Stochastic Processes and their Applications 32 (1989) 253-280
North-Holland

PATHWISE STOCHASTIC INTEGRATION AND APPLICATIONS TO THE THEORY OF CONTINUOUS TRADING

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Received 16 September 1987
Revised 15 August 1988

We develop a pathwise construction of stochastic integrals relative to continuous martingales. The key to the construction is an almost-sure approximation technique which associates a sequence of finitely generated filtrations ("skeleton filtrations") and a sequence of simple stochastic processes ("skeleton processes") to a given continuous martingale and its underlying filtration (which is also assumed to be "continuous"). The pathwise stochastic integral can then be defined along such a "skeleton approximation" and almost-sure convergence follows from a certain completeness property of the skeleton approximation. The limit is the pathwise stochastic integral and it agrees with the integral obtained through the usual Itô approach.

The pathwise stochastic integration theory is applied to the analysis of stochastic models of security markets with continuous trading. A convergence theory for continuous market models with exogenously given equilibrium prices is obtained which enables one to view an idealized economy (i.e. a continuous market model) as an almost-sure limit of "real-life" economies (i.e. finite market models). Since finite market models are well understood, this convergence sheds light on features of the continuous market model, such as completeness.

Pathwise stochastic integration * martingale representation property * fine structure of filtration * continuous trading * complete markets * option pricing

1. Introduction

Stochastic integrals, like ordinary integrals, are defined through limiting procedures. Since probability theory uses different concepts of convergence one has to specify the sense in which a stochastic integral exists. If the "integrator" Z is a stochastic process of bounded variation (i.e. almost all sample paths of Z are functions of finite variation), then for a very general class of "integrands" ϕ, the stochastic integral $\int \phi \, dZ$ is defined *pathwise* as a Riemann-Stieltjes integral; that is, as an almost-sure limit of Riemann-Stieltjes sums. In general, this can no longer be done if Z has sample paths of unbounded variation (e.g. if Z is Brownian motion). By

* Research supported by the National Science Foundation grant ECS-8696-090 at Boston University.

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generalizing a functional-analytic completion procedure of Itô (1944), Kunita and Watanabe (1967) showed that stochastic integrals with good properties can be obtained as limits of *mean-square convergent* sequences with square-integrable martingales as integrators and with the integrands restricted to a certain class of predictable processes. Subsequently, stochastic integrals with respect to local martingales (Meyer, 1967; Doléans-Dade and Meyer, 1970) and semimartingales have been developed (Jacod, 1979; Dellacherie and Meyer, 1982). Semimartingales are, in a well-defined sense, the largest class of stochastic processes with respect to which stochastic integration is reasonable (Dellacherie, 1980; Bichteler, 1981; Protter, 1986). As a consequence of this development, stochastic integrals are now typically defined non-constructively as L^p-limits of Riemann-Stieltjes sums.

In this paper, we present a constructive approach to stochastic integration with respect to continuous, vector-valued martingales with "continuous" filtrations. The idea is not only to discretize time (this alone results in the usual Riemann-Stieltjes approximation) but also to carefully discretize the probability space so that almost-sure (pathwise) convergence of "simple" Riemann-Stieltjes sums can be established. Our construction relies on a general approximation technique for random processes developed in Willinger (1987) and Willinger and Taqqu (1988) and provides a practical method for approximating stochastic integrals.

We also apply pathwise stochastic integration to the theory of security markets with continuous trading. This theory deals with the analysis of stochastic models for the buying and selling of portfolios of securities in continuous time (Harrison and Kreps, 1979; Harrison and Pliska, 1981; Stricker, 1984; Duffie and Huang, 1985; Müller, 1985; Denny and Suchanek, 1986; Föllmer and Sondermann, 1986). For exogenously given equilibrium prices, we characterize convergence of the more "realistic" finite market models to continuous models and thereby explain features of the latter (e.g. completeness, no-arbitrage) via the corresponding well-understood concepts in a finite setting (Taqqu and Willinger, 1987). Such a characterization is of interest from a theoretical as well as practical point of view (Harrison and Pliska, 1983; Kreps, 1982) since it contributes to a better understanding of both the martingale representation theory (Jacod, 1979) and complete security market models. It also solves an open problem stated in Kopp (1984, pp. 168-169).

1.1. Stochastic integration a la Itô

The main ideas in the classical theory of stochastic integration are essentially due to Itô (1944), and concern the definition of stochastic integrals with respect to square-integrable martingales. We briefly describe this method in order to facilitate comparisons with (i) existing pathwise constructions of stochastic integrals discussed in Section 1.2, and (ii) our pathwise approach.

The fundamental concepts of the classical theory are:

(a) predictability,

(b) the Doob-Meyer decomposition,

(c) a functional-analytic completion.

In order to describe these concepts, some notation is necessary. Fix a stochastic base $(\Omega, \mathcal{F}, P, \boldsymbol{F})$ where the filtration $\boldsymbol{F} = (\mathcal{F}_t: 0 \leq t \leq T)$ satisfies the "usual conditions" (see Dellacherie and Meyer, 1978, p. 115; or Section 4 below). Let $\mathcal{M}_0^2$ denote the vector space of all (real-valued) $(\boldsymbol{F}, P)$-martingales $M = (M_t: 0 \leq t \leq T)$ with $\sup_{0 \leq t \leq T} E_P[|M_t|^2] < \infty$ and $M_0 = 0$, and with sample paths that are right continuous and have left limits. Observe that $\mathcal{M}_0^2$ is a Hilbert space under the inner product $(M, N) \to E_P[M_T N_T]$ (by identifying $M \in \mathcal{M}_0^2$ with $M_T \in L^2(\Omega, \mathcal{F}, P)$). For each $M \in \mathcal{M}_0^2$, the *Doob-Meyer decomposition theorem* (Dellacherie and Meyer, 1982, p. 198) applies and yields a unique predictable increasing process $\langle M \rangle = (\langle M \rangle_t: 0 \leq t \leq T)$, called the *predictable quadratic variation of* M such that $M^2 - \langle M \rangle$ is a uniformly integrable martingale. Here, a process is called *predictable* if it is measurable with respect to the *predictable* σ*-algebra* $\mathcal{P}$, where $\mathcal{P}$ is defined to be $\mathcal{P} = \sigma(\{\text{left-continuous, } \boldsymbol{F}\text{-adapted processes}\})$.

