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The quenching behavior of a nonlinear parabolic equation with a singular boundary condition

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Abstract

In this paper, we study the quenching behavior of solution of a nonlinear parabolic equation with a singular boundary condition. We prove finite-time quenching for the solution. Further, we show that quenching occurs on the boundary under certain conditions. Furthermore, we show that the time derivative blows up at quenching point. Also, we get a lower solution and an upper bound for quenching time. Finally, we get a quenching rate and lower bounds for quenching time.

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1. Introduction

In this paper, we study the quenching behavior of solutions of the following nonlinear parabolic equation with a singular boundary condition:

$$(1.1)\qquad \begin{cases} u_t = u_{xx} + (1-u)^{-p}, \ 0 < x < 1, \ 0 < t < T, \\ u_x(0,t) = 0, \ u_x(1,t) = (1-u(1,t))^{-q}, \ 0 < t < T, \\ u(x,0) = u_0(x), \ 0 \le x \le 1, \end{cases}$$

where p, q are positive constants and $T \le \infty$. The initial function $u_0 : [0,1] \to (0,1)$ satisfies the compatibility conditions

$$u_0'(0) = 0, \ u_0'(1) = (1-u_0(1))^{-q}.$$

Throughout this paper, we also assume that the initial function u_0 satisfies the inequalities

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$$u_{xx}(x,0)+(1-u(x,0))^{-p}\geq 0, \tag{1.2}$$

$$u_x(x,0)\geq 0 \tag{1.3}$$

Our main purpose is to examine the quenching behavior of the solutions of problem (1.1) having two singular heat sources. A solution $u(x,t)$ of the problem (1.1) is said to quench if there exists a finite time T such that

$$\lim_{t\to T^-}\max\{u(x,t):0\leq x\leq 1\}\to 1.$$

From now on, we denote the quenching time of the problem (1) with T.

The concept of quenching was first introduced by Kawarada. In [12], Kawarada has considered an initial-boundary value problem for the parabolic equation $u_t=u_{xx}+1/(1-u)$. Then, the quenching problems have been studied extensively by several researchers (cf. the surveys by Chan [1, 2] and Kirk and.Roberts [14] and [3], [4], [6], [8], [9],[10], [13], [15], [16], [17], [19]). In the literature, the quenching problems have been less studied with two nonlinear heat sources. We give as examples two of these papers. Chan and Yuen [5] considered the problem

$$\begin{aligned}&u_t=u_{xx},\ \text{in }\Omega,\\ &u_x(0,t)=(1-u(0,t))^{-p},\ u_x(a,t)=(1-u(a,t))^{-q},\ 0<t<T,\\ &u(x,0)=u_0(x),\ 0\leq u_0(x)<1,\ \text{in }\bar{D},\end{aligned}$$

where $a,p,q>0, T\leq\infty, D=(0,a), \Omega=D\times(0,T)$. They showed that $x=a$ is the unique quenching point in finite time if u_0 is a lower solution, and u_t blows up at quenching. Further, they obtained criteria for nonquenching and quenching by using the positive steady states. Zhi and Mu [20] considered the problem

$$\begin{aligned}&u_t=u_{xx}+(1-u)^{-p},\ 0<x<1,\ 0<t<T\\ &u_x(0,t)=u^{-q}(0,t),\ u_x(1,t)=0,\ 0<t<T,\\ &u(x,0)=u_0(x),\ 0<u_0(x)<1,\ 0\leq x\leq 1,\end{aligned}$$

where $p,q>0$ and $T\leq\infty$. They showed that $x=0$ is the unique quenching point in finite time if u_0 satisfies $u_0''(x)+(1-u_0(x))^{-p}\leq 0$ and $u_0'(x)\geq 0$. Further, they obtained the quenching rate estimates which is $(T-t)^{1/2(q+1)}$ if T denotes the quenching time. Further, the quenching problems have been less studied with combined power-type nonlinearities ([7], [18]) in the literature. In [18], Xu et.al. the studied the following quenching behavior for the solutions of parabolic equation with combined power-type nonlinearities:

$$\begin{aligned}&u_t-\Delta u=\sum_{k=2}^{q}(b-u(x,t))^{-k},\ \text{in }\Omega\times(0,T),\\ &u(x,t)=0,\ \text{on }\partial\Omega\times(0,T),\\ &u(x,0)=u_0(x),\ \text{in }\Omega,\end{aligned}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega, q>2, b=const>0$. The initial data $u_0(x)\in C^1(\overline{\Omega})$ is nonnegative in Ω and $sup_{x\in\Omega}u_0(x)<b$. They showed that the solution of the above problem quenches in a finite time, and estimated its quenching time. Finally, they given numerical examples.

