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Modelling the Dependence between a Wiener Process and Its Running Maxima and Running Minima Processes

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Abstract: We study a triple of stochastic processes: a Wiener process W_t, $t \geq 0$, its running maxima process $M_t = \sup\{W_s : s \in [0,t]\}$, and its running minima process $m_t = \inf\{W_s : s \in [0,t]\}$. We derive the analytical formula for the corresponding copula and show that it is supported on the hemicube, a convex hexahedron with seven vertices. As an application, we draw out an analytical formula for pricing of a double barrier option.

Keywords: copulas; Wiener process; running maxima and minima; strong Markov property; reflection principle; Spearman rho; double-barrier options

MSC: 62H05; 60G18; 60G70; 60E05; 62H20; 91G20

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1. Introduction

The Wiener process is one of the most ubiquitous stochastic processes used to model complex phenomena. One of such areas is financial mathematics, where this process is the foundation for all models of stock, derivatives, or portfolio analysis. But with the development of the market, more complex derivatives have shown up, and a simple approach to the Wiener process was no longer enough. In particular, there was a question as to what is the distribution of the Wiener process adding boundaries on trajectories. It is especially important when one is dealing with barrier options.

The structure of article is as follows. In the first part, there will be provided the main results. We derive the cumulative distribution function and copula for a triple consisting of a Wiener process and its supremum and infimum. The cumulative distribution function is supported on a cone and the copula on a hemicube. Then, we present special cases of this distribution, particularly focusing on marginals. The next part is devoted to proofs of stated theorems. The starting point is the strong Markov property of Wiener processes. First, we study the simple case based on one stopping time. Finally, to the obtain desired results, we move to double stopping time. We provide a recursive dependency, which will lead to the characterization of the cumulative distribution the function and copula in terms of series.

The last parts show applications in financial mathematics, i.e., we calculate the price of a European double-barrier option in the Black–Scholes model. Compare [1–3]. Our approach is based on the Girsanov theorem, which implies the existence a probabilistic measure Q under which the increment of the logarithm of a stock price process divided by the volatility follows the Wiener law. The double change of measure, from the risk neutral to Q and back after some integral calculations, significantly simplifies the proof.

2. Main Results

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Wiener process $(W_t)_{t\geq 0}$ with natural filtration with respect to that process, i.e., $\mathcal{F}_t = \sigma\{ W_s^{-1}(A) \big| s \leq t, A \in \mathcal{B}(\mathbb{R})\}$—where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$.

We denote as

$$M_t := \sup_{0\le s\le t} W_s,$$

and

$$m_t := \inf_{0\le s\le t} W_s$$

the associated running maxima and minima processes.

Note that the triple (W_t, M_t, m_t), $t > 0$ is a self-similar process:

$$\forall t > 0 \quad (W_t, M_t, m_t) \stackrel{d}{=} (\sqrt{t}W_1, \sqrt{t}M_1, \sqrt{t}m_t). \tag{1}$$

Let $F_t(x, y, z)$ be the cumulative distribution function of the joint distribution (W_t, M_t, m_t), where $t > 0$. As follows from the celebrated Sklar theorem, there exists a copula process, C_t, $t > 0$, such that

$$F_t(x, y, z) = C_t\big(\mathbb{P}(W_t \le x), \mathbb{P}(M_t \le y), \mathbb{P}(m_t \le z)\big).$$

For more details about copula theory and some of its applications, we refer to [4–10]. Since the process (W_t, M_t, m_t) is self-similar, the copula C_t of the triple (W_t, M_t, m_t), where $t > 0$, does not depend on t. We will denote it as $C_{W,M,m}$.

Based on the strong Markov property of a Wiener process and the Sklar theorem, we prove what follows, where Φ denotes the cumulative distribution of the $N(0, 1)$ probability law.

Theorem 1. *The joint cumulative distribution function $F_t(x, y, z)$ of (W_t, M_t, m_t), where $t > 0$, is of the form*

$$F_t(x, y, z) = C_{W,M,m}\left(\Phi(\frac{x}{\sqrt{t}}), \left(2\Phi(\frac{y}{\sqrt{t}}) - 1\right)^+, \min(2\Phi(\frac{z}{\sqrt{t}}), 1)\right), \quad t > 0,$$

where

$$C_{W,M,m}(u, v, w) = \begin{cases} u - \Phi\big(\Phi^{-1}(u) - r\big) & \text{for } 2u \le w, \\ 2\Psi\big(\Phi^{-1}(w/2), r, s\big) - \Psi\big(\Phi^{-1}(u) - s, r, s\big) & \\ \quad -\Psi\big(-\Phi^{-1}(u) - s + r, r, s\big) & \text{for } w < 2u \le 1 + v, \\ 2\Psi\big(\Phi^{-1}(w/2), r, s\big)\big) - 2\Psi\big(2\Phi^{-1}(w/2) - \Phi^{-1}((1 + v)/2), r, s\big) & \text{for } 1 + v < 2u, \end{cases}$$

where Ψ is given by formula

$$\Psi(q, r, s) = \sum_{k=0}^{\infty} (\Phi(q - ks) - \Phi(q - r - ks)) \tag{2}$$

with

$$r = 2\Phi^{-1}\big((1 + v)/2\big) \ \text{ and } \ s = 2\big(\Phi^{-1}((1 + v)/2) - \Phi^{-1}(w/2)\big)\big).$$

The proof is provided in Section 3.6.

Note that the copula $C_{W,M,m}$ is absolutely continuous with respect to the Lebesgue measure and is supported on the polyhedron

$$P = \{(u, v, w) \in [0, 1]^3 : \ w \le 2u \le 1 + v\},$$

which has seven vertices

$$(0, 0, 0), \ (0, 1, 0), \ (1, 1, 0), \ (1, 1, 1), \ (1/2, 0, 1), \ (1/2, 0, 0), \ (1/2, 1, 1),$$

six faces, two rectangles, two trapezoids, two triangles, and eleven edges. Such polyhedrons are called hemicubes because they are linearly equivalent to a polyhedron obtained by cutting the unit cube by half by the surface containing two opposite vertices (for example, $(0,0,0)$ and $(1,1,1)$) (Figure 1).

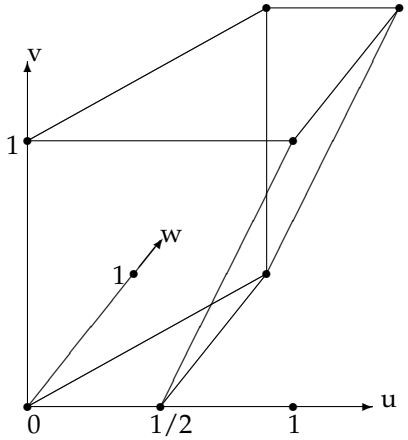


Figure 1. Polyhedron P—the support of the $C_{W,M,m}$ copula.

Furthermore, the copula $C_{W,M,m}$ is invariant with respect to the involution induced by the rotation of the unit cube by an angle π about the line

$$\{(u,v,w) : u = 0.5, v + w = 1\}.$$

For every triple of generators (U,V,W) of the copula $C_{W,M,m}$, the triple $(1-U, 1-W, 1-V)$ is a set of generators of this copula as well. Hence,

$$\begin{aligned} C_{W,M,m}(u,v,w) &= \mathbb{P}(U \le u, V \le v, W \le w) = \mathbb{P}(1-U \le u, 1-W \le v, 1-V \le w) \\ &= u + v + w - 2 + C_{W,M,m}(1, 1-w, 1-v) + C_{W,M,m}(1-u, 1, 1-v) \\ &\quad + C_{W,M,m}(1-u, 1-w, 1) - C_{W,M,m}(1-u, 1-w, 1-v). \end{aligned} \tag{3}$$

The marginal copulas are given by

$$\begin{aligned} C_{W,M}(u,v) = C_{W,M,m}(u,v,1) &= \begin{cases} u - \Phi\big(\Phi^{-1}(u) - 2\Phi^{-1}((1+v)/2)\big) & \text{for } 2u \le 1+v, \\ v & \text{for } 1+v < 2u, \end{cases} \\ C_{W,m}(u,w) = C_{W,M,m}(u,1,w) &= \begin{cases} u & \text{for } 2u \le w, \\ w - \Phi\big(2\Phi^{-1}(w/2) - \Phi^{-1}(u)\big) & \text{for } w < 2u, \end{cases} \\ C_{M,m}(v,w) = C_{W,M,m}(1,v,w) &= 2\Psi\big(\Phi^{-1}(w/2), r, s\big) - 2\Psi\big(2\Phi^{-1}(w/2) - \Phi^{-1}((1+v)/2), r, s\big). \end{aligned}$$

All three of the above bivariate copulas are absolutely continuous. The first two are supported on trapezoids. The first one is supported on

$$\{(u,v) \in [0,1]^2 : 2u \le 1+v\},$$

the second one is supported on

$$\{(u,w) \in [0,1]^2 : w \le 2u\}.$$

The third one has a full support $[0,1]^2$. The scatterplots of the above copulas are shown on Figures 2–4.

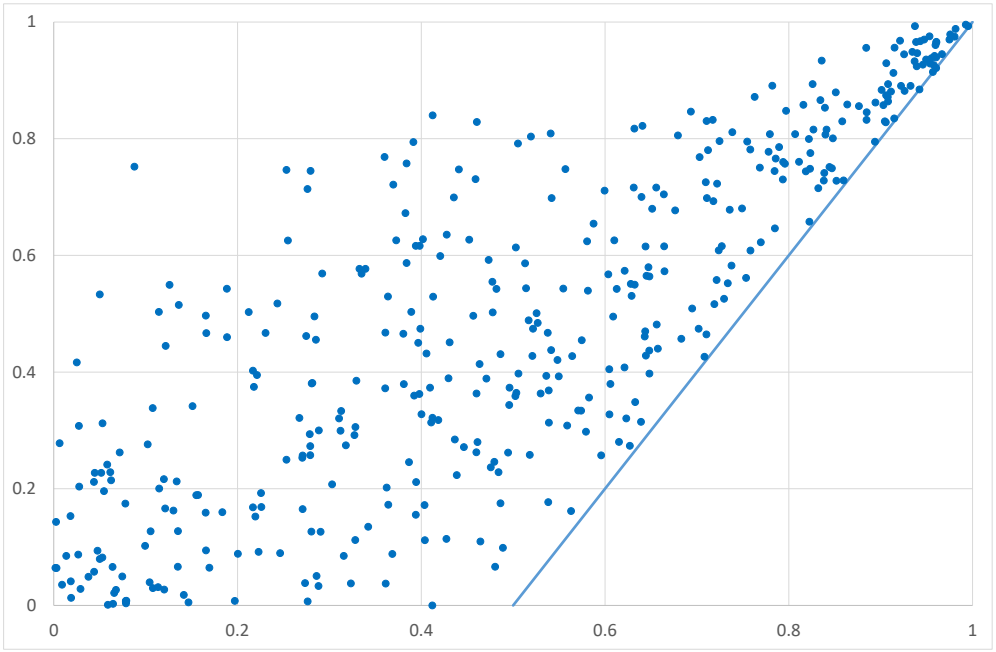


Figure 2. Copula $C_{W,M}$ (scatterplot).