When constructing stochastic integrals with respect to elements in $\mathcal{M}_0^2$ (satisfying some reasonable measurability and integrability properties), it is necessary to restrict potential integrands to predictable processes. (See Rogers, 1981, p. 59, for an example that illustrates this.) Let then $\mathcal{E}$ denote the vector space of all *elementary predictable processes* on $[0, T]$: $\phi \in \mathcal{E}$ if there exists a partition $0 = t(0) < t(1) < \cdots < t(n) = T$ of $[0, T]$ and $\mathcal{F}_{t(k)}$-measurable, bounded random variables $\phi_{t(k)}$ $(k = 0, 1, \ldots, n;\ \phi_0 = 0)$ such that $\phi_t = \phi_{t(k)}$ if $t \in (t(k), t(k+1)]$ for some $0 \leq k < n$. Then, for $M \in \mathcal{M}_0^2$, we define the *stochastic integral* $\int \phi \, dM$ by

$$\int_0^t \phi_s \, dM_s = \sum_{k=0}^{n-1} \phi_{t(k)}(M_{t(k+1)\wedge t} - M_{t(k)\wedge t}), \quad 0 \leq t \leq T.$$

An easy calculation shows that $\int \phi \, dM \in \mathcal{M}_0^2$ and moreover,

$$E_P\left[\left(\int_0^T \phi_s \, dM_s\right)^2\right] = E_P\left[\int_0^T \phi_s^2 \, d\langle M \rangle_s\right].$$

Next, let $\mathcal{L}^2(M)$ denote the space $L^2(\Omega \times [0, T], \mathcal{P}, d(P \times \langle M \rangle))$ of all predictable processes ϕ such that

$$\|\phi\|_M \equiv E_P\left[\int_0^T \phi_s^2 \, d\langle M \rangle_s\right] < \infty.$$

Obviously, $\mathcal{E}$ is *dense* in $\mathcal{L}^2(M)$ in the topology defined by $\|\cdot\|_M$ and therefore, we can conclude:

The map $\phi \to \int \phi \, dM$ from $\mathcal{E}$ into $\mathcal{M}_0^2$ is a *linear isometry* and hence, it has a unique extension (again denoted by $\phi \to \int \phi \, dM$) from $\mathcal{L}^2(M)$ into $\mathcal{M}_0^2$. The image of ϕ under this map is called the *stochastic integral of* ϕ *with respect to* M (denoted by $\int \phi \, dM$).

Observe that this procedure (Itô's method) characterizes the stochastic integral in terms of an isometry from $\mathcal{L}^2(M)$ into $\mathcal{M}_0^2$ rather than in terms of a probabilistically

more satisfying object such as a stochastic process. The latter was achieved by Kunita and Watanabe (1967) who thereby provided the key to the modern development of stochastic integration. They define the stochastic integral as the unique stochastic process satisfying a certain functional relation (see Dellacherie and Meyer, 1982, p. 339).

Neither Kunita and Watanabe's definition nor Itô's enables us to evaluate, for a given $\omega \in \Omega$, the corresponding sample path of the process

$$\int \phi \, dM = \left(\int_0^t \phi_s \, dM_s : 0 \leq t \leq T \right).$$

In fact, one cannot even write

$$\left[\int_0^t \phi_s \, dM_s \right](\omega) = \int_0^t \phi_s(\omega) \, dM_s(\omega),$$

except of course in the case where the sample paths of M are functions of finite variation, making $\int \phi \, dM$ a Riemann-Stieltjes integral. Thus, from a practical (as well as theoretical) point of view, the classical stochastic integration theory is hampered by the fact that, in general, one cannot "construct" and/or approximate stochastic integrals sample path by sample path. However, as is shown in Dellacherie and Meyer (1982, p. 330) (see also Lenglart, 1978; Bichteler, 1981), the paths of $\int \phi \, dM$ do, in a certain sense, depend only on the paths of ϕ and M. In Section 4, we show much more: under well-defined conditions, the stochastic integral $\int \phi \, dM$ can be defined and constructed in a pathwise sense, and moreover, the construction will provide a practical method for "calculating" and approximating stochastic integrals.

1.2. Pathwise "constructions" of stochastic integrals

Since stochastic integrals are typically defined as L^p-limits of Riemann-Stieltjes sums, there always exists a subsequence along which almost-sure convergence holds. Presently available pathwise "constructions" rely in one way or another on the existence of such an almost-sure convergent subsequence. Here we briefly discuss three methods for pathwise constructions of stochastic integrals, due to Wong and Zakai (1965, 1969), Bichteler (1981), and Föllmer (1982), respectively.

Wong and Zakai's basic idea is to replace an integrator such as Brownian motion $W = (W_t : 0 \leq t \leq 1)$ (with typically very erratic sample paths behaviour) with a sequence of "smooth" approximations $(W_t^n : 0 \leq t \leq 1)_{n \geq 0}$ of W. Then the hope is that the corresponding stochastic integrals

$$\int_0^1 \phi_s(\omega) \, dW_s^n(\omega), \quad n \geq 0, \tag{1.2.1}$$

(which are, in fact, ordinary integrals, since W^n is smooth) converge in some sense

to the Itô integral

$$\left[\int_0^1 \phi_s \, \mathrm{d}W_s\right](\omega). \tag{1.2.2}$$

Wong and Zakai (1965, 1969) show that for a restricted class of integrands ϕ, the integrals of the form (1.2.1) converge in mean square (and hence pathwise for a suitable subsequence) along a sequence of partitions of $[0, 1]$ and a corresponding sequence of polygonal approximations W^n of W. But instead of convergence to (1.2.2), one has convergence to

$$\int_0^1 \phi_s \, \mathrm{d}W_s + \text{correction term}.$$

Bichteler (1981) provides, among other things, a method for obtaining almost-sure convergent subsequences by using stopping times. His idea is as follows. Let $\phi = (\phi_t: t \geq 0)$ be a bounded left-continuous (and hence, predictable) stochastic process. Choose a sequence $(c_n)_{n \geq 0}$ of non-negative real numbers and for each $n \geq 0$, define stopping times T_k^n, where

$$T_0^n = 0, \qquad T_{k+1}^n = \inf\{t > T_k^n: |\phi_t - \phi_{T_k^n}| > c_n\}, \quad k \geq 0.$$

Now consider $\phi^n = (\phi_t^n: t \geq 0)$, defined by

$$\phi_t^n = \sum_k \phi_{T_k^n} 1_{(T_k^n, T_{k+1}^n]}(t), \quad t \geq 0,$$

and note that

$$\sup_{t \geq 0} |\phi_t^n - \phi_t| \leq c_n.$$

For $M \in \mathcal{M}_0^2$ (Bichteler allows more general integrators), define the stochastic integral $\int \phi^n \, \mathrm{d}M$ as usual, i.e. set

$$\left[\int_0^t \phi_s^n \, \mathrm{d}M_s\right](\cdot) = \sum_k \phi_{T_k^n(\cdot)}(\cdot)(M_{T_{k+1}^n(\cdot) \wedge t}(\cdot) - M_{T_k^n(\cdot) \wedge t}(\cdot)).$$

Then, Theorem 7.14 in Bichteler (1981) states:

If $\sum_{n \geq 0} c_n < \infty$ then for almost all $\omega \in \Omega$, $\int \phi \, \mathrm{d}M$ is, on every bounded interval, the *uniform* limit of $\int \phi^n \, \mathrm{d}M$.