Here, we would like to study how the reaction term $(1-u)^{-p}$ and the boundary absorption term $(1-u)^{-q}$ affect the quenching behaviour of the solution of the problem (1.1). In Section 2, we first show that quenching occurs in finite time under the condition (1.2). Then, we show that the only quenching point is $x=1$ under the condition (1.2) and (1.3). Further we show that u_t blows up at quenching time. In Section 3, we get a

lower solution and an upper bound for quenching time. In Section 4, we get a quenching rate and lower bounds for quenching time.

2. Quenching on the boundary and blow-up of u_t

2.1. Remark. We assume that the condition (1.2) and (1.3) is proper. Namely, we can easily construct such a initial function satisfying (1.2),(1.3) and compatibility conditions. Let $0 < A < 1, \alpha = \frac{1}{A(1-A)}$ and $u(x,0) = Ax^{\alpha}$. For example, for $q = 1$ and $A = 0.5$, $u(x,0) = \frac{1}{2}x^4$ satisfies compatibility conditions, (1.2) and (1.3).

2.2. Remark. If u_0 satisfies (1.3), then we get $u_x > 0$ in $(0,1] \times (0,T)$ by the maximum principle. Thus we get $u(1,t) = \max\limits_{0 \le x \le 1} u(x,t)$.

2.3. Lemma. If u_0 satisfies (1.2), then $u_t(x,t) \ge 0$ in $[0,1] \times [0,T)$.
Proof. Let us prove it by utilizing Lemma 3.1 of [11]. Let $v = u_t(x,t)$. Then, $v(x,t)$ satisfies

$$\begin{aligned}
&v_t = v_{xx} + p\left(1-u\right)^{-p-1} v,\ 0 < x < 1,\ 0 < t < T,\\
&v_x\left(0,t\right) = 0,\ v_x\left(1,t\right) = q\left(1-u(1,t)\right)^{-q-1} v(1,t),\ 0 < t < T,\\
&v\left(x,0\right) = u_{xx}\left(x,0\right) + \left(1-u\left(x,0\right)\right)^{-p} \ge 0,\ 0 \le x \le 1.
\end{aligned}$$

For any fixed $\tau \in (0,T)$, let

$$\begin{aligned}
L &= \max_{0 \le x \le 1,\ 0 \le t \le \tau} \left(\frac{1}{2} q\left(1-u(x,t)\right)^{-q-1}\right),\\
M &= 2L + 4L^2 + \max_{0 \le x \le 1,\ 0 \le t \le \tau} \left(p\left(1-u(x,t)\right)^{-p-1}\right).
\end{aligned}$$

Set $w(x,t) = e^{-Mt - Lx^2} v(x,t)$. Then w satisfies

$$\begin{aligned}
&w_t = w_{xx} + 4Lxw_x + cw,\ 0 < x < 1,\ 0 < t \le \tau,\\
&w_x(0,t) = 0,\ w_x(1,t) = d(t)w(1,t),\ 0 < t \le \tau,\\
&w(x,0) \ge 0,\ 0 \le x \le 1,
\end{aligned}$$

where

$$c = c(x,t) = 4L^2(x^2-1) + p\left(1-u(x,t)\right)^{-p-1} - \max_{0 \le x \le 1,\ 0 \le t \le \tau} \left(p\left(1-u(x,t)\right)^{-p-1}\right) \le 0$$

and

$$d(t) = -\max_{0 \le x \le 1,\ 0 \le t \le \tau} \left(q\left(1-u(x,t)\right)^{-q-1}\right) + q\left(1-u(1,t)\right)^{-q-1} \le 0.$$