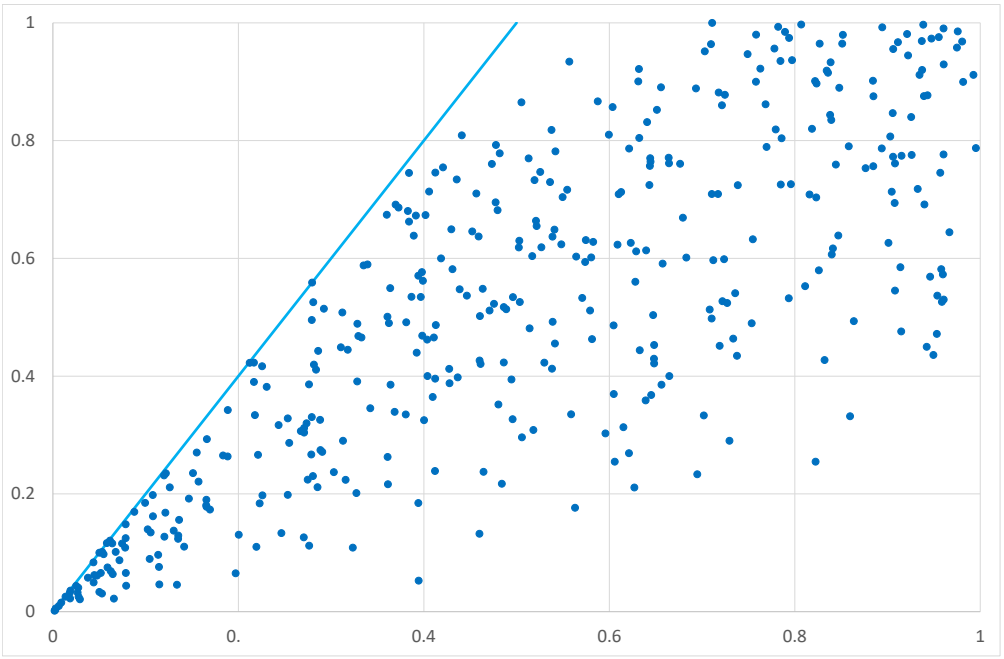


Figure 3. Copula $C_{W,m}$ (scatterplot).

Furthermore, the first and second copula are duals of each other: $C_{W,m}$ is a survival copula of $C_{W,M}$ (and vice versa). As a matter of fact, they are reflections of a copula introduced in [11]. See also [12]. We have

$$C_{W,m}(u,w) = u + w - 1 + C_{W,M}(1-u, 1-w). \tag{4}$$

The third one is self-dual with respect to reflection about the line $v + w = 1$,

$$C_{M,m}(v,w) = u + v - 1 + C_{M,m}(1-w, 1-v). \tag{5}$$

All three marginal copulas are highly dependent.

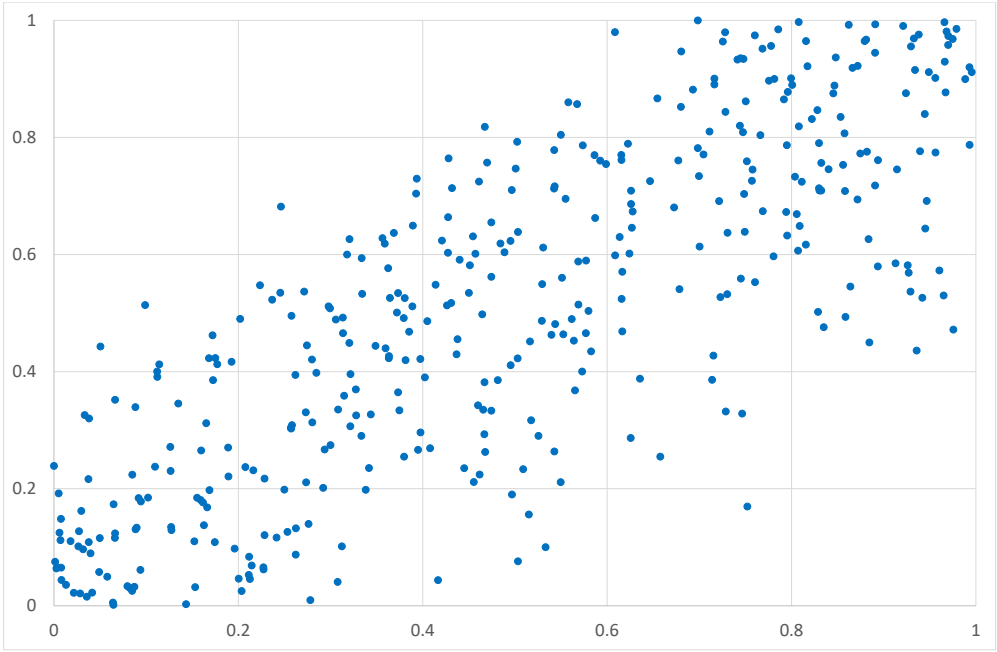


Figure 4. Copula $C_{M,m}$ (scatterplot).

Proposition 1. *The Spearman ρ of the marginal copulas of the copula $C_{W,M,m}$ is given by*

$$\begin{aligned}
\rho(C_{W,M}) &= \rho(C_{W,m}) = 2 - \frac{6}{\pi}\arccos\left(\frac{\sqrt{6}}{3}\right) \approx 0.8245,\\
\rho(C_{M,m}) &= \frac{6}{\pi}\sum_{n=0}^{\infty}(-1)^n\left(\arccos\left(\frac{n}{\sqrt{2(n(n+1)+1)}}\right) + \arccos\left(\frac{n+1}{\sqrt{2(n(n+1)+1)}}\right)\right.\\
&\quad \left. - \arccos\left(\frac{n+1}{\sqrt{2((n+1)(n+2)+1)}}\right) - \arccos\left(\frac{n+2}{\sqrt{2((n+1)(n+2)+1)}}\right)\right)\\
&\approx 0.80649.
\end{aligned}$$

The proofs are provided in Section 3.7.

3. Proofs and Auxiliary Results

3.1. Strong Markov Property

The strong Markov property plays a crucial role in the proof of Theorem 1. We recall the definition.

Definition 1. *Suppose that $X = (X_t : t \geq 0)$ is a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Then, for any stopping time τ on Ω , we can define*

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : \forall t \geq 0, \{\tau \leq t\} \cap A \in \mathcal{F}_t\}.$$

Then, X is said to have the strong Markov property if, for each finite stopping time τ , we have that for each $t \geq 0$, $X_{\tau+t} - X_\tau$ is independent of $\mathcal{F}_\tau$.

It is well known that the Wiener process fulfills the above definition; see, for example, [13] (Section 37, Theorem 37.5), [14] (Section 3.2), or [15] (Theorem 6.5). Furthermore, for a Wiener process $W_t, t \geq 0$, we have

Theorem 2. *Let τ be a finite stopping time and put*

$$W_t^*(\omega) = W_{\tau+t}(\omega) - W_\tau(\omega), \quad t \geq 0, \ \ \omega \in \Omega. \tag{6}$$

Then, $W_t^* : t \geq 0$ is a Brownian motion, and it is independent of $\mathcal{F}_\tau$.

Based on the fact that W_t^* is symmetric

$$\forall t > 0 \quad W_t^* \stackrel{d}{=} -W_t^*$$

we obtain the following version of the reflection principle.

Theorem 3. *Let τ be a finite stopping time and let a set A be an element of $\mathcal{F}_\tau$ of positive probability on which $\tau \leq t$,*

$$A \in \mathcal{F}_\tau, \ \ \mathbb{P}(A) > 0, \ \ A \subset \{\omega : \tau(\omega) \leq t\},$$

such that under the condition A, the probability laws of W_t and $2W_\tau - W_t$ coincide:

$$W_t \,|\, A \ \stackrel{d}{=} 2W_\tau - W_t \,|\, A. \tag{7}$$

Proof. We observe that under the condition $\tau \leq t$,

$$\begin{aligned} W_t &= W_t - W_\tau + W_\tau = W_{t-\tau}^* + W_\tau = W_{|t-\tau|}^* + W_\tau, & (8)\\ 2W_\tau - W_t &= -(W_t - W_\tau) + W_\tau = -W_{t-\tau}^* + W_\tau = -W_{|t-\tau|}^* + W_\tau. & (9) \end{aligned}$$

Next, we compare the conditional distribution functions of the right sides of the above equations. We apply the fact that the random variables τ and W_τ are $\mathcal{F}_\tau$ measurable and

$$\begin{aligned} \mathbb{P}(W_{|t-\tau|}^* \leq x - W_\tau \,|\, \mathcal{F}_\tau) &\stackrel{a.s.}{=} \begin{cases} \mathbf{1}_{W_\tau \leq x} & \text{if } \tau = t \\ \Phi\left(\frac{x - W_\tau}{\sqrt{t-\tau}}\right) & \text{other-ways} \end{cases} & (10)\\ &= \begin{cases} \mathbf{1}_{W_\tau \leq x} & \text{if } \tau = t \\ 1 - \Phi\left(\frac{W_\tau - x}{\sqrt{|t-\tau|}}\right) & \text{other ways} \end{cases} & (11)\\ &\stackrel{a.s.}{=} \mathbb{P}(-W_{|t-\tau|}^* \leq x - W_\tau \,|\, \mathcal{F}_\tau) \end{aligned}$$

where Φ denotes the cumulative distribution function of a standard normal random variable ($N(0,1)$).

To conclude the proof, we apply the second point of the definition of the conditional expected value (see [13] §34).

$$\begin{aligned} \mathbb{P}(W_{t-\tau}^* + W_\tau \leq x \,|\, A) &= \frac{\mathbb{E}\left(\mathbf{1}_{W_{|t-\tau|}^* + W_\tau \leq x} \mathbf{1}_A\right)}{\mathbb{P}(A)} & (12)\\ &= \frac{1}{\mathbb{P}(A)} \int_A \mathbf{1}_{W_{|t-\tau|}^* + W_\tau \leq x} d\mathbb{P} \\ &= \frac{1}{\mathbb{P}(A)} \int_A \mathbb{E}\left(\mathbf{1}_{W_{|t-\tau|}^* + W_\tau \leq x} \,|\, \mathcal{F}_\tau\right) d\mathbb{P} \\ &= \frac{1}{\mathbb{P}(A)} \int_A \mathbb{E}\left(\mathbf{1}_{-W_{|t-\tau|}^* \leq x - W_\tau} \,|\, \mathcal{F}_\tau\right) d\mathbb{P} \\ &= \frac{1}{\mathbb{P}(A)} \int_A \mathbf{1}_{-W_{|t-\tau|}^* \leq x - W_\tau} d\mathbb{P} \\ &= \frac{\mathbb{E}\left(\mathbf{1}_{-W_{|t-\tau|}^* + W_\tau \leq x} \mathbf{1}_A\right)}{\mathbb{P}(A)} \\ &= \mathbb{P}(-W_{t-\tau}^* + W_\tau \leq x \,|\, A). \end{aligned}$$

□

3.2. Marginal Distributions

We consider two families of stopping times. One corresponds to an upper threshold, and the second corresponds to a lower one. In more details, for $y > 0$ and $z < 0$, we put

$$T_y = \inf\{t > 0 : W_t = y\}, \tag{13}$$
$$T_z = \inf\{t > 0 : W_t = z\}. \tag{14}$$

Note that for fixed positive t, the event $T_y \leq t$ is equivalent to the event $M_t \geq y$:

$$\{\omega : T_y(\omega) \leq t\} = \{\omega : M_t(\omega) \geq y\} \tag{15}$$

and the event $T_z \leq t$ is equivalent to the event $m_t \leq z$:

$$\{\omega : T_z(\omega) \leq t\} = \{\omega : m_t(\omega) \leq z\}. \tag{16}$$

We apply the reflection principle for $\tau = T_y$, where $y > 0$. Since

$$W_{T_y} = y,$$

the reflection principle simplifies to

$$W_t \,|T_y \leq t \stackrel{d}{=} 2y - W_t | T_y \leq t. \tag{17}$$

For $y, t > 0$ we obtain

$$\begin{aligned} \mathbb{P}(W_t \leq x, M_t \geq y) &= \mathbb{P}(W_t \leq x, T_y \leq t) \\ &= \mathbb{P}(W_t \leq x \,|T_y \leq t)\mathbb{P}(T_y \leq t) \\ &= \mathbb{P}(2y - W_t \leq x \,|T_y \leq t)\mathbb{P}(T_y \leq t) \\ &= \mathbb{P}(W_t \geq 2y - x, M_t \geq y). \end{aligned} \tag{18}$$

The further assumption $y \geq x$ implies that the event $M_t \geq y$ contains the event $W_t \geq 2y - x$

$$\{\omega : W_t(\omega) \geq 2y - x\} \subset \{\omega : M_t(\omega) \geq y\}.$$

Indeed, for every ω from the first set

$$M_t(\omega) \geq W_t(\omega) \geq 2y - x \geq y.$$

Therefore, for $y \geq x$ and $y, t > 0$,

$$\begin{aligned} \mathbb{P}(W_t \leq x, M_t \geq y) &= \mathbb{P}(W_t \geq 2y - x, M_t \geq y) \\ &= \mathbb{P}(W_t \geq 2y - x) = 1 - \Phi\left(\frac{2y - x}{\sqrt{t}}\right) = \Phi\left(\frac{x - 2y}{\sqrt{t}}\right), \end{aligned} \tag{19}$$

where Φ denotes the cumulative distribution function of a standard normal random variable ($N(0, 1)$).