Note that each ϕ gives rise to its *own* exceptional P-null set for which pathwise convergence does not hold. Although constructive, a practical application of Bichteler's method is hampered by the fact that it involves the use of stopping times. The strength of Bichteler's construction becomes apparent when applied to stochastic differential equations (Bichteler, 1981, Chapter 8).

Föllmer (1982) treats stochastic calculus, namely Itô's formula and stochastic integrals of the form $\int F'(X_{s-})\,\mathrm{d}X_s$ ($F \in C^2$), as an exercise in the analysis of a subset $\mathcal{U}$ of the class of real-valued functions with quadratic variation, and hence it can be understood in a non-probabilistic setting. In particular, he shows that if $x \in \mathcal{U}$ and $F \in C^2$, an Itô formula holds:

$$F(x_t) = F(x_0) + \int_0^t F'(x_{s-})\,\mathrm{d}x_s + \tfrac{1}{2}\int_0^t F''(x_{s-})\,\mathrm{d}[x,x]_s^{\mathrm{c}} + \sum_{s \le t} (F(x_s) - F(x_{s-}) - F'(x_{s-})(x_s - x_{s-})), \tag{1.2.3}$$

where $[x, x]^{\mathrm{c}}$ is the continuous part of the repartition function of

$$\lim_{n\to\infty} \sum_{t_i \in \tau_n} (x_{t_{i+1}} - x_{t_i})^2 \delta_{t_i}. \tag{1.2.4}$$

(Here, δ_{t_i} is the point mass at t_i and $(\tau_n)_{n\ge 0}$ is a sequence of subdivisions whose mesh tends to zero and such that the weak limit (1.2.4) defines a Radon measure on $[0, \infty)$.) The integral

$$\int_0^t F'(x_{s-})\,\mathrm{d}x_s$$

is then defined through the Itô formula (1.2.3), or equivalently, through a limit of (absolutely convergent) Riemann sums, namely

$$\int_0^t F'(x_{s-})\,\mathrm{d}x_s = \lim_{n\to\infty} \sum_{t_i \in \tau_n} F'(x_{t_i})(x_{t_{i+1}} - x_{t_i}).$$

Probability theory reappears by means of semimartingales. Föllmer shows that almost all sample paths of a semimartingale $X = (X_t: t \ge 0)$ belong to the class $\mathcal{U}$. (For example, if x is a typical sample path of Brownian motion then $[x, x]_t^{\mathrm{c}} = t$.) Therefore, Itô's formula holds sample path by sample path and the corresponding stochastic integral in (1.2.3) is obtained as almost-sure limit of discrete stochastic integrals.

Föllmer's approach considers only a subclass of stochastic integrals. Moreover, it cannot be easily used to numerically evaluate stochastic integrals because it requires the simulation of sample paths of X. Note that it typically also requires finding a suitable subsequence of $(\tau_n)_{n\ge 0}$ such that (1.2.4) holds almost surely when x is replaced by X ($[X, X]^{\mathrm{c}}$ exists as a limit in probability). Föllmer's result is, however, of great theoretical interest. It shows that Itô's calculus does not stop with semimartingales but applies for a much larger class of stochastic processes.

1.3. The martingale representation property

Despite their non-constructive nature, stochastic integrals found wide applications in- and outside the field of probability theory. One reason is that stochastic integration

theory has produced its own calculus whose centerpiece is Itô's formula, the stochastic version of the usual change of variable formula. Another reason is the following *martingale representation property* of (say) Brownian motion (usually attributed to Itô, 1951; different proofs of this property are given in Kunita and Watanabe, 1967; Dellacherie, 1975; Clark, 1970):

Let $W=(W_t: 0\leq t\leq 1)$ denote standard Brownian motion and set $\boldsymbol{F}=\boldsymbol{F}^W$, the minimal filtration satisfying the "usual conditions". If $Y\in L^2(\Omega, \mathcal{F}, P)$, then there exists $\phi\in\mathcal{L}^2(W)$ such that

$$Y=E_P[Y|\mathcal{F}_0]+\int_0^1 \phi_s \,\mathrm{d}W_s.$$

Equivalently, if X is a square-integrable $(\boldsymbol{F}, P)$-martingale then there exists $\phi\in\mathcal{L}^2(W)$ such that

$$X_t=E_P[X_1|\mathcal{F}_0]+\int_0^t \phi_s \,\mathrm{d}W_s, \quad 0\leq t\leq 1.$$

The martingale representation property motivated much of our pathwise approach to stochastic integrals. Stochastic processes representing all martingales of a filtration are fundamental to filtering theory and control theory (Liptser and Shiryayev, 1977; Kallianpur, 1980; Elliot, 1982), to the statistics of counting processes in survival analysis (Aalen, 1978; Gill, 1980; Brémaud, 1981), and to the theory of continuous trading in financial economics (Harrison and Kreps, 1979; Harrison and Pliska, 1981, 1983; Stricker, 1984). In the latter context, the martingale representation property is also called *completeness property* and has an intuitive interpretation in terms of complete security markets (see Harrison and Pliska, 1981; Taqqu and Willinger, 1987).

1.4. Outline of the paper

The central concept of our pathwise approach to stochastic integration is presented in Section 2. Here we illustrate the use of the completeness property in connection with stochastic integration in the case of discrete time and a discrete and finite filtration. Freed of technicalities, the discrete-time case allows us to motivate definitions and to explore the ideas behind our results. Section 3 contains a brief description of the approximation technique developed in Willinger and Taqqu (1988). This technique is then used to approximate certain continuous martingales and strictly preserve the completeness property along the approximating sequence. Combining the existence of such approximations with the results of Section 2, we develop in Section 4 the pathwise construction of stochastic integrals relative to continuous martingales. Finally, in Section 5, our pathwise approach is applied to the theory of security markets with continuous trading where equilibrium prices are given exogenously.

2. Stochastic integration and completeness in discrete time

In this section, we illustrate the connection between complete random processes and stochastic integration in discrete time with discrete and finite filtrations.

2.1. The probabilistic setting

Consider a given probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\boldsymbol{F} = (\mathcal{F}_t: t = 0, 1, \ldots, T < \infty)$. For ease of illustration, we assume that to each $\mathcal{F}_t$ there corresponds a minimal partition $\mathcal{P}_t$ of Ω such that $\mathcal{F}_t = \sigma(\mathcal{P}_t)$ $(t = 0, 1, \ldots, T)$, that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$, and that $P[A] > 0$ for each $A \in \mathcal{P}_T$.

Next let $X = (X_t: t = 0, 1, \ldots, T)$ denote an $\mathbb{R}^d$-valued martingale $(d \geq 1)$ with respect to $\boldsymbol{F}$ and P (an $(\boldsymbol{F}, P)$-martingale, for short) and without loss of generality we assume that $X_0 = 0$. Note that because of our assumptions on $\boldsymbol{F}$, X is a *simple* process, i.e. for each $t = 0, 1, \ldots, T$, X_t takes finitely many different values in $\mathbb{R}^d$. X gives rise to the so-called *minimal* filtration $\boldsymbol{F}^X = (\mathcal{F}_t^X: t = 0, 1, \ldots, T)$, where $\mathcal{F}_t^X$ denotes the σ algebra generated by $X_0, X_1, \ldots, X_t$.