By the maximum principle and Hopf lemma, we obtain that $w \ge 0$ in $[0,1] \times [0,\tau]$. Thus, $u_t(x,t) \ge 0$ in $[0,1] \times [0,T)$. □

2.4. Theorem. If u_0 satisfies (1.2), then there exist a finite time T, such that the solution u of the problem (1.1) quenches at time T.
Proof. Assume that u_0 satisfies (1.2). Then there exist

$$w = \left(1-u\left(1,0\right)\right)^{-q} + \int_0^1 \left(1-u\left(x,0\right)\right)^{-p} dx > 0.$$

Introduce a mass function; $m\left(t\right) = \int_0^1 \left(1-u\left(x,t\right)\right) dx, 0 < t < T$. Then

$$m'\left(t\right) = -\left(1-u\left(1,t\right)\right)^{-q} - \int_0^1 \left(1-u\left(x,t\right)\right)^{-p} dx \le -w,$$

by Lemma 2.3. Thus, $m\left(t\right) \le m(0) - wt$; which means that $m\left(T_0\right) = 0$ for some $T_0, (0 < T \le T_0)$. Then u quenches in finite time. □

2.5. Theorem. If u_0 satisfies (1.2) and (1.3), then $x = 1$ is the only quenching point.
Proof. Define

$$J(x,t) = u_x - \varepsilon\,(x - (1-\eta)) \text{ in } [1-\eta, 1] \times [\tau, T),$$

where $\eta \in (0,1), \tau \in (0,T)$ and ε is a positive constant to be specified later. Then, $J(x,t)$ satisfies

$$J_t - J_{xx} = p(1-u)^{-p-1} u_x > 0 \text{ in } (1-\eta, 1) \times (\tau, T),$$

since $u_x(x,t) > 0$ in $(0,1] \times (0,T)$. Thus, $J(x,t)$ cannot attain a negative interior minimum by the maximum principle. Further, if ε is small enough, $J(x,\tau) > 0$ since $u_x(x,t) > 0$ in $(0,1] \times (0,T)$. Furthermore, if ε is small enough,

$$\begin{aligned} J(1-\eta, t) &= u_x(1-\eta, t) > 0, \\ J(1,t) &= (1-u(1,t))^{-q} - \varepsilon\eta > 1 - \varepsilon\eta > 0 \end{aligned}$$

for $t \in (\tau, T)$. By the maximum principle, we obtain that $J(x,t) > 0$, i.e. $u_x > \varepsilon\,(x - (1-\eta))$ for $(x,t) \in [1-\eta, 1] \times [\tau, T)$. Integrating this with respect to x from $(1-\eta)$ to 1, we have

$$u(1-\eta, t) < u(1,t) - \frac{\varepsilon\eta^2}{2} < 1 - \frac{\varepsilon\eta^2}{2}.$$

So u does not quench in $[0,1)$. The theorem is proved. □

2.6. Theorem. u_t blows up at the quenching point $x = 1$.
Proof. We will prove that u_t blows up at quenching, as in [5]. Suppose that u_t is bounded on $[0,1] \times [0,T)$. Then, there exists a positive constant M such that $u_t < M$. We have $u_{xx} + (1-u)^{-p} < M \Rightarrow u_{xx} < M$. Integrating this twice with respect to x from x to 1, and then from 0 to 1, we have

$$\frac{1}{(1-u(1,t))^q} < \frac{M}{2} + u(1,t) - u(0,t).$$

As $t \to T^-$, the left-hand side tends to infinity, while the right-hand side is finite. This contradiction shows that u_t blows up somewhere. □

3. A lower solution and an upper bound for the quenching time

3.1. Definition. μ is called a lower solution of problem (1.1) if μ satisfies the following conditions:

$$\begin{aligned} &\mu_t - \mu_{xx} \le (1-\mu)^{-p},\ 0 < x < 1,\ 0 < t < T, \\ &\mu_x(0,t) = 0,\ \mu_x(1,t) \le (1-\mu(1,t))^{-q},\ 0 < t < T, \\ &\mu(x,0) \le u_0(x),\ 0 \le x \le 1. \end{aligned}$$

It is an upper solution when the inequalities are reversed.