Similarly, for $t > 0$, $z < 0$, and $z \leq x$, putting $\tau = T_z$, we obtain

$$\begin{aligned} \mathbb{P}(W_t \geq x, m_t \leq z) &= \mathbb{P}(W_t \leq 2z - x, m_t \leq z) \\ &= \mathbb{P}(W_t \leq 2z - x) = \Phi\left(\frac{2z - x}{\sqrt{t}}\right). \end{aligned} \tag{20}$$

The above, together with inequalities

$$m_t \leq W_t \leq M_t, \;\; m_t \leq 0 \leq M_t,$$

allows us to determine the joint distribution functions for the pairs (W_t, M_t) and (W_t, m_t).

Theorem 4. *The joint cumulative distribution function $F_{W,M,t}(x,y)$ of (W_t, M_t), where $t > 0$, is of the form*

$$F_{W,M,t}(x,y) = F_{W,M}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right), \quad t > 0,$$

where

$$F_{W,M}(x,y) = \begin{cases} 0 & \text{for} \quad y < 0, \\ 2\Phi(y) - 1 & \text{for} \quad 0 \le y < x, \\ \Phi(x) - \Phi(x - 2y) & \text{for} \quad x^+ \le y. \end{cases}$$

Furthermore,

$$\mathbb{P}(M_t \le y) = \max\left(0, 2\Phi\left(\frac{y}{\sqrt{t}}\right) - 1\right).$$

Proof. Case $y < 0$.

Since $M_t \ge 0$,

$$\forall t > 0 \quad \mathbb{P}(M_t < 0) = 0.$$

Case $y \ge \max(0, x)$.

We apply Formula (19).

$$\begin{aligned} \mathbb{P}(W_t \le x, M_t < y) &= \mathbb{P}(W_t \le x) - \mathbb{P}(W_t \le x, M_t \ge y) \\ &= \Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{x - 2y}{\sqrt{t}}\right). \end{aligned} \tag{21}$$

Due to the continuity of Φ, we obtain

$$F_{WM}(x,y) = \Phi(x) - \Phi(x - 2y).$$

Case $x = y$.

The event $\{\omega : M_t(\omega) \le y\}$ is contained in the event $\{\omega : W_t(\omega) \le y\}$. Indeed, for ω belonging to the first set, we have

$$W_t(\omega) \le M_t(\omega) \le y.$$

Hence, from the previous step, we obtain

$$\mathbb{P}(M_t \le y) = \mathbb{P}(W_t \le y, M_t \le y) = \Phi\left(\frac{y}{\sqrt{t}}\right) - \Phi\left(\frac{y - 2y}{\sqrt{t}}\right) = 2\Phi\left(\frac{y}{\sqrt{t}}\right) - 1.$$

Due to the continuity of Φ, we obtain

$$F_{WM}(x,y) = \Phi(x) - \Phi(x - 2y).$$

Case $0 \le y < x$.

We apply the previous case and the inequality

$$\forall t > 0 \quad W_t \le M_t.$$

This leads to

$$\begin{aligned} \mathbb{P}(W_t \le x, M_t \le y) &= \mathbb{P}(W_t \le y, M_t \le y) \\ &= \Phi\left(\frac{y}{\sqrt{t}}\right) - \Phi\left(\frac{y - 2y}{\sqrt{t}}\right) = 2\Phi\left(\frac{y}{\sqrt{t}}\right) - 1. \end{aligned} \tag{22}$$

□

Theorem 5. *The joint cumulative distribution function $F_{W,m,t}(x,z)$ of (W_t, m_t), where $t > 0$, is of the form*

$$F_{W,m,t}(x,z) = F_{W,m}\left(\frac{x}{\sqrt{t}}, \frac{z}{\sqrt{t}}\right), \quad t > 0,$$

where

$$F_{W,m}(x,z) = \begin{cases} 2\Phi(z) - \Phi(2z-x) & \text{for} \quad z \leq min(0,x), \\ \Phi(x) & \text{for} \quad z > \min(0,x). \end{cases}$$

Furthermore,

$$\mathbb{P}(m_t \leq z) = \min\left(1, 2\Phi\left(\frac{z}{\sqrt{t}}\right)\right).$$

Proof. Case $x = z$. We apply Formula (20).

$$\begin{aligned} \mathbb{P}(W_t \leq z, m_t \leq z) &= \mathbb{P}(m_t \leq z) - \mathbb{P}(W_t \geq z, m_t \leq z) \\ &= \mathbb{P}(m_t \leq z) - \Phi\left(\frac{2z-z}{\sqrt{t}}\right). \end{aligned}$$

Since the event $\{W_t \leq z\}$ is contained in the event $\{m_t \leq z\}$,

$$\mathbb{P}(m_t \leq z) = \Phi\left(\frac{z}{\sqrt{t}}\right) + \mathbb{P}(W_t \leq z, m_t \leq z) = 2\Phi\left(\frac{z}{\sqrt{t}}\right).$$

Case $z \leq min(0,x)$.
We apply Formula (20).

$$\begin{aligned} \mathbb{P}(W_t \leq x, m_t \leq z) &= \mathbb{P}(m_t \leq z) - \mathbb{P}(W_t \geq x, m_t \leq z) \\ &= 2\Phi\left(\frac{z}{\sqrt{t}}\right) - \Phi\left(\frac{2z-x}{\sqrt{t}}\right). \end{aligned} \tag{23}$$

Case $z > x$.
The event $\{W_t \leq x\}$ is contained in the event $\{m_t \leq z\}$. Indeed, for every ω from the first set,

$$m_t(\omega) \leq W_t(\omega) \leq x \leq z.$$

Thus,

$$\mathbb{P}(W_t \leq x, m_t \leq z) = \mathbb{P}(W_t \leq x) = \Phi\left(\frac{x}{\sqrt{t}}\right).$$

Case $z > 0$.
Since $m_t \leq 0$,

$$\{\omega : m_t(\omega) \leq z\} = \Omega.$$

Thus,

$$\mathbb{P}(W_t \leq x, m_t \leq z) = \mathbb{P}(W_t \leq x) = \Phi\left(\frac{x}{\sqrt{t}}\right).$$

□

Note that, since the distributions of running maxima and minima are closely related with stopping times

$$\mathbb{P}(M_t \leq y) = 1 - \mathbb{P}(T_y \leq t),$$

$$\mathbb{P}(m_t \leq z) = \mathbb{P}(T_z \leq t),$$

they might be drawn from the formulas describing exit times (from the half-line) provided, for example, in [16] (Theorem 4.4.5).

3.3. The Two-Barrier Stopping Time

In this section, we apply the reflection principle for the two-barrier stopping time

$$\tau = \min(T_y, T_z), \;\; z < 0 < y,$$

and for the set A, which are equal, respectively, to

$$\{\omega : W_\tau(\omega) = y, \tau(\omega) \le t\} \;\text{ or }\; \{\omega : W_\tau(\omega) = z, \tau(\omega) \le t\}.$$

Note that the range of W_τ consists only of two points y and z.

Lemma 1. *For any $x \le y$*

$$\mathbb{P}(W_t \le x, \;\; W_\tau = y, \;\; \tau \le t) = \sum_{k=1}^{\infty} \left(\Phi\left(\frac{x + 2(k-1)z - 2ky}{\sqrt{t}} \right) - \Phi\left(\frac{x + 2kz - 2ky}{\sqrt{t}} \right) \right).$$

Proof. Applying twice the reflection principle, we obtain

$$\begin{aligned}
\mathbb{P}(W_t \le x, \;\; W_\tau = y, \;\; \tau \le t) &= \mathbb{P}(2W_\tau - W_t \le x, \;\; W_\tau = y, \;\; \tau \le t) \\
&= \mathbb{P}(2y - W_t \le x, \;\; W_\tau = y, \;\; \tau \le t) \\
&= \mathbb{P}(W_t \ge 2y - x, \;\; W_\tau = y, \;\; \tau \le t) \\
&= \mathbb{P}(W_t \ge 2y - x, \;\; \tau \le t) - \mathbb{P}(W_t \ge 2y - x, \;\; W_\tau = z, \;\; \tau \le t) \\
&= \mathbb{P}(W_t \ge 2y - x, \;\; \tau \le t) - \mathbb{P}(2W_\tau - W_t \ge 2y - x, \;\; W_\tau = z, \;\; \tau \le t) \\
&= \mathbb{P}(W_t \ge 2y - x, \;\; \tau \le t) - \mathbb{P}(W_t \le 2z - 2y + x, \;\; W_\tau = z, \;\; \tau \le t) \\
&= \mathbb{P}(W_t \ge 2y - x, \;\; \tau \le t) - \mathbb{P}(W_t \le 2z - 2y + x, \;\; \tau \le t) \\
&\quad + \mathbb{P}(W_t \le 2z - 2y + x, \;\; W_\tau = y, \;\; \tau \le t).
\end{aligned} \tag{24}$$

Since $x \le y$, the event $W_t \ge 2y - x$ is contained in the event $\tau \le t$, and

$$W_t(\omega) \ge 2y - x \ge y \Longrightarrow M_t(\omega) \ge y \Longrightarrow T_y(\omega) \le t \Longrightarrow \tau(\omega) = \min(T_y(\omega), T_z(\omega)) \le t.$$

Therefore,

$$\mathbb{P}(W_t \ge 2y - x, \;\; \tau \le t) = \mathbb{P}(W_t \ge 2y - x) = \Phi_t(x - 2y),$$

where Φ_t denotes the distribution function of an $N(0, t)$ random variable:

$$\Phi_t(x) = \Phi\left(\frac{x}{\sqrt{t}} \right).$$

Similarly, since $z < 0 < y$ and $x \le y$, the event $W_t \le 2z - 2y + x$ is contained in the event $\tau \le t$, so

$$W_t(\omega) \le 2z - 2y + x \le z \Longrightarrow m_t(\omega) \le z \Longrightarrow T_z(\omega) \le t \Longrightarrow \tau(\omega) = \min(T_y(\omega), T_z(\omega)) \le t.$$

Therefore,

$$\mathbb{P}(W_t \le 2z - 2y + x, \;\; \tau \le t) = \mathbb{P}(W_t \le 2z - 2y + x) = \Phi_t(x + 2z - 2y).$$