Finally, let $\psi = (\psi_t: t = 1, 2, \ldots, T)$ denote an $\mathbb{R}^d$-valued stochastic process on $(\Omega, \mathcal{F}, P)$ with component processes $\psi^0, \psi^1, \ldots, \psi^d$. ψ is said to be ***F*-predictable** if $\psi_t \in \mathcal{F}_{t-1}$ $(t = 1, 2, \ldots, T)$. For a predictable process ψ, the *discrete stochastic integral with respect to* X, $\psi \circ X = ((\psi \circ X)_t: t = 0, 1, \ldots, T)$ (also called the *martingale-transform of* X *by* ψ) is defined by

$$(\psi \circ X)_t(\cdot) = \psi_1 \cdot X_0 + \sum_{s=1}^{t} \psi_s(\cdot) \cdot (X_s(\cdot) - X_{s-1}(\cdot)) \quad P\text{-a.s.} \tag{2.1.1}$$

(with $(\psi \circ X)_0 \equiv \psi_1 \cdot X_0 = 0$). It is easy to see that for each predictable ψ, $\psi \circ X$ is a real-valued $(\boldsymbol{F}, P)$-martingale.

2.2. Discrete stochastic integrals and completeness

By identifying $\psi \circ X$ with its terminal value $Y \equiv (\psi \circ X)_T$, we can view the operation of forming $\psi \circ X$ as a mapping from the set of all $\boldsymbol{F}$-predictable processes into $L^1(\Omega, \mathcal{F}, P)$. For example, for $d = 1$, the image of $\psi = (\psi_t: t = 1, 2, \ldots, T)$ with $\psi_t \equiv 1$ $(t = 1, 2, \ldots, T)$ is the random variable X_T.

In general, stochastic integrals with respect to X do not span $L^1(\Omega, \mathcal{F}, P)$. However, if the pair $(\boldsymbol{F}, X)$ is such that

$$\{c + (\psi \circ X)_T: \psi \text{ predictable}, c \in \mathbb{R}\} = L^1(\Omega, \mathcal{F}, P), \tag{2.2.1}$$

then any $Y \in L^1(\Omega, \mathcal{F}, P)$ can be written as $E_P[Y]$ plus a discrete stochastic integral $\psi \circ X$ for some predictable integrand ψ. ψ is obtained through an explicit pathwise construction (see below) and satisfies for P-almost all $\omega \in \Omega$,

$$Y(\omega) = E_P[Y] + \sum_{s=1}^{T} \psi_s(\omega) \cdot (X_s(\omega) - X_{s-1}(\omega)). \tag{2.2.2}$$

In this case, ψ is said to *generate* or *represent* Y, or equivalently, the real-valued martingale $(E_P[Y \mid \mathcal{F}_t]: t = 0, 1, \ldots, T)$.

In order to characterize the pairs (F, X) for which (2.2.1) holds, it is convenient to add to X the constant component process $X^0 = (X_t^0: t = 0, 1, \ldots, T)$ with $X_t^0 = 1, (t = 0, 1, \ldots, T)$. Clearly, the resulting $\mathbb{R}^{d+1}$-valued stochastic process $Z = (1, X)^T = ((1, X_t)^T: t = 0, 1, \ldots, T)$ is also an (F, P)-martingale and $F^Z = F^X$. Subsequently, Z will always denote an $\mathbb{R}^{d+1}$-valued (F, P)-martingale with component processes $Z^0 \equiv 1, Z^1 = X^1, \ldots, Z^d = X^d$. Define the class Φ of all $\mathbb{R}^{d+1}$-valued stochastic processes $\phi = (\phi_t: t = 1, 2, \ldots, T)$ satisfying

$$\phi \text{ is } F\text{-predictable}, \tag{2.2.3}$$

and

$$\phi_t \cdot Z_t = \phi_{t+1} \cdot Z_t \quad P\text{-a.s.} \quad (t = 1, 2, \ldots, T-1). \tag{2.2.4}$$

Property (2.2.4) implies that $\phi \circ Z$ (as defined in (2.1.1)) agrees with $(\phi_t \cdot Z_t: t = 0, 1, \ldots, T)$ (where we set $\phi_0 \equiv \phi_1$) because for $t = 1, 2, \ldots, T$,

$$\begin{aligned}(\phi \circ Z)_t &= \phi_1 \cdot Z_0 + \sum_{s=1}^{t} \phi_s \cdot (Z_s - Z_{s-1}) \quad \text{(by (2.1.1))} \\ &= \phi_1 \cdot Z_0 + \sum_{s=1}^{t} \phi_s \cdot Z_s - \sum_{s=1}^{t} \phi_s \cdot Z_{s-1}.\end{aligned}$$

Then by (2.2.4),

$$(\phi \circ Z)_t = \phi_t \cdot Z_t \quad P\text{-a.s.} \quad (t = 0, 1, \ldots, T). \tag{2.2.5}$$

Further, the following result shows that the class of discrete stochastic integrals of $\phi \in \Phi$ with respect to Z agrees with the set of all martingale-transforms of X.

Lemma 2.2.1. $\{\phi \circ Z: \phi \in \Phi\} = \{c + \psi \circ X: \psi \text{ is } F\text{-predictable}, c \in \mathbb{R}\}$.

Proof. (1) ("$\supseteq$") Given $c \in \mathbb{R}$ and $\psi = (\psi_t: t = 1, 2, \ldots, T)$, ψ $\mathbb{R}^d$-valued and F-predictable, set

$$\phi_t^0 = c + (\psi \circ X)_{t-1} - \psi_t \cdot X_{t-1}, \tag{2.2.6}$$

and

$$\phi_t^k = \psi_t^k \quad (k = 1, 2, \ldots, d), \tag{2.2.7}$$

for $t = 1, 2, \ldots, T$. Clearly, $\phi = (\phi_t: t = 1, 2, \ldots, T)$ is $\mathbb{R}^{d+1}$-valued and F-predictable; in fact, ϕ satisfies (2.2.4) since for $t = 1, 2, \ldots, T-1$,

$$\begin{aligned}\phi_{t+1} \cdot Z_t &= \phi_{t+1}^0 Z_t^0 + \sum_{k=1}^{d} \phi_{t+1}^k Z_t^k \\ &= (c + (\psi \circ X)_t - \psi_{t+1} \cdot X_t)1 + \sum_{k=1}^{d} \psi_{t+1}^k X_t^k \quad \text{(by (2.2.6) and (2.2.7))} \\ &= c + (\psi \circ X)_t \\ &= (c + (\psi \circ X)_{t-1} - \psi_t \cdot X_{t-1})1 + \sum_{k=1}^{d} \psi_t^k X_t^k \quad \text{(by (2.1.1))} \\ &= \phi_t \cdot Z_t \quad P\text{-a.s.} \quad \text{(by (2.2.6) and (2.2.7))}.\end{aligned}$$

Moreover, the reduction property (2.2.5) yields

$$(\phi \circ Z)_t = \phi_t \cdot Z_t$$

$$= (c + (\psi \circ X)_{t-1} - \psi_t \cdot X_{t-1})1 + \sum_{k=1}^{d} \psi_t^k X_t^k \quad \text{(by (2.2.6) and (2.2.7))}$$

$$= c + (\psi \circ X)_t \quad P\text{-a.s.} \quad (t = 0, 1, \ldots, T),$$

and thus, $\{c + \psi \circ X: \psi \text{ is } \boldsymbol{F}\text{-predictable}, c \in \mathbb{R}\} \subseteq \{\phi \circ Z: \phi \in \boldsymbol{\Phi}\}$.

