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On Convergence of Double Series

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For the convergence of double series and iterated series, a sufficient condition is obtained. This result provides a test for the convergence of double series and iterated series. © 1999 Academic Press

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For $(n,m),(n',m') \in \mathbb{N}^2$, let $(n,m) \le (n',m')$ if and only if $n \le n'$ and $m \le m'$. Then $\mathbb{N}^2$ is a directed set and for elements x_{ij} of an abelian topological group, $\{\sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} : (n,m) \in \mathbb{N}^2\}$ is a net. If $\lim_{(n,m)} \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}$ exists, then we say that the double series $\sum_{i,j} x_{ij}$ converges and

$$\sum_{i,j} x_{ij} = \lim_{(n,m)} \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}.$$

For $\sum_{i,j} x_{ij}$, there is an interesting result owing to Antosik:

THEOREM 1 [1, 6, 7]. *Let G be a Hausdorff abelian topological group and let $x_{ij} \in G$ for $i,j \in \mathbb{N}$. If*

(I) $\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} x_{ij_k}$ *converges for every $j_1 < j_2 < \cdots$, then*

(II) *the three series $\sum_{i,j} x_{ij}$, $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_{ij}$, and $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij}$ all converge and are equal.*

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Using this result Swartz [7] established a perfect Stiles-type Orlicz–Pettis theorem. However, as was stated in [6], condition (I) is so strong that the matrix

$$\left[\frac{1}{ij}(-1)^{i+j}\right]_{i,j}$$

fails to satisfy it, though

$$\sum_{i,j}\frac{(-1)^{i+j}}{ij}=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{(-1)^{i+j}}{ij}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{i+j}}{ij}=\left[\sum_{j=1}^{\infty}\frac{(-1)^{j}}{j}\right]^2.$$

COROLLARY 2. *Condition* (I) *is equivalent to*

(III) *For every sequence* $j_1 < j_2 < \cdots$ *in* $\mathbb{N}$, *the three series* $\sum_{i,k} x_{ij_k}$, $\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} x_{ij_k}$, *and* $\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} x_{ij_k}$ *all converge and are equal.*

Usually, we are interested in (II) but, indeed, (III) is too much and, equivalently, (I) is too strong for (II). Therefore, in this note we will find a weaker condition which implies (II).

Recall that a net $\{x_d : d \in D\}$ is Cauchy if for every neighborhood U of $0 \in G$ there is a $d_0 \in D$ such that $x_d - x_{d'} \in U$ whenever $d \geq d_0$ and $d' \geq d_0$. G is complete if every Cauchy net in G converges to a point in G. For complete metric spaces, there is a Moore lemma [3, p. 28] which gives a sufficient condition for the existence of double and iterated limits. It is clear that the Moore lemma is also valid for complete abelian topological groups.

LEMMA 3 (Moore lemma). *Let* D_1 *and* D_2 *be directed sets, and suppose that* $D_1 \times D_2$ *is directed by the relation* $(d_1, d_2) \leq (d_1', d_2')$, *which is defined to mean that* $d_1 \leq d_1'$ *and* $d_2 \leq d_2'$. *Let* f: $D_1 \times D_2 \to G$ *be a net in the complete Hausdorff abelian topological group* G. *Suppose that*

(a) *for each* $d_2 \in D_2$, $\lim_{d_1} f(d_1, d_2)$ *exists, and*

(b) $\lim_{d_2} f(d_1, d_2)$ *exists uniformly for* $d_1 \in D_1$.

Then the three limits $\lim_{(d_1, d_2)} f(d_1, d_2)$, $\lim_{d_1}\lim_{d_2} f(d_1, d_2)$, *and* $\lim_{d_2}\lim_{d_1} f(d_1, d_2)$ *all exist and are equal.*

THEOREM 4. *Let* G *be a complete Hausdorff abelian topological group and let* $x_{ij} \in G$ *for* $i, j \in \mathbb{N}$. *Suppose that*