In such a way, we obtain the recurrence for $x \le y$:

$$\mathbb{P}(W_t \le x, \; W_\tau = y, \; \tau \le t) = \mathbb{P}(W_t \le 2z - 2y + x, \; W_\tau = y, \; \tau \le t) + \Phi_t(x - 2y) - \Phi_t(x + 2z - 2y). \tag{25}$$

Since $2z - 2y < 0$, we may repeat it. After n repetitions, we obtain

$$\begin{aligned}\mathbb{P}(W_t \le x,\ W_\tau = y,\ \tau \le t) &= \mathbb{P}(W_t \le n(2z-2y)+x,\ W_\tau = y,\ \tau \le t) \\ &\quad + \sum_{k=1}^{n}(\Phi_t(x+2(k-1)z-2ky) - \Phi_t(x+2kz-2ky)).\end{aligned} \tag{26}$$

Since

$$\mathbb{P}(W_t \le n(2z-2y)+x,\ W_\tau = y,\ \tau \le t) \le \mathbb{P}(W_t \le n(2z-2y)+x) \overset{n\to\infty}{\longrightarrow} 0,$$

and passing to the limit $n \to \infty$, we obtain

$$\mathbb{P}(W_t \le x,\ W_\tau = y,\ \tau \le t) = \sum_{k=1}^{\infty}(\Phi_t(x+2(k-1)z-2ky) - \Phi_t(x+2kz-2ky)). \tag{27}$$

□

Lemma 2. *For any $x \ge z$*

$$\mathbb{P}(W_t \ge x,\ \ W_\tau = z,\ \ \tau \le t) = \sum_{k=1}^{\infty}\left(\Phi\left(\frac{x-2kz+2ky}{\sqrt{t}}\right) - \Phi\left(\frac{x-2kz+2(k-1)y}{\sqrt{t}}\right)\right).$$

Proof. We base our proof on the symmetry of the Wiener process W_t. We put $\tilde{W}_t = -W_t$. Then, we have, for $z < 0 < y$,

$$\begin{aligned}\tilde{M}_t &= \sup_{0\le s\le t}\tilde{W}_s = -\inf_{0\le s\le t}W_s = -m_t, \\ \tilde{m}_t &= \inf_{0\le s\le t}\tilde{W}_s = -\sup_{0\le s\le t}W_s = -M_t, \\ \tilde{T}_{-z} &= \inf\{t>0 : \tilde{W}_t = -z\} = T_z, \\ \tilde{T}_{-y} &= \inf\{t>0 : \tilde{W}_t = -y\} = T_y, \\ \tilde{\tau} &= \min(\tilde{T}_{-z}, \tilde{T}_{-y}) = \min(T_z, T_y) = \tau.\end{aligned}$$

Therefore,

$$\mathbb{P}(W_t \ge x,\ \ W_\tau = z,\ \ \tau \le t) = \mathbb{P}\big(\tilde{W}_t \le -x,\ \ \tilde{W}_{\tilde{\tau}} = -z,\ \ \tilde{\tau} \le t\big).$$

Since $\tilde{W}_t$ is as well a Wiener process, we apply Lemma 1 and obtain

$$\begin{aligned}\mathbb{P}\big(\tilde{W}_t \le -x,\ \ \tilde{W}_{\tilde{\tau}} = -z,\ \ \tilde{\tau} \le t\big) &= \sum_{k=1}^{\infty}\left(\Phi\left(\frac{-x+2kz-2(k-1)y}{\sqrt{t}}\right) - \Phi\left(\frac{-x+2kz-2ky}{\sqrt{t}}\right)\right) \\ &= \sum_{k=1}^{\infty}\left(\Phi\left(\frac{x-2kz+2ky}{\sqrt{t}}\right) - \Phi\left(\frac{x-2kz+2(k-1)y}{\sqrt{t}}\right)\right).\end{aligned}$$

□

Together, the above lemmas give us

Corollary 1. *For $z < 0 < y$ and $z \le x \le y$,*

$$\begin{aligned}\mathbb{P}(W_t \le x,\ \tau \le t) &= \mathbb{P}(W_\tau = z,\ \tau \le t) \\ &\quad + \sum_{k=-\infty}^{\infty}\left(\Phi\left(\frac{x+2(k-1)z-2ky}{\sqrt{t}}\right) - \Phi\left(\frac{x+2kz-2ky}{\sqrt{t}}\right)\right) + \Phi(x)\end{aligned} \tag{28}$$

where

$$\mathbb{P}(W_\tau = z,\ \tau \le t) \quad = \quad 2\sum_{k=1}^{\infty}\left(\Phi\left(\frac{(2k-1)z-2(k-1)y}{\sqrt{t}}\right)-\Phi\left(\frac{(2k-1)z-2ky}{\sqrt{t}}\right)\right). \tag{29}$$

Furthermore,

$$\mathbb{P}(W_\tau = y,\ \tau \le t) \quad = \quad 2\sum_{k=1}^{\infty}\left(\Phi\left(\frac{2(k-1)z-(2k-1)y}{\sqrt{t}}\right)-\Phi\left(\frac{2kz-(2k-1)y}{\sqrt{t}}\right)\right) \tag{30}$$

and

$$\begin{aligned}\mathbb{P}(\tau \le t) \quad = \quad & 2\sum_{k=1}^{\infty}\left(\Phi\left(\frac{(2k-1)z-2(k-1)y}{\sqrt{t}}\right)-\Phi\left(\frac{(2k-1)z-2ky}{\sqrt{t}}\right)\right) \\ & +2\sum_{k=1}^{\infty}\left(\Phi\left(\frac{2(k-1)z-(2k-1)y}{\sqrt{t}}\right)-\Phi\left(\frac{2kz-(2k-1)y}{\sqrt{t}}\right)\right).\end{aligned} \tag{31}$$

Proof. We observe that

$$\begin{aligned}\mathbb{P}(W_t \le x,\ \tau \le t) \quad &= \quad \mathbb{P}(W_t \le x,\ W_\tau = y,\ \tau \le t)+\mathbb{P}(W_t \le x,\ W_\tau = z,\ \tau \le t) \\ &= \quad \mathbb{P}(W_t \le x,\ W_\tau = y,\ \tau \le t) \\ &\quad +\mathbb{P}(W_\tau = z,\ \tau \le t)-\mathbb{P}(W_t \ge x,\ W_\tau = z,\ \tau \le t).\end{aligned}$$

Thus, applying Lemmas 1 and 2, we obtain

$$\begin{aligned}\mathbb{P}(W_t \le x,\ \tau \le t) \quad &= \quad \mathbb{P}(W_\tau = z,\ \tau \le t) \\ &\quad +\sum_{k=1}^{\infty}(\Phi_t(x+2(k-1)z-2ky)-\Phi_t(x+2kz-2ky)) \\ &\quad +\sum_{k=1}^{\infty}(\Phi_t(x-2kz+2(k-1)y)-\Phi_t(x-2kz+2ky)). \\ &= \quad \mathbb{P}(W_\tau = z,\ \tau \le t)+\Phi(x) \\ &\quad +\sum_{k=-\infty}^{\infty}(\Phi_t(x+2(k-1)z-2ky)-\Phi_t(x+2kz-2ky))\end{aligned} \tag{32}$$

Since the event $W_t \le z$ implies the event $\tau \le t$, we have

$$\mathbb{P}(W_t \le z,\ \tau \le t) = \mathbb{P}(W_t \le z) = \Phi_t(z).$$

Therefore, substituting $x = z$ in the above formula, we obtain

$$\begin{aligned}\mathbb{P}(W_\tau = z,\ \tau \le t) \quad &= \quad \mathbb{P}(W_t \le z,\ \tau \le t)-\Phi(z) \\ &\quad -\sum_{k=-\infty}^{\infty}(\Phi_t(z+2(k-1)z-2ky)-\Phi_t(z+2kz-2ky)) \\ &= \quad \Phi_t(z)-2\sum_{k=1}^{\infty}\Phi_t((2k-1)z-2ky) \\ &\quad +\sum_{k=2}^{\infty}\Phi_t((2k-1)z-2(k-1)y)+\sum_{k=1}^{\infty}\Phi_t((2k-1)z-2(k-1)y) \\ &= \quad 2\sum_{k=1}^{\infty}(\Phi_t((2k-1)z-2(k-1)y)-\Phi_t((2k-1)z-2ky)).\end{aligned}$$

This concludes the proof of the first two formulas of the corollary. The third one follows from the symmetry of the Wiener process. We apply the notation from the proof of Lemma 2.

Since $\tilde{W}_t = -W_t$ is a Wiener process,

$$\begin{aligned}\mathbb{P}(W_\tau = y,\ \tau \le t) &= \mathbb{P}\big(\tilde{W}_{\tilde{\tau}} = -y,\ \tilde{\tau} \le t\big)\\ &= 2\sum_{k=1}^{\infty}(\Phi_t((2k-1)(-y) - 2(k-1)(-z)) - \Phi_t((2k-1)(-y) - 2k(-z)))\\ &= 2\sum_{k=1}^{\infty}(\Phi_t(2(k-1)z - (2k-1)y) - \Phi_t(2kz - (2k-1)y)).\end{aligned}$$

Since

$$\mathbb{P}(\tau \le t) = \mathbb{P}(W_\tau = y,\ \tau \le t) + \mathbb{P}(W_\tau = z,\ \tau \le t),$$

the last equality of the corollary is a consequence of the two previous ones. □

Note that, since the Formula (31) is describing an exit time from a segment, it might be obtained as a solution of a partial differential equation (see [16] Theorem 4.4.5).

3.4. Joint Cumulative Distribution

Theorem 6. *The joint cumulative distribution function $F_t(x,y,z)$ of (W_t, M_t, m_t), where $t > 0$, is of the form*

$$F_t(x,y,z) = F\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}, \frac{z}{\sqrt{t}}\right), \quad t > 0,$$

where

$$F(x,y,z) = \begin{cases} 0 & \text{for } y \le 0,\\ \Phi(x) - \Phi(x-2y) & \text{for } z \ge 0,\ 0 < y,\ x < y,\\ 2\Phi(y) - 1 & \text{for } z \ge 0,\ 0 < y \le x,\\ \Phi(x) - \Phi(x-2y) & \text{for } x < z < 0 < y,\\ 2\Psi(z, 2y, 2(y-z)) - \Psi(x + 2z - 2y, 2y, 2(y-z)) & \\ \quad -\Psi(-x + 2z, 2y, 2(y-z)) & \text{for } z < 0 < y,\ z \le x \le y,\\ 2\Psi(z, 2y, 2(y-z)) - 2\Psi(2z - y, 2y, 2(y-z)) & \text{for } z < 0 < y < x, \end{cases} \tag{33}$$

where

$$\Psi(q,r,s) = \sum_{k=0}^{\infty}(\Phi(q - ks) - \Phi(q - r - ks)) \tag{34}$$

and Φ denotes the distribution function of the standard normal probability law N(0, 1).