(2) ("$\subseteq$") Given $\phi \in \boldsymbol{\Phi}$, set

$$\psi_t^k = \phi_t^k \quad (k = 1, 2, \ldots, d) \quad \text{for } t = 1, 2, \ldots, T.$$

and $c = \phi_1^0$. Clearly, $\psi = (\psi_t: t = 1, 2, \ldots, T)$ is $\mathbb{R}^d$-valued and $\boldsymbol{F}$-predictable and satisfies, for all $t = 0, 1, \ldots, T$,

$$c + (\psi \circ X)_t = \phi_1^0 + \psi_1 \cdot X_0 + \sum_{s=1}^{t} \psi_s \cdot (X_s - X_{s-1}) \quad (X_0 = 0)$$

$$= \phi_1 \cdot Z_0 + \sum_{s=1}^{t} \phi_s \cdot (Z_s - Z_{s-1}) \quad (Z_t^0 = 1 \text{ for all } t)$$

$$= (\phi \circ Z)_t \quad P\text{-a.s.} \qquad \square$$

Because of Lemma 2.2.1, we study the representation problem (2.2.1) in terms of Z and $\boldsymbol{\Phi}$. When we discuss the construction of integrands generating a given $Y \in L^1(\Omega, \mathcal{F}, P)$ in Section 2.3, the advantages of the $(Z, \boldsymbol{\Phi})$-setup over the one involving X are apparent. Below, we formally define the completeness of Z.

Definition 2.2.1. The $(\boldsymbol{F}, P)$-martingale Z is said to be *complete* (for $(\Omega, \boldsymbol{F}, P)$) if for each $Y \in L^1(\Omega, \mathcal{F}, P)$ there exists $\phi \in \boldsymbol{\Phi}$ such that for P-almost all $\omega \in \Omega$,

$$Y(\omega) = (\phi \circ Z)_T(\omega),$$

or equivalently (using (2.2.5)),

$$Y(\omega) = \phi_T(\omega) \cdot Z_T(\omega).$$

A complete process Z allows us to represent anything probabilistically interesting as a discrete stochastic integral. In particular, every real-valued $(\boldsymbol{F}, P)$-martingale $M = (M_t: t = 0, 1, \ldots, T)$ can be written in the form $\phi \circ Z$ for some $\phi \in \boldsymbol{\Phi}$; in fact, setting $Y = M_T$ gives

$$M_T = (\phi \circ Z)_T \quad P\text{-a.s.} \quad \text{for some } \phi \in \boldsymbol{\Phi},$$

which implies, by taking conditional expectations with respect to $\mathcal{F}_t$,

$$M_t = (\phi \circ Z)_t \quad P\text{-a.s.} \quad (t = 0, 1, \ldots, T).$$

Because of the above, the "Strasbourg" terminology (Dellacherie and Meyer, 1982) for completeness of Z is *Z has the martingale representation property.*

We now characterize complete processes.

Definition 2.2.2. A probability measure Q on $(\Omega, \mathcal{F})$ is called an *equivalent martingale measure* (for $(\boldsymbol{F}, Z)$) if

(i) $Q \sim P$, i.e. $Q[A]=0$ iff $P[A]=0$ $(A \in \mathcal{F})$;

(ii) Z is an $(\boldsymbol{F}, Q)$-martingale.

Let $\boldsymbol{P}=\{Q\colon Q \sim P, Z \text{ an } (\boldsymbol{F}, Q)\text{-martingale}\}$ denote the set of all equivalent martingale measures for $(\boldsymbol{F}, Z)$ and note that $\boldsymbol{P} \neq \emptyset$ since $P \in \boldsymbol{P}$. The results below follow immediately from Theorem 4.2. and Corollaries 4.1. and 4.2 of Taqqu and Willinger (1987).

Theorem 2.2.1. *The following are equivalent.*

(1) *Z is complete.*

(2) *Z satisfies condition* (C_z').

(C_z') *for each* $t=1, 2, \ldots, T$, *and for each* $A \in \mathcal{P}_{t-1}$,

$$\dim(\operatorname{span}(\{Z_t(\omega)-Z_{t-1}(\omega)\colon \omega \in A\})) = \text{cardinality}(A' \in \mathcal{P}_t\colon A' \subseteq A) - 1.$$

(3) *$\boldsymbol{P}$ is a singleton, i.e.* $\boldsymbol{P}=\{P\}$. □

Corollary 2.2.1. *If Z is complete then $\boldsymbol{F}$ is necessarily minimal, i.e.* $\boldsymbol{F}=\boldsymbol{F}^Z$. □

Observe that condition (C_z') imposes extremely stringent constraints on the dynamic nature of Z, i.e. on the flow of information and the possible changes of the values of Z from $t-1$ to t. Corollary 2.2.1 states that the minimal filtration is necessary for completeness, and (C_z') explicitly indicates the central role of the fine structure of the filtration in this context.

We conclude this subsection with examples of a complete and an incomplete process. See Section 2.4 for another complete process.

Example 2.2.1. Let $S=(S_t\colon t=0, 1, \ldots, T<\infty)$ be a one-dimensional random walk; that is, let $X_1, X_2, \ldots$, be i.i.d. random variables with $P[X_1=+1]=P[X_1=-1]=\frac{1}{2}$, and $S_0=0$, $S_t=\sum_{i=1}^{t} X_i$ $(t=1, 2, \ldots)$. Set $Z=(1, S)^{\mathrm{T}}$ and $\boldsymbol{F}=\boldsymbol{F}^S$. It is easy to check that condition (C_z') is satisfied and, therefore, Z is complete.

Example 2.2.2. Let $S=(S_t\colon t=0, 1, \ldots, T<\infty)$ denote a d-dimensional symmetric random walk $(d \geq 2)$, and set $Z=(1, S)^{\mathrm{T}}$ and $\boldsymbol{F}=\boldsymbol{F}^S$. Then Z is not complete, since for each $A \in \mathcal{P}_t$ $(t=0, 1, \ldots)$,

$$\dim(\operatorname{span}(\{Z_{t+1}(\omega)-Z_t(\omega)\colon \omega \in A\})) = d$$

and

$$\text{cardinality}(A' \in \mathcal{P}_{t+1}\colon A' \subseteq A) = 2^d,$$

i.e. (C_z') fails for each $d \geq 2$.