(1) $\sum_{i=1}^{\infty} x_{ij}$ *converges for each* j,

(2) $\sum_{j=1}^{\infty} x_{ij}$ *converges for each* i, *and*

(3) *for every sequence* $p_1 < q_1 < p_2 < q_2 < \cdots$ *in* $\mathbb{N}$ *there is a sequence* $k_1 < k_2 < \cdots$ *in* $\mathbb{N}$ *such that the iterated series* $\sum_{i=1}^{\infty}\sum_{r=1}^{\infty}\sum_{j=p_{k_r}}^{q_{k_r}} x_{ij}$ *converges.*

Then the three series $\sum_{i,j} x_{ij}, \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij}$, *and* $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_{ij}$ *all converge and are equal.*

Proof. By (1) and (2),

$$\lim_{n} \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{\infty} x_{ij}$$

exists for each m and

$$\lim_{m} \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{\infty} x_{ij}$$

exists for each n. If the series $\sum_{i=1}^{n}\sum_{j=1}^{\infty} x_{ij}$ converges uniformly for $n \in \mathbb{N}$, then the desired conclusion follows from Lemma 3.

Suppose that the convergence of $\sum_{i=1}^{n}\sum_{j=1}^{\infty}$ is not uniform with respect to $n \in \mathbb{N}$. Then there is a neighborhood U of $0 \in G$ for which the following holds:

$$\text{for every } m_0 \in \mathbb{N}, \text{ there is } m > m_0 \text{ and } n \in \mathbb{N} \text{ with } \sum_{i=1}^{n}\sum_{j=m}^{\infty} x_{ij} \notin U. \quad (*)$$

Pick a neighborhood V of $0 \in G$ with $V + V \subseteq U$. By $(*)$, there exist $p_1 > 1$ and $n_1 \in \mathbb{N}$ such that

$$\sum_{i=1}^{n_1}\sum_{j=p_1}^{\infty} x_{ij} \notin U \qquad \text{but} \qquad \sum_{i=1}^{n_1}\sum_{j=q_1+1}^{\infty} x_{ij} \in V$$

for sufficiently large $q_1 > p_1$ and hence,

$$\sum_{i=1}^{n_1}\sum_{j=p_1}^{q_1} x_{ij} \notin V.$$

Let $p_0 > q_1$ be such that $\sum_{i=1}^{n}\sum_{j=p}^{\infty} x_{ij} \in U$ for all $1 \le n \le n_1$ and $p > p_0$. Then, by $(*)$ again, there exist $p_2 > p_0$ $(> q_1)$ and $n_2 > n_1$ such that

$$\sum_{i=1}^{n_2}\sum_{j=p_2}^{\infty} x_{ij} \notin U$$

and, hence,

$$\sum_{i=1}^{n_2}\sum_{j=p_2}^{q_2} x_{ij} \notin V$$

for some $q_2 > p_2$. Continuing this construction we have integer sequences $n_1 < n_2 < \cdots$ and $p_1 < q_1 < p_2 < q_2 < \cdots$ such that

$$\sum_{i=1}^{n_k}\sum_{j=p_k}^{q_k} x_{ij}h \notin V, \qquad k = 1, 2, 3, \ldots . \qquad (**)$$

Consider the matrix $[\sum_{i=1}^{n_k}\sum_{j=p_m}^{q_m} x_{ij}]_{k,m}$. For each m,

$$\lim_k \sum_{i=1}^{n_k}\sum_{j=p_m}^{q_m} x_{ij} = \sum_{j=p_m}^{q_m}\sum_{i=1}^{\infty} x_{ij}$$

exists by (1). Let $m_1 < m_2 < \cdots$ in $\mathbb{N}$. Then $p_{m_1} < q_{m_1} < p_{m_2} < q_{m_2} < \cdots$ and, by condition (3), there is a sequence $\nu_1 < \nu_2 < \cdots$ in $\mathbb{N}$ such that the iterated series $\sum_{i=1}^{\infty}\sum_{r=1}^{\infty}\sum_{j=p_{m_{\nu_r}}}^{q_{m_{\nu_r}}} x_{ij}$ converges and, hence,

$$\lim_k \sum_{i=1}^{n_k}\sum_{r=1}^{\infty}\sum_{j=p_{m_{\nu_r}}}^{q_{m_{\nu_r}}} x_{ij} = \sum_{i=1}^{\infty}\sum_{r=1}^{\infty}\sum_{j=p_{m_{\nu_r}}}^{q_{m_{\nu_r}}} x_{ij}.$$

Now, by the Antosik–Mikusinski matrix theorem [2, 4],

$$\lim_k \sum_{i=1}^{n_k}\sum_{j=p_k}^{q_k} x_{ij} = 0.$$

This contradicts $(**)$ and hence

$$\lim_m \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{\infty} x_{ij}$$

uniformly for $n \in \mathbb{N}$.