Note that the distribution of (W_t, M_t, m_t) is absolutely continuous with respect to the Lebesgue measure (Section 3.5) and is supported on the set

$$\{(x,y,z) \in \mathbb{R}^3 :\ z \le 0 \le y,\ z \le x \le y\}.$$

Furthermore, the marginal distributions of F are given by the following formulas:

$$F(x,+\infty,+\infty) = \mathbb{P}(W_1 \le x) = \Phi(x), \tag{35}$$

$$F(+\infty,y,+\infty) = \mathbb{P}(M_1 \le y) = \begin{cases} 0 & \text{if } y \le 0, \\ 2\Phi(y)-1 & \text{if } y > 0, \end{cases} \tag{36}$$

$$F(+\infty,+\infty,z) = \mathbb{P}(m_1 \le z) = \begin{cases} 2\Phi(z) & \text{if } z < 0, \\ 1 & \text{if } z \ge 0. \end{cases} \tag{37}$$

$$F(x,y,+\infty) = \mathbb{P}(W_1 \le x, M_1 \le y) = \begin{cases} 0 & \text{if } y \le 0, \\ 2\Phi(y)-1 & \text{if } 0 < y \le x, \\ \Phi(x)-\Phi(x-2y) & \text{if } \max(0,x) < y, \end{cases} \tag{38}$$

$$F(x,+\infty,z) = \mathbb{P}(W_1 \le x, m_1 \le z) = \begin{cases} 2\Phi(z)-\Phi(2z-x) & \text{if } z \le \min(0,x), \\ \Phi(x) & \text{if } x < z \le 0 \vee 0 < z, \end{cases} \tag{39}$$

$$F(+\infty,y,z) = \mathbb{P}(M_1 \le y, m_1 \le z) = \begin{cases} 0 & \text{if } y \le 0, \\ 2\Psi(z,2y,2(y-z)) & \\ -2\Psi(2z-y,2y,2(y-z)) & \text{if } z \le 0 < y, \\ 2\Phi(y)-1 & \text{if } 0 < z, 0 < y. \end{cases} \tag{40}$$

We prove Theorem 6 case by case.

Proof. Case $y \le 0$.
Since W_t is almost surely continuous, and $W_0 = 0$, M_t is almost surely positive:

$$F_t(x,y,z) = \mathbb{P}(W_t \le x,\ M_t \le y,\ m_t \le z) \le \mathbb{P}(M_t \le y) = 0.$$

Case $z \ge 0$ and $y > 0$.
Since m_t is almost surely negative,

$$F_t(x,y,z) = \mathbb{P}(W_t \le x,\ M_t \le y,\ m_t \le z) = \mathbb{P}(W_t \le x,\ M_t \le y).$$

Following Theorem 4, we obtain

$$F_t(x,y,z) = \begin{cases} 2\Phi_t(y)-1 & \text{for} \quad y < x, \\ \Phi_t(x)-\Phi_t(x-2y) & \text{for} \quad x \le y. \end{cases}$$

Case $z < 0 < y$ and $x \le z$.
Since the event $W_t \le x \le z$ implies the event $m_t \le z$,

$$F_t(x,y,z) = \mathbb{P}(W_t \le x,\ M_t \le y,\ m_t \le z) = \mathbb{P}(W_t \le x,\ M_t \le y).$$

As above, following Theorem 4, we obtain

$$F_t(x,y,z) = \Phi_t(x) - \Phi_t(x-2y).$$

Case $z < 0 < y$ and $z < x \le y$.
We observe that

$$\begin{aligned} F_t(x,y,z) &= \mathbb{P}(W_t \le x,\ M_t \le y,\ m_t \le z) \\ &= \mathbb{P}(W_t \le x,\ (M_t \ge y \vee m_t \le z)) \\ &\quad -\mathbb{P}(W_t \le x,\ M_t \ge y) \\ &= \mathbb{P}(W_t \le x,\ \tau \le t) \\ &\quad -\mathbb{P}(W_t \le x) + \mathbb{P}(W_t \le x,\ M_t \le y) \end{aligned}$$

where τ is the two-barrier stopping time discussed in Section 3.3. Following Corollary 1 and Theorem 4, we obtain

$$\begin{aligned} F_t(x,y,z) &= 2\sum_{k=1}^{\infty}(\Phi_t((2k-1)z-2ky)-\Phi_t((2k+1)z-2ky)) \\ &\quad +\sum_{k=1}^{\infty}(\Phi_t(x+2(k-1)z-2ky)-\Phi_t(x+2kz-2ky)) \\ &\quad -\sum_{k=1}^{\infty}(\Phi_t(-x+2kz-2(k-1)y)-\Phi_t(-x+2kz-2ky)) \\ &\quad -\Phi_t(x)+\Phi_t(x)-\Phi_t(x-2y) \\ &= 2\sum_{k=1}^{\infty}(\Phi_t((2k-1)z-2ky)-\Phi_t((2k+1)z-2ky)) \\ &\quad +\sum_{k=1}^{\infty}(\Phi_t(x+2kz-2(k+1)y)-\Phi_t(x+2kz-2ky)) \\ &\quad -\sum_{k=1}^{\infty}(\Phi_t(-x+2kz-2(k-1)y)-\Phi_t(-x+2kz-2ky)). \end{aligned}$$

Case $z<0<y$ and $y\le x$.
Since $W_t\le M_t$,

$$\begin{aligned} F_t(x,y,z) &= \mathbb{P}(W_t\le x,\ M_t\le y,\ m_t\le z)=\mathbb{P}(M_t\le y,\ m_t\le z) \\ &= \mathbb{P}(M_t\ge y\vee m_t\le z)+\mathbb{P}(M_t\le y)-1 \\ &= \mathbb{P}(\tau\le t)+\mathbb{P}(M_t\le y)-1 \\ &= 2\sum_{k=1}^{\infty}(\Phi_t((2k-1)z-2(k-1)y)-\Phi_t((2k-1)z-2ky)) \\ &\quad +2\sum_{k=1}^{\infty}(\Phi_t(2(k-1)z-(2k-1)y)-\Phi_t(2kz-(2k-1)y)) \\ &\quad -2\Phi(-y) \\ &= 2\sum_{k=1}^{\infty}(\Phi_t((2k-1)z-2(k-1)y)-\Phi_t((2k-1)z-2ky)) \\ &\quad -2\sum_{k=1}^{\infty}(\Phi_t(2kz-(2k-1)y)-\Phi_t(2kz-(2k+1)y)) \end{aligned}$$

□

As a corollary of Theorem 6, we prove the 80-year-old characterization of the Wiener process called the "Lévy Triple Law" ([15] Theorem 6.18), which deals with the trajectories lying between the upper barrier u and lower barrier l.

Corollary 2. *For $l<0<u$, $l\le x\le u$, and $t>0$,*

$$\frac{\partial}{\partial x}\mathbb{P}(W_t\le x, M_t\le u, m_t\ge l)=\sum_{k=-\infty}^{+\infty}(\varphi_t(x+2k(u-l))-\varphi_t(x-2l-2k(u-l)), \quad (41)$$

where φ_t denotes the density of $N(0,t)$.

Proof. We observe that

$$\begin{aligned}
&\mathbb{P}(W_t \le x, M_t \le u, m_t \ge l) = F_t(x,u,0) - F_t(x,u,l) \qquad (42)\\
&\quad = \Phi_t(x) - \Phi_t(x-2u)\\
&\qquad - \sum_{k=0}^{+\infty} (\Phi_t(l-2k(u-l)) - \Phi_t(l-2u-2k(u-l)))\\
&\qquad + \sum_{k=0}^{+\infty} (\Phi_t(x+2l-2u-2k(u-l)) - \Phi_t(x+2l-2u-2u-2k(u-l)))\\
&\qquad + \sum_{k=0}^{+\infty} (\Phi_t(-x+2l-2k(u-l)) - \Phi_t(-x+2l-2u-2k(u-l))).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\partial}{\partial x}\mathbb{P}(W_t \le x, M_t \le u, m_t \ge l) \qquad (43)\\
&\quad = \varphi_t(x) - \varphi_t(x-2u)\\
&\qquad + \sum_{k=0}^{+\infty} (\varphi_t(x+2l-2u-2k(u-l)) - \varphi_t(x+2l-2u-2u-2k(u-l)))\\
&\qquad - \sum_{k=0}^{+\infty} (\varphi_t(-x+2l-2k(u-l)) - \varphi_t(-x+2l-2u-2k(u-l))).\\
&\quad = \varphi_t(x) - \varphi_t(x-2u)\\
&\qquad + \sum_{k=0}^{+\infty} (\varphi_t(x-2(k+1)(u-l)) - \varphi_t(x-2u-2(k+1)(u-l)))\\
&\qquad - \sum_{k=0}^{+\infty} (\varphi_t(x-2u+2(k+1)(u-l)) - \varphi_t(x+2(k+1)(u-l))).\\
&\quad = \sum_{k=-\infty}^{+\infty} \varphi_t(x-2k(u-l)) - \sum_{k=-\infty}^{+\infty} \varphi_t(x-2u-2k(u-l))\\
&\quad = \sum_{k=-\infty}^{+\infty} \varphi_t(x-2k(u-l)) - \sum_{k=-\infty}^{+\infty} \varphi_t(x-2l-2(k+1)(u-l)).
\end{aligned}$$

After shifting by one the parameter k in the second sum, we obtain the required Formula (41). □

3.5. Properties of the Cumulative Distribution Function F and the Marginal Distributions

We start with the absolute continuity of the marginal distributions of random pairs (W_1, M_1) and (W_1, m_1):

$$F_{W,M}(x,y) = F(x,y,\infty) \quad \text{and} \quad F_{W,m}(x,z) = F(x,\infty,z).$$

Proposition 2. *The bivariate cumulative distributions $F_{W,M}$ and $F_{W,m}$ are absolutely continuous with respect to the Lebesgue measure.*

Proof. We start with the function $F_{W,m}(x,z)$. We show that a function

$$g(\xi,\eta) = -2(2\eta-\xi)\varphi(2\eta-\xi)\mathbf{1}_{\eta\le 0}\mathbf{1}_{\eta\le\xi}$$

is a corresponding density. Let $I(x,z)$ denote an integral of g over the set $(-\infty,x]\times(-\infty,z]$.

$$\begin{aligned} I(x,y) &= \int_{-\infty}^{z}\int_{-\infty}^{x} g(\xi,\eta)d\eta d\xi \qquad (44)\\ &= \int_{-\infty}^{\min(z,x,0)}\int_{\eta}^{x}(-2)(2\eta-\xi)\varphi(2\eta-\xi)d\eta d\xi\\ &= \int_{-\infty}^{\min(z,x,0)} 2\Big(\varphi(\eta)-\varphi(2\eta-x)\Big)d\xi\\ &= 2\Phi(\min(z,x,0))-\Phi(2\min(z,x,0)-x)\\ &= \begin{cases} 2\Phi(z)-\Phi(2z-x) & \text{when } \min(z,x,0)=z,\\ 2\Phi(x)-\Phi(2x-x)=\Phi(x) & \text{when } \min(z,x,0)=x,\\ 1-\Phi(0-x)=\Phi(x) & \text{when } \min(z,x,0)=0. \end{cases} \end{aligned}$$

Since all points (x,z), $I(x,z)$, and $F_{W,m}(x,z)$ are equal each other, we conclude that $F_{W,m}$ is absolutely continuous.

Next, since $\tilde{W}_t=-W_t$ is a Wiener process, the pairs (W_1,M_t) and $(-W_1,-m_1)$ have the same distribution

$$(W_1,M_t)\overset{d}{=}(-W_1,-m_1).$$

Therefore, $F_{W,M}$ is also absolutely continuous. □

Now, we are in position to show the absolute continuity of the cumulative distribution function F of the random triple (W_1,M_1,m_1).

Proposition 3. *The function F is absolutely continuous with respect to the Lebesgue measure. Furthermore,*

$$\mathbb{P}\Big((W_1,M_1,m_1)\in\{(x,y,z)\in\mathbb{R}^3 : z<0<y, z<x<y\}\Big)=1.$$

Proof. Since the definition of running maxima and minima implies that

$$m_1\le W_1\le M_1, \text{ and } m_1\le 0\le M_1,$$

we have

$$\mathbb{P}\Big((W_1,M_1,m_1)\in\{(x,y,z)\in\mathbb{R}^3 : z\le 0\le y, z\le x\le y\}\Big)=1.$$

We show that the probability that the random triple (W_1,M_1,m_1) belongs to the boundary and is equal to 0.