3.2. Lemma. Let u be a solution and μ be a lower solution of problem (1.1) in $[0,1] \times [0,T)$. Then $u \ge \mu$ in $[0,1] \times [0,T)$.
Proof. Let $v(x,t) = u(x,t) - \mu(x,t)$. Then $v(x,t)$ satisfies

$$\begin{aligned} &v_t \ge v_{xx} + p\,(1-\eta)^{-p-1} v,\ 0 < x < 1,\ 0 < t < T, \\ &v_x(0,t) = 0,\ v_x(1,t) \ge q\,(1-\xi(1,t))^{-q-1} v(1,t),\ 0 < t < T, \\ &v(x,0) \ge 0,\ 0 \le x \le 1, \end{aligned}$$

where $\eta(x,t)$ lies between $u(x,t)$ and $\mu(x,t)$ and $\xi(1,t)$ lies between $u(1,t)$ and $\mu(1,t)$.

For any fixed $\tau \in (0,T)$, let

$$\begin{aligned} L &= \max_{0\le x\le 1,\ 0\le t\le \tau} \left(\frac{1}{2} q\left(1-\xi(x,t)\right)^{-q-1}\right), \\ M &= 2L+4L^2+\max_{0\le x\le 1,\ 0\le t\le \tau} \left(p\left(1-\eta(x,t)\right)^{-p-1}\right). \end{aligned}$$

Set $w(x,t)=e^{-Mt-Lx^2}v(x,t)$. Then w satisfies

$$\begin{aligned} &w_t \ge w_{xx}+4Lxw_x+cw,\ 0<x<1,\ 0<t\le\tau, \\ &w_x(0,t)=0,\ w_x(1,t)\ge d(t)w(1,t),\ 0<t\le\tau, \\ &w(x,0)\ge 0,\ 0\le x\le 1, \end{aligned}$$

where $c=c(x,t)\le 0$ and $d=d(t)\le 0$. By the maximum principle, we obtain that $w\ge 0$ in $[0,1]\times[0,\tau]$. Thus, $u\ge\mu$ in $[0,1]\times[0,T)$. □

3.3. Theorem. $x=1$ is a quenching point.

Proof. Let $\min_{x\in[0,1]} u_0(x)=c\ge 0$. Define

$$\mu(x,t)=1-\left(\frac{(q+1)\left(1-x^2+\tau-t\right)}{2}\right)^{1/(q+1)} \quad \text{in } [0,1]\times[0,\tau],$$

where $\tau=2(1-c)^{q+1}/(q+1)$. We have

$$\begin{aligned} \mu_t-\mu_{xx} &= \frac{-1}{2}\left(\frac{(q+1)\left(1-x^2+\tau-t\right)}{2}\right)^{-q/(q+1)} \\ &\quad -x^2 q\left(\frac{(q+1)\left(1-x^2+\tau-t\right)}{2}\right)^{(-2q-1)/(q+1)} \\ &\le 0 \end{aligned}$$

for $x\in(0,1), t\in(0,\tau]$. Further,

$$\begin{aligned} \mu_x(0,t) &= 0, \\ \mu_x(1,t) &= \left(1-\mu(1,t)\right)^{-q} \end{aligned}$$

for $t\in(0,\tau]$. Furthermore,

$$\mu(x,0)=1-\left(\frac{(q+1)\left(1-x^2+\tau\right)}{2}\right)^{1/(q+1)} \le 1-\left(\frac{(q+1)\,\tau}{2}\right)^{1/(q+1)}=c,$$

for $x\in[0,1]$. Thus, $\mu(x,t)$ is a lower solution of the problem (1.1). In addition, at $t=\tau$ and $x=1$, we get

$$\mu(1,\tau)=1.$$

Hence, we have

$$u(1,\tau)\ge\mu(1,\tau)=1$$

by Lemma 3.2. Thus, $x=1$ is a quenching point. □

3.4. Remark. We can calculate an upper bound for the quenching time. From Theorem 3.3, maximum upper bound is $T=2/(q+1)$ (for $c=0$). Also, as in Remark 2.1, $u_0(x)=\frac{1}{2}x^4$ (for $q=1$), then we have $T=1$.