Note that, without completeness of G, conditions (1), (2), and (3) imply that $\sum_{i=1}^{n}\sum_{j=1}^{\infty} x_{ij}$ converges uniformly for $n \in \mathbb{N}$. Consider the condition

(2′) the iterated series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij}$ converges.

Clearly, (2′) is stronger than (2).

THEOREM 5. *Let G be a Hausdorff abelian topological group which need not be complete, $x_{ij} \in G$ for $i, j \in \mathbb{N}$. If conditions* (1), (2′), *and* (3) *hold, then the three series $\sum_{i,j} x_{ij}$, $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_{ij}$, and $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij}$ all converge and are equal.*

Proof. Since (2′) $\Rightarrow$ (2), as in the proof of Theorem 4,

$$\lim_m \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{\infty} x_{ij}$$

uniformly for $n \in \mathbb{N}$.

Let U and V be neighborhoods of $0 \in G$ such that V is symmetric and $V + V \subseteq U$. There is an $m_0 \in \mathbb{N}$ for which

$$\sum_{i=1}^{n}\sum_{j=1}^{\infty} x_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} \in V$$

for all $n \in \mathbb{N}$ and $m \geq m_0$. Since the iterated series $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}$ converges, there is an $n_0 \in \mathbb{N}$ such that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{\infty} x_{ij} \in V$$

for all $n \geq n_0$. Therefore, if $n \geq n_0$ and $m \geq m_0$, then

$$\begin{aligned} &\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \\ &\quad = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{\infty} x_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{\infty} x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \\ &\quad \in V + V \\ &\quad \subseteq U. \end{aligned}$$

This shows that

$$\sum_{i,j} x_{ij} = \lim_{(n,m)} \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}.$$

Note that condition (I) implies (1), (2), (2′), and (3). Hence Theorem 4 implies Theorem 1.

COROLLARY 6. *Let G be a Hausdorff abelian topological group and let $x_{ij} \in G$ for $i, j \in \mathbb{N}$ be such that conditions* (1) *and* (3) *hold. Then* (2′) *and* (II) *are equivalent, i.e., the three series $\sum_{i,j} x_{ij}$, $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}$, and $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} x_{ij}$ all converge and are equal if and only if the iterated series $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}$ converges.*

A pseudometric is a real function d on a set $X \times X$ such that $d(x, y) = d(y, x) \geq 0$, $d(x, x) = 0$, $d(x, z) \leq d(x, y) + d(y, z)$ [9, p. 10]. For a vector space X, a function $\|\cdot\|: X \to [0, +\infty)$ is a paranorm on X if and only if there is a pseudometric d on X such that $d(x, y) = d(x - y, 0)$ for $x, y \in X$ and $\|x\| = d(x, 0)$ for $x \in X$ [9, p. 16]. A series $\sum x_j$ on a paranormed space $(X, \|\cdot\|)$ is said to be absolutely convergent if $\sum_{j=1}^{\infty} \|x_j\| < +\infty$. A paranormed space X is complete if and only if every absolutely convergent series on X is convergent in X [9, p. 57].

Consider the condition

(3′) for every sequence $p_1 < q_1 < p_2 < q_2 < \cdots$ in $\mathbb{N}$ there is a sequence $k_1 < k_2 < \cdots$ in $\mathbb{N}$ such that $\lim_{\nu} \sum_{i=1}^{n} \sum_{j=p_{k_\nu}}^{q_{k_\nu}} x_{ij} = 0$ uniformly with respect to $n \in \mathbb{N}$.