Since the distribution functions of M_1 and m_1 are continuous, we have

$$\mathbb{P}(M_1=0)=0,\ \ \mathbb{P}(m_1=0)=0.$$

Since the distribution functions of the random pairs (W_1,M_1) and (W_1,m_1) are absolutely continuous,

$$\mathbb{P}(M_1=W_1)=0=\mathbb{P}(m_1=W_1).$$

To conclude the proof, it is enough to observe that the function F is C^∞ on the set

$$\{(x,y,z)\in\mathbb{R}^3 : z<0<y, z<x<y\}.$$

Hence, it is absolutely continuous with respect to the Lebesgue measure. □

3.6. Proof of Theorem 1

We recall that a *copula* is the restriction to the unit n-cube $[0,1]^n$ of a distribution function whose univariate margins are uniformly distributed on $[0,1]$. Specifically, a

function $C : [0,1]^n \to [0,1]$ is a copula if it has the following properties. For every $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ such that $0 \le u_i \le v_i \le 1$ for $i = 1, \ldots, n$, we have the following:

(C1) $(\exists i \;\; u_i = 0) \Rightarrow C(\mathbf{u}) = 0$;
(C2) $\forall j \in \{1, \ldots, n\} \quad (\forall i \neq j \;\; u_i = 1) \Rightarrow C(\mathbf{u}) = u_j$;
(C3) C is *n-nondecreasing*, that is, the C-volume $V_C(\mathbf{u}, \mathbf{v})$ of any n-rectangle with lower vertex $\mathbf{u}$ and upper vertex $\mathbf{v}$ is non-negative.

We recall that the C-volume is a signed sum of the values of C at the vertices of the n-rectangle,

$$V_C(\mathbf{u}, \mathbf{v}) = C(\mathbf{w})|_{w_1=u_1}^{v_1} \ldots |_{w_n=u_n}^{v_n} = \sum_{j_1=1}^{2} \cdots \sum_{j_n=1}^{2} (-1)^{j_1+\cdots+j_n} C(w_{1,j_1}, \ldots, w_{n,j_n}),$$

where $w_{i,1} = u_i$, and $w_{i,2} = v_i$.

According to the celebrated *Sklar's theorem*, the joint distribution function F of any n-tuple $\mathbf{X} = (X_1, \ldots, X_n)$ of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ can be written as a composition of a copula C and the univariate marginals F_i, i.e., for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$F(\mathbf{x}) = C(F_1(x_1), \ldots, F_n(x_n)).$$

Moreover, if the distribution functions F_k are continuous, then the copula C is uniquely determined and can be described in terms of the quantile functions, i.e., the generalized inverses of F_k. Since $F_k(F_k^{\leftarrow}(u_k) = u_k$, we obtain

$$C(u_1, \ldots, u_n) = C(F_1(F_1^{\leftarrow}(u_1)), \ldots, F_n(F_n^{\leftarrow}(u_n))) = F(F_1^{\leftarrow}(u_1), \ldots, F_n^{\leftarrow}(u_n).$$

In our case where $n = 3$ and for $u, v, w \in (0,1)$, the quantile functions of W_1, M_1, and m_1 are given by

$$F_1^{\leftarrow}(u) = \Phi^{-1}(u), \tag{45}$$

$$F_2^{\leftarrow}(v) = \Phi^{-1}\left(\frac{1+v}{2}\right), \tag{46}$$

$$F_3^{\leftarrow}(w) = \Phi^{-1}\left(\frac{w}{2}\right). \tag{47}$$

To conclude the proof of Theorem 1, we substitute the above quantile functions into Formula (33).

3.7. Properties of the Copula $C_{W,M,m}$ and Its Marginal Copulas

The copulas mentioned in Section 2 regarding the involution of the copula $C_{W,M,m}$ and its marginal copulas follows from the fact that $-W_t$ is a Wiener process as well. Hence, the triple $(-W_1, -m_1, -M_1)$ has the same distribution as the triple (W_1, M_1, m_1). We put

$$U = \Phi(W_1), \; V = 2\Phi(M_1) - 1, \; W = 2\Phi(m_1).$$

Obviously, (U, V, W) are the generators of the copula $C_{W,M,m}$

$$C_{W,M,m}(u, v, w) = \mathbb{P}(U \le u, V \le v, W \le w).$$

Since

$$\begin{aligned} \Phi(-W_1) &= 1 - \Phi(W_1) = 1 - U, \\ 2\Phi(-m_1) - 1 &= 2(1 - \Phi(m_1)) - 1 = 1 - W, \\ 2\Phi(-M_1) &= 2(1 - \Phi(M_1)) = 1 - V, \end{aligned}$$

we apply the inclusion/exclusion principle and obtain

$$\begin{aligned} C_{W,M,m}(u,v,w) &= \mathbb{P}(\Phi(-W_1) \le u, 2\Phi(-m_1) - 1 \le v, 2\Phi(-M_1) \le w) \\ &= \mathbb{P}(1-U \le u, 1-W \le v, 1-V \le w) \\ &= \mathbb{P}(U \ge 1-u, W \ge 1-v, V \ge 1-w) \\ &= C_{W,M,m}(1,1,1) - C_{W,M,m}(1,1,1-v) - C_{W,M,m}(1-u,1,1) - C_{W,M,m}(1,1-w,1) \\ &\quad + C_{W,M,m}(1,1-w,1-v) + C_{W,M,m}(1-u,1,1-v) + C_{W,M,m}(1-u,1-w,1) \\ &\quad - C_{W,M,m}(1-u,1-w,1-v) \\ &= u+v+w-2 + C_{W,M,m}(1,1-w,1-v) + C_{W,M,m}(1-u,1,1-v) \\ &\quad + C_{W,M,m}(1-u,1-w,1) - C_{W,M,m}(1-u,1-w,1-v). \end{aligned} \tag{48}$$

The involutions of the marginal copulas are obtained by substitution in the preceding formulas, respectively, where $u = 1$, $v = 1$, or $w = 1$.

Proof of Proposition 1. We recall that the Spearman ρ of the copula C is given by the formula ([10] Theorem 5.1.6)

$$\rho(C) = 12 \int_0^1 \int_0^1 C(u,v)dudv - 3. \tag{49}$$

Due to the involution (4), the Spearman ρ of the copula $C_{W,M}$ is the same as the Spearman ρ of the copula $C_{W,m}$. We observe that

$$\begin{aligned} \int_0^1 \int_0^1 C_{W,m}(u,w)dudw &= \int_0^1 \int_0^{w/2} ududw + \int_0^1 \int_{w/2}^1 wdudw \\ &\quad - \int_0^1 \int_{w/2}^1 \Phi(2\Phi^{-1}(w/2) - \Phi^{-1}(u))dudw. \end{aligned} \tag{50}$$

The first two integrals are obvious:

$$\int_0^1 \int_0^{w/2} ududw = \int_0^1 \frac{1}{8} w^2 dw = \frac{1}{24},$$

$$\int_0^1 \int_{w/2}^1 wdudw = \int_0^1 w\Big(1 - \frac{w}{2}\Big)dw = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$$

To calculate the third integral, we substitute

$$u = \Phi(x), \quad w = 2\Phi(z).$$

We obtain

$$\begin{aligned} \int_0^1 \int_{w/2}^1 \Phi(2\Phi^{-1}(w/2) - \Phi^{-1}(u))dudw &= 2\int_{-\infty}^0 \int_z^{+\infty} \Phi(2z-x)\phi(x)\phi(z)dxdz \\ &= 2\int_{-\infty}^0 \int_z^{+\infty} \int_{-\infty}^{2z-x} \phi(\xi)\phi(x)\phi(z)d\xi dxdz. \end{aligned} \tag{51}$$

Note that the product $\phi_3(\xi, x, z) = \phi(\xi)\phi(x)\phi(z)$ is a density of the three-dimensional standard normal distribution $N(0, Id)$, which is a spherical distribution. Therefore, the above integral can be expressed in terms of the volume of a spherical triangle. We recall that such a volume equals

$$Vol = A + B + C - \pi, \tag{52}$$

where A, B, C are inner angles (in radians) of the triangle (see, for example, [17] Formula (1.12.2) in Section 1.12). Note that the cosine of the inner angle of the intersection of two half-spaces is given by the scalar product of normal vectors:

$$\cos(\angle(a_1\xi + b_1x + c_1z \le 0,\, a_2\xi + b_2x + c_2z \le 0)) = -\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}\sqrt{a_1^2 + b_1^2 + c_1^2}}. \quad (53)$$

$$\begin{aligned}
\int_{-\infty}^{0}\int_{z}^{+\infty}\int_{-\infty}^{2z-x}\phi(\xi)\phi(x)\phi(z)d\xi dxdz &= \iiint_{\{z\le 0, z\le x, \xi\le 2z-x\}}\phi_3(\xi,x,z)d\xi dxdz \quad (54)\\
&= \frac{1}{\mu_{S^2}(S^2)}\mu_{S^2}(\{(\xi,x,z)\in S^2 : z\le 0,\, z\le x,\, \xi\le 2z-x\}\\
&= \frac{1}{4\pi}\Big(\angle(z\le 0, z-x\le 0) + \angle(z\le 0, \xi+x-2z\le 0)\\
&\quad + \angle(z-x\le 0, \xi+x-2z\le 0) - \pi\Big)\\
&= \frac{1}{4\pi}\left(\frac{3}{4}\pi + \frac{1}{6}\pi + \arccos\left(\frac{\sqrt{6}}{3}\right) - \pi\right)\\
&= \frac{1}{4\pi}\arccos\left(\frac{\sqrt{6}}{3}\right) - \frac{1}{48}.
\end{aligned}$$

Finally, we conclude the following:

$$\rho(C_{W,m}) = 12\left(\frac{1}{3} + \frac{1}{24} - 2\left(\frac{1}{4\pi}\arccos\left(\frac{\sqrt{6}}{3}\right) - \frac{1}{48}\right)\right) - 3 = 2 - \frac{6}{\pi}\arccos\left(\frac{\sqrt{6}}{3}\right) \approx 0.82452.$$

The case of the copula $C_{M,m}$ can be handled in a similar way. We substitute

$$v = 2\Phi(y) - 1, \quad w = 2\Phi(z).$$

$$\begin{aligned}
I &= \int_0^1\int_0^1 C_{M,m}(v,w)dvdw \quad (55)\\
&= 2\int_{-\infty}^{0}\int_{0}^{+\infty}\Big(\Psi(z,2y,2(y-z)) - \Psi(2z-y,2y,2(y-z))\Big)\varphi(y)\varphi(z)dydz\\
&= 2\sum_{k=0}^{\infty}\int_{-\infty}^{0}\int_{0}^{+\infty}\Big(\Phi(z-2k(y-z)) - \Phi(z-2y-2k(y-z))\\
&\quad - \Phi(2z-y-2k(y-z)) + \Phi(2z-3y-2k(y-z))\Big)\varphi(y)\varphi(z)dydz\\
&= 2\sum_{k=0}^{\infty}\left(\int_{-\infty}^{0}\int_{0}^{+\infty}\int_{-\infty}^{(2k+1)z-2ky}\varphi_3(\xi,y,z)d\xi dydz - \int_{-\infty}^{0}\int_{0}^{+\infty}\int_{-\infty}^{(2k+1)z-(2k+2)y}\varphi_3(\xi,y,z)d\xi dydz\right.\\
&\quad \left. - \int_{-\infty}^{0}\int_{0}^{+\infty}\int_{-\infty}^{(2k+2)z-(2k+1)y}\varphi_3(\xi,y,z)d\xi dydz + \int_{-\infty}^{0}\int_{0}^{+\infty}\int_{-\infty}^{(2k+2)z-(2k+3)y}\varphi_3(\xi,y,z)d\xi dydz\right).
\end{aligned}$$