2.3. A duality structure

The equivalence between statements (1), (2), and (3) of Theorem 2.2.1 is the result of a duality structure between completeness of Z and uniqueness of an equivalent

martingale measure for Z. In fact, there is a genuine linear programming duality between these two problems and we explicitly provide the corresponding primal-dual pair of LP's. This LP formulation is used to construct the integrands $\phi \in \mathbf{\Phi}$ that generate a given $Y \in L^1(\Omega, \mathcal{F}, P)$ (see Section 2.4).

Consider, for simplicity, a finite probability space $(\Omega, \boldsymbol{F}, P)$ with $P[\{\omega_i\}] > 0$ for all $i = 1, 2, \ldots, N = |\Omega|$ and take $\mathcal{F} = 2^{\Omega}$. For ease of exposition, we consider only the simple case of a single-period stochastic process $Z = (Z_0, Z_1)$, and let B denote the $(d+1) \times N$ matrix with columns formed by the vectors $Z_1(\omega_1), Z_1(\omega_2), \ldots, Z_1(\omega_N)$. Let Y be an arbitrary random variable and set $\nu = (Y(\omega_1), Y(\omega_2), \ldots, Y(\omega_N))$. Then define the following primal-dual pair of linear programs:

(P)	min	$\nu \cdot x$	(D)	max	$y \cdot Z_0$
	such that	$B \cdot x = Z_0$,		such that	$y \cdot B = \nu$,
		$x > 0$,			y unrestricted.

Observe that the (P)-feasible region consists of all probability measures Q on $(\Omega, \mathcal{F})$ with $Q \sim P$ and such that $E_Q[Z_1 | \mathcal{F}_0] = Z_0$ (set $x = (Q[\{\omega_1\}], Q[\{\omega_2\}], \ldots, Q[\{\omega_N\}])^{\mathrm{T}}$). Also note that the (P)-feasible region is non-empty because Z is an $(\boldsymbol{F}, P)$-martingale. Secondly, by identifying each element y in the (D)-feasible region with an element $\phi = (\phi_1) \in \mathbf{\Phi}$ (set $y = \phi_1 \in \mathcal{F}_0$), we recognize (D) as the linear programming formulation of the completeness problem for Z (and for the particular ν at hand). Observe that since $T = 1$, property (2.2.4) is not required for ϕ; this property is only of interest in the multi-period case (see below). The result of Theorem 2.2.1 can then be understood in terms of this linear programming formulation.

Focus next on the objective functions of (P) and (D). When Z is complete, equivalently, $\boldsymbol{P} = \{P\}$, the primal objective function directly yields $E_P[Y]$ with P the unique equivalent martingale measure for Z. The dual objective function provides the exact same value (by the strong duality theorem of linear programming), although (D) is formulated without probabilities ((D) is a typical pathwise statement). Subsequently, we shall call this connection between completeness of Z and uniqueness of an equivalent martingale measure for Z *stochastic duality*.

Stochastic duality for multi-period processes ($T > 1$) generally involves different primal-dual pairs of LP's $\mathrm{P}(t, A)$ and $\mathrm{D}(t, A)$ for each $t = 0, 1, \ldots, T-1$ and for each $A \in \mathcal{P}_t$ (see Section 2.4 for an explicit formulation.) Corollary 2.3.1 below explicitly relates the LP-formulation to condition (C'_z). It follows directly from the finite sample space results of Taqqu and Willinger (1987, Theorem 5.1 and the proof of Theorem 4.1). This result is of practical importance since it makes LP techniques available for the pathwise construction of stochastic integrals.

Corollary 2.3.1. *The following are equivalent.*

(1) *For each* $t = 0, 1, \ldots, T-1$, *and each* $A \in \mathcal{P}_t$, $\mathrm{D}(t, A)$ *has an optimal feasible solution* (*for any given right-hand side*).

(2) *For each* $t=0,1,\ldots,T-1$, *and each* $A\in\mathscr{P}_t$, $\mathrm{P}(t,A)$ *has an optimal feasible solution.*

(3) Z *satisfies* (C'_z). □

2.4. A simple example

We illustrate here how the LP formulation is used to obtain $\phi\in\mathbf{\Phi}$ that generates a given $Y\in L^1(\Omega,\mathscr{F},P)$. Consider the following numerical example of a two-step martingale $Z=(Z_0,Z_1,Z_2)$ with $d=2$ portrayed in Figure 1. Assume that Ω is arbitrary and that the filtration $\boldsymbol{F}$ is given by the tree diagram of the process, namely

$$\mathscr{P}_0=\{\Omega\},$$

$$\mathscr{P}_1=\{\mathrm{B}_1,\mathrm{B}_2,\mathrm{B}_3\},$$

$$\mathscr{P}_2=\{\mathrm{B}_{11},\mathrm{B}_{12},\mathrm{B}_{13},\mathrm{B}_{21},\mathrm{B}_{22},\mathrm{B}_{31},\mathrm{B}_{32},\mathrm{B}_{33}\}.$$

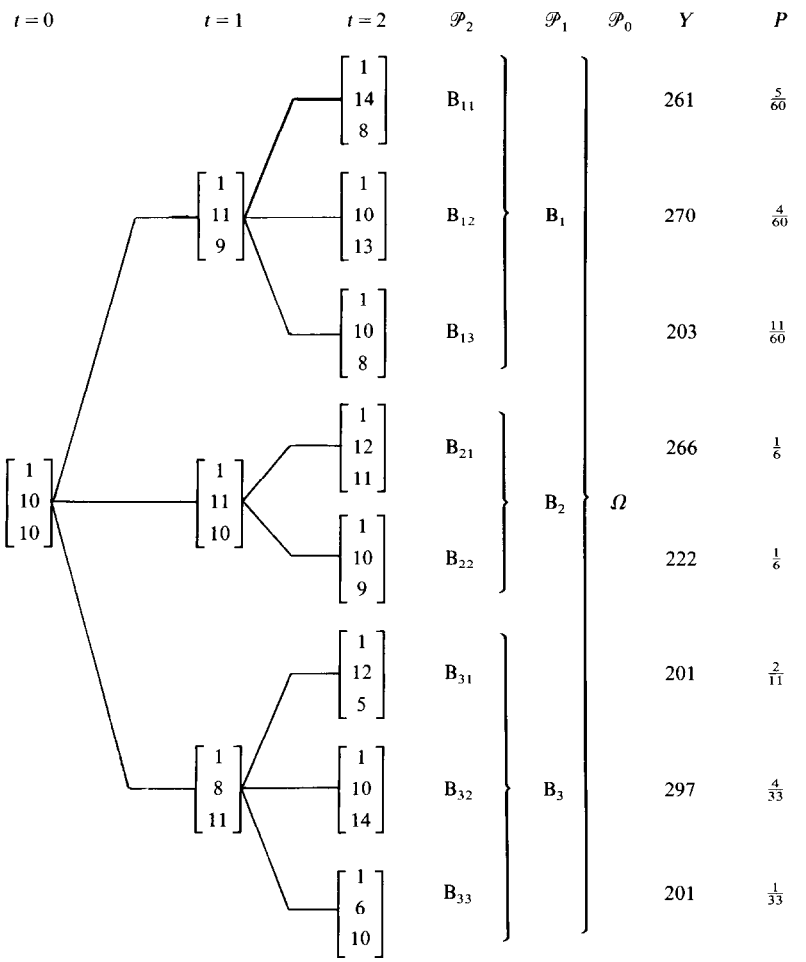


Fig. 1. A discrete-time example ($d=2$, $T=2$).