THEOREM 7. *Let $(X, \|\cdot\|)$ be a complete Hausdorff paranormed space and let $x_{ij} \in X$ for $i, j \in \mathbb{N}$. If conditions* (1), (2), *and* (3′) *hold, then the three series $\sum_{i,j} x_{ij}$, $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij}$, and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_{ij}$ all converge and are equal.*

Proof. Let $p_1 < q_1 < p_2 < q_2 < \cdots$ in $\mathbb{N}$. There is a sequence $k_1 < k_2 < \cdots$ in $\mathbb{N}$ such that

$$\lim_{\nu} \left\| \sum_{i=1}^{n} \sum_{j=p_{k_\nu}}^{q_{k_\nu}} x_{ij} \right\| = 0$$

uniformly for $n \in \mathbb{N}$. Hence there exists a sequence $\nu_1 < \nu_2 < \cdots$ in $\mathbb{N}$ for which

$$\sup_{n} \left\| \sum_{i=1}^{n} \sum_{j=p_{k_{\nu_r}}}^{q_{k_{\nu_r}}} x_{ij} \right\| \le 2^{-r}, \qquad r = 1, 2, 3, \ldots .$$

Observing that $(X, \|\cdot\|)$ is complete,

$$\sum_{r=1}^{\infty} \sup_{n} \left\| \sum_{i=1}^{n} \sum_{j=p_{k_{\nu_r}}}^{q_{k_{\nu_r}}} x_{ij} \right\| \le \sum_{r=1}^{\infty} 2^{-r} = 1$$

shows that the series $\sum_{r=1}^{\infty}\sum_{i=1}^{n}\sum_{j=p_{k_{\nu_r}}}^{q_{k_{\nu_r}}} x_{ij}$ converges uniformly for $n \in \mathbb{N}$. This shows that the series

$$\sum_{i=1}^{n}\sum_{r=1}^{\infty}\sum_{j=p_{k_{\nu_r}}}^{q_{k_{\nu_r}}} x_{ij} \left(= \sum_{r=1}^{\infty}\sum_{i=1}^{n}\sum_{j=p_{k_{\nu_r}}}^{q_{k_{\nu_r}}} x_{ij} \right)$$

converges uniformly for $n \in \mathbb{N}$, i.e.,

$$\lim_{m} \sum_{i=1}^{n}\sum_{r=1}^{m}\sum_{j=p_{k_{\nu_r}}}^{q_{k_{\nu_r}}} x_{ij} = \sum_{i=1}^{n}\sum_{r=1}^{\infty}\sum_{j=p_{k_{\nu_r}}}^{q_{k_{\nu_r}}} x_{ij}$$

uniformly for $n \in \mathbb{N}$.

On the other hand, by condition (1),

$$\lim_{n} \sum_{i=1}^{n}\sum_{r=1}^{m}\sum_{j=p_{k_{\nu_r}}}^{q_{k_{\nu_r}}} x_{ij} = \sum_{r=1}^{m}\sum_{j=p_{k_{\nu_r}}}^{q_{k_{\nu_r}}}\sum_{i=1}^{\infty} x_{ij}$$

exists for each $m \in \mathbb{N}$.

Therefore, by Lemma 3, the iterated series $\sum_{i=1}^{\infty}\sum_{r=1}^{\infty}\sum_{j=p_{k_r}}^{q_{k_r}} x_{ij}$ converges; i.e., condition (3) holds. Now the desired conclusion follows from Theorem 4.

EXAMPLE 1. Let

$$e_j = \left(\overbrace{0,\ldots,0}^{j-1},1,0,0,\ldots\right) \qquad \text{for all } j.$$

Then the double series $\sum_{i,j}((-1)^i/\sqrt{i}\ln(j+1))e_{i+j}$ converges in $(c_0,\|\cdot\|_\infty)$. In fact, if $p_1 < q_1 < p_2 < q_2 < \cdots$ in $\mathbb{N}$, then for each $n \in \mathbb{N}$, we have that