Next, we observe that for $a, b \ge 0$, we have

$$
\begin{aligned}
I_{a,b} &= \int_{-\infty}^{0}\int_{0}^{+\infty}\int_{-\infty}^{az-by}\varphi_3(\xi,y,z)d\xi dydz \qquad (56)\\
&= \frac{1}{4\pi}\left(\angle(z\le 0,-y\le 0)+\angle(z\le 0,\xi-az+by\le 0)+\angle(-y\le 0,\xi-az+by\le 0)-\pi\right)\\
&= \frac{1}{4\pi}\left(\frac{1}{2}\pi+\arccos\left(\frac{a}{\sqrt{1+a^2+b^2}}\right)+\arccos\left(\frac{b}{\sqrt{1+a^2+b^2}}\right)-\pi\right)\\
&= \frac{1}{4\pi}\left(\arccos\left(\frac{a}{\sqrt{1+a^2+b^2}}\right)+\arccos\left(\frac{b}{\sqrt{1+a^2+b^2}}\right)-\frac{1}{2}\pi\right).
\end{aligned}
$$

Note that $I_{a,b}=I_{b,a}$. Therefore, putting respectively $n=2k$ and $n=2k+1$, we obtain

$$
\begin{aligned}
I &= 2\sum_{k=0}^{\infty}\left(I_{2k+1,2k}-I_{2k+2,2k+1}-I_{2k+2,2k+1}+I_{2k+3,2k+2}\right) \qquad (57)\\
&= 2\sum_{n=0}^{\infty}(-1)^n\left(I_{n+1,n}-I_{n+2,n+1}\right)\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{1}{2\pi}\left(\arccos\left(\frac{n}{\sqrt{2(n^2+n+1)}}\right)+\arccos\left(\frac{n+1}{\sqrt{2(n^2+n+1)}}\right)\right.\\
&\quad\left.-\arccos\left(\frac{n+1}{\sqrt{2(n^2+3n+3)}}\right)-\arccos\left(\frac{n+2}{\sqrt{2(n^2+3n+3)}}\right)\right).
\end{aligned}
$$

Finally, we conclude the following:

$$
\begin{aligned}
\rho(C_{M,m}) &= 12I-3=\frac{6}{\pi}\sum_{n=0}^{\infty}(-1)^n\left(\arccos\left(\frac{n}{\sqrt{2(n(n+1)+1)}}\right)+\arccos\left(\frac{n+1}{\sqrt{2(n(n+1)+1)}}\right)\right.\\
&\quad\left.-\arccos\left(\frac{n+1}{\sqrt{2((n+1)(n+2)+1)}}\right)-\arccos\left(\frac{n+2}{\sqrt{2((n+1)(n+2)+1)}}\right)\right) \qquad (58)\\
&\approx 0.80649.
\end{aligned}
$$

□

4. Application of Results: Double-Barrier Option Pricing

In this section, we present one of the possible applications of the derived results. We consider probability space $(\Omega,\mathcal{F},\mathbb{P})$ and the Black–Scholes–Merton market model [18] or [19] in which the asset price S_t dynamic follows the geometric Brownian motion, i.e.,

$$dS_t=\mu S_t dt+\sigma S_t dW_t^1,$$

where $\mu\in\mathbb{R}$, $\sigma>0$, and $(W_t^1)_{t\ge 0}$ is a Wiener process. Additionally, the bank account $(B_t)_{t\ge 0}$ is described with the following differential equation:

$$dB_t=rB_t dt,\ \ B_0=1.$$

The solution to this system of equations is well known in the literature, and we present only a final result without proof. The reader can check that

$$S_t=S_0e^{\sigma W_t},\text{where } W_t=(\frac{\mu}{\sigma}-\frac{1}{2}\sigma)t+W_t^1,$$

and

$$B_t=e^{rt}$$

indeed satisfy the above stochastic differential equations.

To valuate in this model a derivative with a payoff X, which is $\mathcal{F}_T$ measurable ($(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of W_t^1), we need to calculate the expected value of the form

$$V = \mathbb{E}_{\mathbb{P}^*}\left(\frac{X}{B_T}\right) = e^{-rT}\mathbb{E}_{\mathbb{P}^*}(X),$$

where $\mathbb{P}^*$ is so-called risk neutral measure, i.e., the probability measure equivalent to $\mathbb{P}$ and described by the following Radon–Nikodym derivative

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(\frac{r-\mu}{\sigma}W_T^1 - \frac{1}{2}(\frac{r-\mu}{\sigma})^2 T\right).$$

Note that, for $t \leq T$, $W_t^2 = W_t^1 - \frac{r-\mu}{\sigma}t$ coincides with a Wiener process in $\mathbb{P}^*$ measure. For example the Black–Scholes formula for the price of an European call option is

$$C_{BS}(S_0, K, T, r, \sigma) = e^{-rT}\mathbb{E}_{\mathbb{P}^*}((S_T - K)^+).$$

Notice that

$$W_t = \left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)t + W_t^2.$$

This means that under the risk neutral measure $\mathbb{P}^*$, the stochastic differential equation of $S_t = S_0 e^{\sigma W_t}$ is of the form

$$dS_t = rS_t dt + \sigma S_t dW_t^2. \tag{59}$$

We continue to apply the Girsanov theorem. Let $\mathbb{Q}$ be the probability measure given by the Radon–Nikodym derivative of the form

$$\frac{d\mathbb{Q}}{d\mathbb{P}^*} = \exp\left(-\left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)W_T^2 - \frac{1}{2}\left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)^2 T\right).$$

In the measure Q, for $t \leq T$, $W_t = (\ln S_t - \ln S_0)/\sigma$ coincides with a Wiener process. The reader is referred to [20] Theorem 8.6.4 for more details. As in Section 2, we will denote by M_t and m_t the running maxima and minima of W_t.

Note that the inverse Radon–Nikodym derivative is given by the formula

$$\frac{d\mathbb{P}^*}{d\mathbb{Q}} = \exp\left(\left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)W_T - \frac{1}{2}\left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)^2 T\right).$$

Now, we can move on to the valuation of the double-barrier options.

A double-barrier option is a derivative whose payoff is path-dependent. There are few possibilities, like starting between arbitrary values $L < S_0 < U$, and the option becomes worthless whenever S_t crosses at least one of these two barriers. To be precise, we restrict ourselves to the double-barrier European call option with the payoff X at T of the form

$$X = (S_T - K)^+ \mathbb{1}_{L<\min_{0\leq t\leq T} S_t} \mathbb{1}_{\max_{0\leq t\leq T} S_t<U}, \tag{60}$$

where $\mathbb{1}_{\mathcal{A}}$ is the indicator of the event $\mathcal{A}$, and $0 < L < K, S_0 < U$.

First of all, notice that

$$L < \min_{0\leq s\leq t} S_s$$

is equivalent to

$$m_t \geq z, \quad z = \frac{1}{\sigma}\ln\left(\frac{L}{S_0}\right).$$

Analogously,

$$\max_{0\leq s\leq t} S_s < U$$

is equivalent to

$$M_t \leq y, \quad y = \frac{1}{\sigma}\ln\left(\frac{U}{S_0}\right).$$

Now, we are ready to move on toward the final price of the derivative.

Theorem 7. *The price of a European double-barrier option maturing at time T with a strike K and barriers L, U satisfying $0 < L < K, S_0 < U$ with the payoff (60) of the form*

$$C_{2B}(S_0, L, U, K, T, r, \sigma) = \sum_{k=-\infty}^{+\infty} C_{2B*}(k),$$

where

$$\begin{aligned} C_{2B*}(k) &= \exp\left(\left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)2k(y-z)\right)\Big(C_{BS}(e^{2k\sigma(y-z)}S_0, K, T, r, \sigma) \\ &\quad - C_{BS}(e^{2k\sigma(y-z)}S_0, U, T, r, \sigma) - e^{-rT}(U-K)\Phi\left(\frac{2k(y-z) - y + (r/\sigma - \sigma/2)T}{\sqrt{T}}\right)\Big) \\ &- \exp\left(\left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)(2k(y-z) + 2y)\right)\Big(C_{BS}(e^{2\sigma(k(y-z)+y)}S_0, K, T, r, \sigma) \\ &\quad - C_{BS}(e^{2\sigma(k(y-z)+y)}S_0, U, T, r, \sigma) - e^{-rT}(U-K)\Phi\left(\frac{2k(y-z) + y + (r/\sigma - \sigma/2)T}{\sqrt{T}}\right)\Big), \\ &\quad y = \frac{1}{\sigma}\ln\left(\frac{U}{S_0}\right), \quad z = \frac{1}{\sigma}\ln\left(\frac{L}{S_0}\right). \end{aligned} \tag{61}$$

Note that since, for k tending to $\pm\infty$, $C_{2B}(k)$ are converging to 0, for practical purposes, it is enough to consider only a couple of terms with small $|k|$ values. The similar formulas were provided by Haug ([3]) and by Barker ([1]).

Proof. The price of a European double-barrier option is given by a formula:

$$\begin{aligned} C_{2B} &= C_{2B}(S_0, L, U, K, T, r, \sigma) \\ &= e^{-rT}\mathbb{E}_{P^*}\Big((S_T - K)^+ \mathbb{1}_{L < \min_{0\leq s\leq T} S_s} \mathbb{1}_{\max_{0\leq s\leq T} S_s < U}\Big) \\ &= e^{-rT}\mathbb{E}_Q\left(\frac{dP^*}{dQ}(S_T - K)^+ \mathbb{1}_{m_T \geq z} \mathbb{1}_{M_T \leq y}\right) \\ &= \mathbb{E}_Q\big(H(W_T)\mathbb{1}_{m_T \geq z}\mathbb{1}_{M_T \leq y}\big). \end{aligned} \tag{62}$$

where

$$H(W_T) = e^{-rT}\frac{dP^*}{dQ}(S_T - K)^+ \mathbb{1}_{W_T \leq y},$$

and

$$\begin{aligned} H(\xi) &= e^{-rT}\exp\left(\left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)\xi - \frac{1}{2}\left(\left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)^2 T\right)(S_0 exp(\sigma\xi) - K)^+ \mathbb{1}_{\xi \leq y} \\ &= e^{-rT}\exp\left(\left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)\xi - \frac{1}{2}\left(\left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)^2 T\right)\left(\left(S_0 e^{\sigma\xi} - K\right)^+ - \left(S_0 e^{\sigma\xi} - U\right)^+ - (U-K)\mathbb{1}_{\xi \geq y}\right). \end{aligned} \tag{63}$$

