$ X. This property is also referred as connectedness or comparability.

Special Binary Relations

Some binary relations possessing compatible combinations of the above properties are frequently encountered in decision theory and have therefore been given special names. A number of these are defined hereafter.

1. Equivalence Relations: An equivalence relation E defined as a set X is a binary relation on X which has the following properties:

   - reflexivity
   - symmetry
   - transitivity

. Order Relations: An order relation usually refers to a transitive binary relation. Order relations are usually classified in

two categories:

- asymmetric orders; these are necessarily irreflexive, and include preference relations.

- reflexive orders; these are not asymmetric, and include indifference relations.

Letting P denote an asymmetric order relation, the following are common special cases of asymmetric orders:

a. strict partial order: P is irreflexive and transitive

b. weak order: P is asymmetric and negatively transitive

c. linear order: P is irreflexive, transitive and complete. The prime example of a linear order is the relation > on the set of real numbers.

Reflexive orders are usually discussed in conjunction with preference-or-indifference relations, consisting of the union of a preference relation P (usually an asymmetric order) and an indifference relation I (which is usually reflexive). The resulting preference-or-indifference relation R is defined as [x R y] if and only if [x P y] or [x I y]. Common types of reflexive orders include the following combinations of properties:

a. preorder: R is reflexive and transitive

b. weak order: R is transitive and complete

c. partial order: R is reflexive transitive and antisymmetric

d. complete (or total) order: R is transitive, complete and antisymmetric.

A frequently encountered example of a complete order is $\leq$, defined over the set of real numbers.

The above were only basic mathematical definitions.