$$\begin{aligned}
&\left\|\sum_{i=1}^{n}\sum_{j=p_k}^{q_k}\frac{(-1)^i}{\sqrt{i}\ln(j+1)}e_{i+j}\right\|_\infty \\
&\quad=\left\|\sum_{j=p_k}^{q_k}\frac{1}{\ln(j+1)}\sum_{i=1}^{n}\frac{(-1)^i}{\sqrt{i}}e_{i+j}\right\|_\infty \\
&\quad=\left\|\left(\overbrace{0,\ldots,0}^{p_k},\frac{-1}{\ln(p_k+1)},\frac{1}{\sqrt{2}\ln(p_k+1)},\ldots,\right.\right. \\
&\qquad\qquad\left.\frac{(-1)^n}{\sqrt{n}\ln(p_k+1)},0,0,\ldots\right) \\
&\qquad+\left(\overbrace{0,\ldots,0,0}^{p_k+1},\frac{-1}{\ln(p_k+2)},\frac{1}{\sqrt{2}\ln(p_k+2)},\ldots,\right. \\
&\qquad\qquad\left.\frac{(-1)^n}{\sqrt{n}\ln(p_k+2)},0,0,\ldots\right) \\
&\qquad+\cdots \\
&\qquad+\left(\overbrace{0,\ldots,0}^{q_k},\frac{-1}{\ln(q_k+1)},\frac{1}{\sqrt{2}\ln(q_k+1)},\ldots,\right. \\
&\qquad\qquad\left.\left.\frac{(-1)^n}{\sqrt{n}\ln(q_k+1)},0,0,\ldots\right)\right\|_\infty \\
&\quad\le 3\sup_m\left|\sum_{i=1}^{m}\frac{(-1)^i}{\sqrt{i}}\right|\frac{1}{\ln(p_k+1)}
\end{aligned}$$

by the Abel transformation. Therefore,

$$\lim_k \left\| \sum_{i=1}^{n} \sum_{j=p_k}^{q_k} \frac{(-1)^i}{\sqrt{i}\,\ln(j+1)} e_{i+j} \right\|_\infty = 0$$

uniformly for $n \in \mathbb{N}$.

Finally, we show that, sometimes, condition (3′) is necessary for the convergence of $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}$.

EXAMPLE 2. Let X be a Banach space with an unconditional basis $\{e_j\}$ and $x_i = \sum_{j=1}^{\infty} t_{ij} e_j \in X$ with $t_{ij} \geq 0$ for all $i, j \in \mathbb{N}$. If the series $\sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} t_{ij} e_j$ converges, then for every sequence $p_1 < q_1 < p_2 < q_2 < \cdots$ in $\mathbb{N}$,

$$\lim_k \left\| \sum_{i=1}^{n} \sum_{i=p_k}^{q_k} t_{ij} e_j \right\| = 0$$

uniformly for $n \in \mathbb{N}$; i.e., (3′) holds and $\sum_{i,j} t_{ij} e_j = \sum_{i=1}^{\infty} x_i$.

To see this, observe that the continuity of the coordinate functionals with respect to the basis $\{e_j\}$ implies that

$$\sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} t_{ij} e_j = \lim_n \sum_{i=1}^{n} \sum_{j=1}^{\infty} t_{ij} e_j$$

$$= \lim_n \sum_{j=1}^{\infty} \left(\sum_{i=1}^{n} t_{ij} \right) e_j = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} t_{ij} \right) e_j$$

because if $\sum_{i=1}^{\infty} x_i = \sum_{j=1}^{\infty} t_j e_j$, then $t_j = \lim_n \sum_{i=1}^{n} t_{ij} = \sum_{i=1}^{\infty} t_{ij}$ for all j.

Since $\{e_j\}$ is unconditional, $\sum_{j=1}^{\infty} a_j e_j$ converges if $|a_j| \leq \sum_{i=1}^{\infty} t_{ij}$ for all j and, in particular, $\sum_{k=1}^{\infty} (\sum_{i=1}^{n} t_{ij_k}) e_{j_k}$ converges for all n and $j_1 < j_2 < \cdots$. Let $D_j = \{z \in \mathbb{C} : |z| \leq \sum_{i=1}^{\infty} t_{ij}\}$, $j = 1, 2, 3, \ldots$. If the convergence of $\sum_{j=1}^{\infty} a_j e_j$ is not uniform for $\{a_j\} \in \prod_{j=1}^{\infty} D_j$, then there exist an $\epsilon > 0$ and $\{(a_{k_j})_{j=1}^{\infty} : k = 1, 2, 3, \ldots\} \subseteq \prod_{j=1}^{\infty} D_j$ and an integer sequence $m_1 < n_1 < m_2 < n_2 < \cdots$ such that

$$\left\| \sum_{j=m_k}^{n_k} a_{kj} e_j \right\| \geq \epsilon, \qquad k = 1, 2, 3, \ldots.$$