Next, we apply Theorem 6. Basing on the fact that the density of the joint distribution of (W_T, M_T, m_T), $F_T(\xi, \eta, \theta)$, is vanishing outside the set

$$\{(\xi, \eta, \theta) \in \mathbb{R}^3 : \ \theta \leq -\xi^-, \ \xi^+ \leq \eta\}, \quad \xi^{\pm} = \max(0, \pm\xi),$$

we obtain

$$
\begin{aligned}
C_{2B} &= \int_{-\infty}^{+\infty} H(\xi) \int_{\xi^+}^{y} \int_{z}^{-\xi^-} \frac{\partial^3 F_T}{\partial\xi\partial\eta\partial\theta}(\xi,\eta,\theta) d\theta d\eta d\xi \qquad (64)\\
&= \int_{-\infty}^{+\infty} H(\xi) \Big(\partial_{\xi} F_T(\xi, y, -\xi^-) + \partial_{\xi} F_T(\xi, \xi^+, z) - \partial_{\xi} F_T(\xi, y, z)\\
&\quad -\partial_{\xi} F_T(\xi, \xi^+, -\xi^-) \Big) d\xi\\
&= \int_{-\infty}^{+\infty} H(\xi) \Big(\varphi_T(\xi) - \varphi_T(\xi - 2y)\\
&\quad + \sum_{k=0}^{\infty} \varphi_T(\xi - 2(k+1)(y-z)) - \sum_{k=0}^{\infty} \varphi_T(\xi - 2(k+2)y + 2(k+1)z)\\
&\quad - \sum_{k=0}^{\infty} \varphi_T(-\xi - 2ky + 2(k+1)z) + \sum_{k=0}^{\infty} \varphi_T(-\xi - 2(k+1)(y-z)) \Big) d\xi\\
&= \int_{-\infty}^{+\infty} H(\xi) \Big(\sum_{k=0}^{\infty} \varphi_T(\xi - 2k(y-z)) - \sum_{k=0}^{\infty} \varphi_T(\xi - 2(k+1)y + 2kz)\\
&\quad - \sum_{k=0}^{\infty} \varphi_T(\xi + 2ky - 2(k+1)z) + \sum_{k=0}^{\infty} \varphi_T(\xi + 2(k+1)(y-z)) \Big) d\xi\\
&= \int_{-\infty}^{+\infty} H(\xi) \Big(\sum_{k=-\infty}^{\infty} \varphi_T(\xi - 2k(y-z)) - \sum_{k=-\infty}^{\infty} \varphi_T(\xi - 2(k+1)y + 2kz) \Big) d\xi,
\end{aligned}
$$

where φ_T denotes the density of the normal distribution $N(0,T)$, and

$$\varphi_T(x) = \frac{1}{\sqrt{2T\pi}} \exp\Big(-\frac{x^2}{2T}\Big).$$

Note that the formula in the last brackets may be obtained as well from the Lévy Triple Law; see Corollary 2 or [15] Theorem 6.18.

Next, we return to the probability $\mathbb{P}^*$.

$$
\begin{aligned}
C_{2B} &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{+\infty} H(\xi)\varphi_T(\xi - 2k(y-z)) d\xi - \sum_{k=-\infty}^{\infty} \int_{-\infty}^{+\infty} H(\xi)\varphi_T(\xi - 2(k+1)y + 2kz)) d\xi\\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{+\infty} H(\xi + 2k(y-z))\varphi_T(\xi) d\xi - \sum_{k=-\infty}^{\infty} \int_{-\infty}^{+\infty} H(\xi + 2(k+1)y - 2kz)\varphi_T(\xi)) d\xi\\
&= \sum_{k=-\infty}^{\infty} \mathbb{E}_Q\Big(H(W_T + 2k(y-z)) \Big) - \sum_{k=-\infty}^{\infty} \mathbb{E}_Q\Big(H(W_T + 2(k+1)y - 2kz) \Big) \qquad (65)
\end{aligned}
$$

Since

$$
\begin{aligned}
H(W_T + \Delta) &= e^{-rT} \exp\Big(\Big(\frac{r}{\sigma} - \frac{1}{2}\sigma \Big)(W_T + \Delta) - \frac{1}{2}\Big(\Big(\frac{r}{\sigma} - \frac{1}{2}\sigma \Big)^2 T \Big) \qquad (66)\\
&\quad \times \Big(\Big(S_0 e^{\sigma(W_T + \Delta)} - K \Big)^+ - \Big(S_0 e^{\sigma(W_T + \Delta)} - U \Big)^+ - (U-K)\mathbb{1}_{W_T \geq y - \Delta} \Big)\\
&= \frac{d\mathbb{P}^*}{d\mathbb{Q}} e^{-rT} \exp\Big(\Big(\frac{r}{\sigma} - \frac{1}{2}\sigma \Big)\Delta \Big)\\
&\quad \times \Big(\Big(S_0 e^{\sigma\Delta} e^{\sigma W_T} - K \Big)^+ - \Big(S_0 e^{\sigma\Delta} e^{\sigma W_T} - U \Big)^+ - (U-K)\mathbb{1}_{W_T \geq y - \Delta} \Big),
\end{aligned}
$$

we have

$$\begin{aligned}\mathbb{E}_Q(H(W_T+\Delta)) &= \mathbb{E}_{P^*}\left(e^{-rT}\exp\left(\left(\frac{r}{\sigma}-\frac{1}{2}\sigma\right)\Delta\right)\right. \\ &\quad \left.\times\left(\left(S_0e^{\sigma\Delta}e^{\sigma W_T}-K\right)^+-\left(S_0e^{\sigma\Delta}e^{\sigma W_T}-U\right)^+-(U-K)\mathbb{1}_{W_T\geq y-\Delta}\right)\right) \\ &= \exp\left(\left(\frac{r}{\sigma}-\frac{1}{2}\sigma\right)\Delta\right) \\ &\quad \times\Big(C_{BS}(e^{\sigma\Delta}S_0,K,T,r,\sigma)-C_{BS}(e^{\sigma\Delta}S_0,U,T,r,\sigma) \\ &\quad -e^{-rT}(U-K)\Phi\left(\frac{\Delta-y+(r/\sigma-\sigma/2)T}{\sqrt{T}}\right)\Big)\end{aligned} \tag{67}$$

To conclude, for the proof, it is enough to put, respectively, $\Delta = 2k(y-z)$ or $\Delta = 2k(y-z)+2y$. □

The alternative method of pricing the double-barrier options is via simulations in the risk-neutral measure. In the Table 1, we present the results. The simulation is provided by an approximation of the Wiener process by a Random Walk process with 100,000 steps. We iterated the procedure 1534 times. The analytic formula for the price is approximated by the sum of the seven terms $C_{2B*}(k)$, where $k = -3,\ldots,3$.

Table 1. The comparison of simulated (S) and approximated (A) values of the two-barrier call options with varying strike K and risk-free interest rate r (in %). The initial value of the stock, barriers, volatility, and time to maturity are fixed: $S_0 = 100$, $L = 80$, $U = 130$, $\sigma = 0.5$, and $T = 1/4$.

r	**1**		**3**		**6**		**9**		**12.5**	
K	**S**	**A**	**S**	**A**	**S**	**A**	**S**	**A**	**S**	**A**
80	7.318	7.373	7.434	7.39	7.486	7.412	7.556	7.428	7.507	7.439
81	6.981	7.037	7.095	7.055	7.149	7.078	7.219	7.095	7.178	7.109
82	6.645	6.703	6.757	6.722	6.813	6.745	6.883	6.764	6.849	6.779
83	6.31	6.371	6.42	6.39	6.477	6.415	6.547	6.435	6.521	6.452
84	5.977	6.042	6.086	6.062	6.144	6.088	6.214	6.109	6.195	6.127
85	5.649	5.718	5.755	5.738	5.814	5.765	5.884	5.787	5.871	5.807
86	5.327	5.4	5.432	5.42	5.49	5.447	5.557	5.47	5.55	5.491
87	5.012	5.087	5.113	5.108	5.173	5.135	5.239	5.159	5.233	5.182
88	4.703	4.782	4.803	4.802	4.862	4.83	4.926	4.855	4.925	4.878
89	4.403	4.484	4.501	4.504	4.56	4.533	4.62	4.557	4.622	4.582
90	4.113	4.194	4.207	4.215	4.265	4.243	4.323	4.268	4.326	4.293
91	3.834	3.913	3.924	3.934	3.98	3.962	4.035	3.987	4.04	4.012
92	3.568	3.642	3.654	3.662	3.707	3.69	3.757	3.715	3.764	3.741
93	3.311	3.38	3.395	3.4	3.446	3.428	3.492	3.453	3.497	3.478
94	3.065	3.129	3.146	3.148	3.194	3.175	3.24	3.2	3.244	3.225
95	2.828	2.888	2.908	2.907	2.955	2.933	2.998	2.957	3.002	2.982
96	2.601	2.658	2.678	2.676	2.725	2.701	2.768	2.725	2.77	2.749
97	2.388	2.438	2.461	2.455	2.504	2.48	2.546	2.503	2.549	2.527
98	2.183	2.229	2.255	2.246	2.296	2.269	2.334	2.291	2.337	2.315
99	1.988	2.031	2.056	2.047	2.097	2.07	2.136	2.091	2.137	2.113
100	1.806	1.844	1.871	1.859	1.906	1.881	1.945	1.901	1.949	1.923
101	1.633	1.668	1.696	1.682	1.73	1.703	1.763	1.722	1.768	1.742
102	1.47	1.503	1.529	1.516	1.561	1.535	1.595	1.553	1.598	1.573
103	1.318	1.348	1.374	1.361	1.403	1.378	1.434	1.395	1.441	1.414
104	1.176	1.204	1.229	1.215	1.256	1.232	1.284	1.248	1.291	1.265
105	1.045	1.07	1.095	1.08	1.119	1.096	1.146	1.11	1.151	1.126
106	0.923	0.946	0.972	0.956	0.993	0.97	1.017	0.983	1.023	0.998
107	0.813	0.832	0.857	0.84	0.877	0.853	0.9	0.865	0.904	0.879

Table 1. *Cont.*

r	1		3		6		9		12.5	
K	S	A	S	A	S	A	S	A	S	A
108	0.712	0.727	0.753	0.735	0.77	0.746	0.792	0.757	0.795	0.769
109	0.621	0.631	0.658	0.638	0.673	0.648	0.693	0.658	0.696	0.669
110	0.54	0.544	0.573	0.55	0.585	0.559	0.604	0.568	0.606	0.578
111	0.465	0.465	0.496	0.471	0.507	0.479	0.524	0.487	0.525	0.495
112	0.395	0.394	0.424	0.399	0.436	0.406	0.452	0.413	0.452	0.421
113	0.338	0.331	0.362	0.335	0.37	0.341	0.387	0.347	0.387	0.354
114	0.285	0.275	0.308	0.279	0.315	0.284	0.325	0.289	0.328	0.295
115	0.237	0.226	0.257	0.229	0.266	0.233	0.275	0.237	0.273	0.242
116	0.195	0.183	0.213	0.185	0.22	0.189	0.23	0.193	0.228	0.197
117	0.159	0.146	0.175	0.148	0.18	0.151	0.188	0.154	0.189	0.157
118	0.127	0.114	0.141	0.116	0.146	0.118	0.152	0.12	0.151	0.123
119	0.097	0.087	0.111	0.089	0.115	0.091	0.121	0.092	0.119	0.095
120	0.073	0.065	0.085	0.066	0.088	0.068	0.093	0.069	0.093	0.071
121	0.054	0.047	0.063	0.048	0.065	0.049	0.069	0.05	0.07	0.051
122	0.038	0.033	0.046	0.034	0.047	0.034	0.049	0.035	0.049	0.036
123	0.025	0.022	0.032	0.022	0.033	0.023	0.034	0.023	0.033	0.024
124	0.014	0.014	0.02	0.014	0.021	0.014	0.023	0.015	0.021	0.015
125	0.007	0.008	0.011	0.008	0.012	0.008	0.014	0.008	0.013	0.009
126	0.002	0.004	0.005	0.004	0.005	0.004	0.008	0.004	0.007	0.004
127	0	0.002	0.001	0.002	0.002	0.002	0.008	0.002	0.004	0.002
128	0	0	0	0.001	0	0.001	0.001	0.001	0.001	0.001
129	0	0	0	0	0	0	0	0	0	0
130	0	0	0	0	0	0	0	0	0	0

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