4. A quenching rate and lower bounds for the quenching time

In this section, we get a quenching rate and lower bounds for quenching time. Throughout this section, we assume that

$$u_x(x,0) \geq x(1-u(x,0))^{-q}, 0<x<1, \tag{4.1}$$

$$u_t(1,t) = u_{xx}(1,t) + (1-u(1,t))^{-p}, 0<t<T. \tag{4.2}$$

4.1. Theorem. If u_0 satisfies (1.2), (1.3), (4.1) and (4.2), then there exists positive constants C_1 and C_2 such that

$$\begin{aligned} \text{if } p &> 2q+1, \text{ then } u(1,t) \geq 1 - C_1(T-t)^{1/(p+1)}, \\ \text{if } q &\leq p \leq 2q+1, \text{ then } u(1,t) \geq 1 - C_2(T-t)^{1/(2q+2)}, \end{aligned}$$

for t sufficiently close to T.

Proof. Define

$$J(x,t) = u_x - x(1-u)^{-q} \text{ in } [0,1]\times[0,T).$$

Then, $J(x,t)$ satisfies

$$\begin{aligned} J_t - J_{xx} - p(1-u)^{-p-1}J &= 2q(1-u)^{-q-1}u_x + (p-q)x(1-u)^{-p-q-1} \\ &\quad + xq(q+1)(1-u)^{-q-2}u_x^2, \end{aligned}$$

since $u_x > 0$ and $p \geq q$, $J(x,t)$ cannot attain a negative interior minimum. On the other hand, $J(x,0) \geq 0$ by (4.1) and

$$J(0,t) = 0, J(1,t) = 0,$$

for $t \in (0,T)$. By the maximum principle, we obtain that $J(x,t) \geq 0$ for $(x,t) \in [0,1]\times[0,T)$. Therefore

$$J_x(1,t) = \lim_{h\to 0^+} \frac{J(1,t) - J(1-h,t)}{h} = \lim_{h\to 0^+} \frac{-J(1-h,t)}{h} \leq 0.$$

From (4.2), we get

$$\begin{aligned} J_x(1,t) &= u_{xx}(1,t) - (1-u(1,t))^{-q} - q(1-u(1,t))^{-2q-1} \\ &= u_t(1,t) - (1-u(1,t))^{-p} - (1-u(1,t))^{-q} - q(1-u(1,t))^{-2q-1} \leq 0 \end{aligned}$$

and

$$\begin{aligned} \text{if } p &> 2q+1, \text{ then } u_t(1,t) \leq (q+2)(1-u(1,t))^{-p}, \\ \text{if } q &\leq p \leq 2q+1, \text{ then } u_t(1,t) \leq (q+2)(1-u(1,t))^{-2q-1}. \end{aligned}$$

Integrating for t from t to T we get

$$\begin{aligned} \text{if } p &> 2q+1, \text{ then } u(1,t) \geq 1 - C_1(T-t)^{1/(p+1)}, \\ \text{if } q &\leq p \leq 2q+1, \text{ then } u(1,t) \geq 1 - C_2(T-t)^{1/(2q+2)}, \end{aligned}$$

where $C_1 = [(q+2)(p+1)]^{1/(p+1)}$ and $C_2 = [(q+2)(2q+2)]^{1/(2q+2)}$. □

4.2. Remark. We can calculate a lower bound for the quenching time. From Theorem 4.1, lower bounds are

$$\begin{aligned} \text{if } p &> 2q+1, \text{ then } T = (1-u_0(1))^{p+1}/(q+2)(p+1), \\ \text{if } q &\leq p \leq 2q+1, \text{ then } T = (1-u_0(1))^{2q+2}/(q+2)(2q+2), \end{aligned}$$

for quenching time T. If we choose, as Remark 1, $u_0(x) = \frac{1}{2}x^4$ (for $q=1$), then we have

$$\begin{aligned} T &\approx 0.0021 \text{ for } p=4, q=1, \\ T &\approx 0.0052 \text{ for } 1 \leq p < 3, q=1. \end{aligned}$$

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