The values of Z at times $t=0,1,2$ are given in Figure 1 as nodes of the tree, e.g. $Z_2(\omega)=(1,12,11)^{\mathrm{T}}$ if $\omega\in \mathrm{B}_{21}$. The measure P is recorded in the last column in Figure 1 and it is easy to see that Z is an $(\boldsymbol{F}, P)$-martingale. In fact, P is the unique equivalent martingale measure for $(\boldsymbol{F}, Z)$ (direct verification or check (C_z')) and, therefore, Z is complete.

The purpose of this example is to illustrate the pathwise construction of discrete stochastic integrals relative to Z for a given $Y\in L^1(\Omega, \mathcal{F}, P)$. Suppose $Y(\cdot)=\sup_{t=0,1,2}|Z_t(\cdot)|^2$. In order to construct $\phi=(\phi_1,\phi_2)\in\boldsymbol{\Phi}$ that generates Y, we proceed pathwise and apply Corollary 2.3.1 "backwards". That is, we set $t=1$ and solve for ϕ_2 and then set $t=0$ to obtain ϕ_1. More precisely, for $t=1$, we consider the following three linear programs.

$$
\begin{aligned}
A=\mathrm{B}_1\colon\quad &\max \quad y_{11}+11y_{12}+9y_{13}\\
&\text{s.t.}\quad y_{11}+14y_{12}+8y_{13}=261,\\
&\qquad\ y_{11}+10y_{12}+13y_{13}=270,\\
&\qquad\ y_{11}+10y_{12}+8y_{13}=203,\\
&\qquad\ y_{11}, y_{12}, y_{13}\ \text{unrestricted.}
\end{aligned}
\tag{2.4.1}
$$

$$
\begin{aligned}
A=\mathrm{B}_2\colon\quad &\max \quad y_{21}+11y_{22}+10y_{23}\\
&\text{s.t.}\quad y_{21}+12y_{22}+11y_{23}=266,\\
&\qquad\ y_{21}+10y_{22}+9y_{23}=222,\\
&\qquad\ y_{21}, y_{22}, y_{23}\ \text{unrestricted.}
\end{aligned}
\tag{2.4.2}
$$

$$
\begin{aligned}
A=\mathrm{B}_3\colon\quad &\max \quad y_{31}+8y_{32}+11y_{33}\\
&\text{s.t.}\quad y_{31}+12y_{32}+5y_{33}=201,\\
&\qquad\ y_{31}+10y_{32}+14y_{33}=297,\\
&\qquad\ y_{31}+6y_{32}+10y_{33}=201,\\
&\qquad\ y_{31}, y_{32}, y_{33}\ \text{unrestricted.}
\end{aligned}
\tag{2.4.3}
$$

Solutions to (2.4.1), (2.4.2), and (2.4.3) are given by $y_1=(-\frac{246}{5}, \frac{29}{2}, \frac{67}{5})$, $y_2=(2,22,0)$, and $y_3=(\frac{51}{11}, \frac{120}{11}, \frac{144}{11})$, respectively, and define ϕ_2 by:

$$\phi_2(\omega)=y_i \quad \text{if } \omega\in B_i \qquad (i=1,2,3).$$

Since ϕ must satisfy (2.2.4), i.e.

$$\phi_1\cdot Z_1=\phi_2\cdot Z_1 \quad P\text{-a.s.}, \tag{2.4.4}$$

and the right-hand side of (2.4.4) is known, we obtain the following linear program

for $t=0$ and $A=\Omega$:

$$\begin{aligned} \max \quad & y_1+10y_2+10y_3 \\ \text{s.t.} \quad & y_1+11y_2+9y_3=\tfrac{2309}{10}, \\ & y_1+11y_2+10y_3=244, \\ & y_1+8y_2+11y_3=\tfrac{2595}{11}, \\ & y_1, y_2, y_3 \text{ unrestricted.} \end{aligned} \tag{2.4.5}$$

Then $y_0=(\frac{353}{10}, \frac{777}{110}, \frac{131}{10})$ solves (2.4.5) and we set $\phi_1(\omega)=y_0$ for all $\omega\in\Omega$ completing the construction of ϕ.

3. From discrete to continuous time: the skeleton approach

This section contains a brief description of the "skeleton approach" which approximates a continuous martingale by "skeleton martingales", i.e. simple martingales of the type considered in the previous section. Besides exhibiting strong convergence properties, the skeleton technique is also capable of preserving structural properties such as completeness along the entire approximating sequence and, therefore, permits extension of the pathwise construction of stochastic integrals (see Section 2.2) to the case of continuous martingales. For a more detailed discussion of the skeleton approach, we refer the interested reader to Willinger (1987) and Willinger and Taqqu (1988).

3.1. Probabilistic assumptions

Fix a finite time horizon T and consider a stochastic base $(\Omega, \mathcal{F}, P, \boldsymbol{F})$ with a complete probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\boldsymbol{F}=(\mathcal{F}_t: 0\leq t\leq T)$ satisfying the *usual conditions*, i.e., $\mathcal{F}_0$ contains all P-null sets and $\mathcal{F}_t=\mathcal{F}_{t+}\equiv\bigcap_{s>t}\mathcal{F}_s$ $(0\leq t\leq T)$. We also require $\boldsymbol{F}$ to be *continuous*, that is, to satisfy:

> For every $B\in\mathcal{F}$, the $(\boldsymbol{F}, P)$-martingale $(P[B\mid\mathcal{F}_t]: 0\leq t\leq T)$ has a continuous modification. (3.1.1)

Definition (3.1.1) of a continuous information structure is originally due to Harrison (1982) and formalizes the idea that there are no events "that can take us by surprise" (for more details, see Huang, 1985).