Therefore, letting

$$a_j = \begin{cases} a_{kj}, & m_k \leq j \leq n_k \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases}$$

$\{a_j\} \in \prod_{j=1}^{\infty} D_j$ but $\sum_{j=1}^{\infty} a_j e_j$ diverges. This contradiction shows that $\sum_{j=1}^{\infty} a_j e_j$ converges uniformly with respect to $\{a_j\} \in \prod_{j=1}^{\infty} D_j$ and hence, if $(a_{nj})_{j=1}^{\infty}, (a_j)_{j=1}^{\infty} \in \prod_{j=1}^{\infty} D_j$ are such that $\lim_n a_{nj} = a_j$ for all j, then

$$\lim_n \left\| \sum_{j=1}^{\infty} a_j e_j - \sum_{j=1}^{\infty} a_{nj} e_j \right\| = 0.$$

In particular,

$$\lim_n \left\| \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_{ij_k} \right) e_{j_k} - \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n} t_{ij_k} \right) e_{j_k} \right\| = 0$$

for every $j_1 < j_2 < \cdots$.

Now let $p_1 < q_1 < p_2 < q_2 < \cdots$ in $\mathbb{N}$. If

$$\lim_k \left\| \sum_{i=1}^{n} \sum_{j=p_k}^{q_k} t_{ij} e_j \right\| = 0$$

is not uniform with respect to $n \in \mathbb{N}$, then there exist $\epsilon > 0$ and integer sequences $n_1 < n_2 < \cdots$ and $k_1 < k_2 < \cdots$ such that

$$\left\| \sum_{i=1}^{n_r} \sum_{j=p_{k_r}}^{q_{k_r}} t_{ij} e_j \right\| \geq \epsilon, \qquad r = 1, 2, 3, \ldots . \tag{α}$$

Consider the matrix $[\sum_{i=1}^{n_r} \sum_{j=p_{k_s}}^{q_{k_s}} t_{ij} e_j]_{r,s}$. For each $s \in \mathbb{N}$,

$$\lim_r \sum_{i=1}^{n_r} \sum_{j=p_{k_s}}^{q_{k_s}} t_{ij} e_j = \sum_{j=p_{k_s}}^{q_{k_s}} \left(\sum_{i=1}^{\infty} t_{ij} \right) e_j$$

exists. Suppose that $s_1 < s_2 < \cdots$ in $\mathbb{N}$. Then the series

$$\sum_{\nu=1}^{\infty} \sum_{j=p_{k_{s_\nu}}}^{q_{k_{s_\nu}}} \left(\sum_{i=1}^{\infty} t_{ij} \right) e_j$$

converges and

$$\lim_r \left\| \sum_{\nu=1}^{\infty} \sum_{j=p_{k_{s_\nu}}}^{q_{k_{s_\nu}}} \left(\sum_{i=1}^{\infty} t_{ij} \right) e_j - \sum_{\nu=1}^{\infty} \sum_{j=p_{k_{s_\nu}}}^{q_{k_{s_\nu}}} \left(\sum_{i=1}^{n_r} t_{ij} \right) e_j \right\| = 0.$$

Therefore, the Antosik–Mikusinski matrix theorem shows that

$$\lim_r \left\| \sum_{i=1}^{n_r} \sum_{j=p_{k_r}}^{q_{k_r}} t_{ij} e_j \right\| = 0.$$

This contradicts (α) and hence

$$\lim_k \left\| \sum_{i=1}^{n} \sum_{j=p_k}^{q_k} t_{ij} e_j \right\| = 0$$

uniformly for $n \in \mathbb{N}$.

Note that Stuart [5] has also given a generalization of Antosik's result but this generalization belongs to a different direction [8, p. 99].

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