Next let $Z=(Z_t: 0\leq t\leq T)$ denote an $\mathbb{R}^{d+1}$-valued $(\boldsymbol{F}, P)$-martingale with continuous sample paths. We assume that $Z^0=(Z_t^0: 0\leq t\leq T)$, the 0th component process of Z, is deterministic and constant with $Z_t^0=1$ $(0\leq t\leq T)$ and take $Z_0^k=0$ P-a.s. $(k=1,\ldots,d)$. Z gives rise to the *minimal* filtration $\boldsymbol{F}^Z$; that is, the smallest filtration satisfying the usual conditions and with respect to which Z is adapted. In the sequel, we restrict ourselves to minimal filtrations, i.e. we assume

$$\boldsymbol{F}=\boldsymbol{F}^Z. \tag{3.1.2}$$

Note that $\mathscr{F}_0$ is almost trivial (i.e. $\mathscr{F}_0$ contains only sets of P-measure zero or one) and for convenience, we also require $\mathscr{F}_T = \mathscr{F}$. Requiring (3.1.2) does limit the generality of our set-up; however, Willinger and Taqqu (1988) show that the minimal filtration is necessary for the skeleton approach. Also, in many practical examples, assumption (3.1.2) holds in the first place. Examples of filtrations $\boldsymbol{F}$ satisfying (3.1.1) and (3.1.2) include, among others, Brownian filtrations in one and higher dimensions (see Huang, 1985) and, more generally, filtrations generated by continuous processes having the strong Markov property (see Meyer, 1963).

3.2. Completeness as a structure-preserving property

Recalling Example 2.2.1, we note that, under proper rescaling, a one-dimensional symmetric random walk converges weakly to a one-dimensional standard Brownian motion. In this case, both the approximating processes and the limiting process are complete. In contrast, a d-dimensional symmetric random walk ($2 \leq d < \infty$) is not complete (see Example 2.2.2) although its weak limit, d-dimensional standard Brownian motion, is continuous and complete (Jacod, 1979). Thus weak convergence is not appropriate for preserving structural properties such as completeness.

The skeleton approach developed in Willinger and Taqqu (1988) is capable of preserving such structural properties. This method deals with pathwise approximations of continuous sample paths processes and is based on a natural convergence concept for filtrations. The latter enables us to deal with the fine structure of the filtration, the central role of which can be seen in condition (C_z') of Theorem 2.2.1.

The skeleton approach relies on the notions of skeletons and skeleton approximations both of which are given below.

Definition 3.2.1. A *continuous-time skeleton of* $(\boldsymbol{F}, Z)$ is a triplet $(I^\zeta, \boldsymbol{F}^\zeta, \zeta)$ consisting of a deterministic index set I^ζ, a filtration $\boldsymbol{F}^\zeta = (\mathscr{F}_t^\zeta : 0 \leq t \leq T)$ (the *skeleton filtration*), and an $\mathbb{R}^{d+1}$-valued stochastic process $\zeta = (\zeta_t : 0 \leq t \leq T)$ (the *skeleton process*) with the following properties.

(a) $I^\zeta = \{t(\zeta, 0), t(\zeta, 1), \ldots, t(\zeta, N^\zeta)\}$, $0 = t(\zeta, 0) < \cdots < t(\zeta, N^\zeta) = T$, $N^\zeta < \infty$.

(b) For each $t \in I^\zeta$, $\mathscr{F}_t^\zeta$ is a finitely generated sub-σ-algebra of $\mathscr{F}_t$, i.e. there exists a minimal partition $\mathscr{P}_t$ of Ω with $\mathscr{F}_t^\zeta = \sigma(\mathscr{P}_t)$.

(c) For $t \notin I^\zeta$, $0 \leq t \leq T$, set $\mathscr{F}_t^\zeta = \mathscr{F}_{t(\zeta,k)}^\zeta$ if $t \in [t(\zeta, k), t(\zeta, k+1))$ for some $0 \leq k < N^\zeta$.

(d) For each $t \in I^\zeta$, let $\zeta_t \in \mathscr{F}_t^\zeta$.

(e) For $t \notin I^\zeta$, $0 \leq t \leq T$, let $\zeta_t = \zeta_{t(\zeta,k)}$ if $t \in [t(\zeta, k), t(\zeta, k+1))$ for some $0 \leq k < N^\zeta$.

Definition 3.2.2. A *sequence* $(I^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})_{n \geq 0}$ of continuous-time skeletons of $(\boldsymbol{F}, Z)$ is called a *continuous-time skeleton approximation of* $(\boldsymbol{F}, Z)$ if the following properties hold.

(a) *Dense subset property.* The sequence $(I^{(n)})_{n\geq 0}$ of finite index sets satisfies:
(i) $|I^{(n)}| \equiv \max_{1\leq k\leq N^{\zeta^{(n)}}} |t(\zeta^{(n)}, k) - t(\zeta^{(n)}, k-1)| \to 0$ as $n \to \infty$.
(ii) $I = \bigcup_{n\geq 0} I^{(n)}$ is a dense subset of $[0, T]$.

(b) *Convergence of information.* $\boldsymbol{F}^{(n)} \uparrow \boldsymbol{F}$ as $n \to \infty$, that is, for each $0 \leq t \leq T$,

$$\mathcal{F}_t = \sigma\left(\bigcup_{k\geq 0} \mathcal{F}_t^{(k)}\right) \supseteq \cdots \supseteq \mathcal{F}_t^{(n)} \supseteq \mathcal{F}_t^{(n-1)} \supseteq \cdots \supseteq \mathcal{F}_t^{(0)}$$

(up to P-null sets), $n \geq 1$.

(c) *Pathwise approximation.* $\zeta^{(n)} \to Z$ as $n \to \infty$ (uniformly in t) P-a.s.; that is,

$$P\left[\left\{\omega \in \Omega: \lim_{n\to\infty} \sup_{0\leq t\leq T} |Z_t(\omega) - \zeta_t^{(n)}(\omega)| = 0\right\}\right] = 1.$$

Note that because of (c) and (e) of Definition 3.2.1, skeletons can be considered as stochastic processes in continuous time or discrete time; in the sequel, we take whichever setting is most convenient.

The existence of skeletons and skeleton-approximations for the given pair $(\boldsymbol{F}, Z)$ is established in Willinger and Taqqu (1988), and several explicit constructions are provided. Based on the completeness characterization for skeletons (Theorem 2.2.1) and the so called Special Construction applied to $(\boldsymbol{F}, Z)$ (see Sections 4.4 and 5.3 of Willinger and Taqqu, 1988) we obtain the following result concerning the existence of skeleton-approximations which preserve completeness.

Theorem 3.2.1. *There exists a continuous-time skeleton approximation* $(I^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})_{n\geq 0}$ *of the* $(\boldsymbol{F}, P)$*-martingale* Z *such that for each* $n \geq 0$:
(1) $(I^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})$ *is a complete* $(\boldsymbol{F}^{(n)}, P)$*-martingale*;
(2) $(I^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})$ *is an* $(\boldsymbol{F}^{(n)}, P)$*-martingale satisfying condition* $(C'_{\zeta^{(n)}})$;
(3) $(I^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})$ *admits a unique equivalent martingale measure* $P^{(n)}$ *where* $P^{(n)} = P|_{\mathcal{F}_{T^{(n)}}^{(n)}}$ *is the restriction of* P *to* $\mathcal{F}_{T^{(n)}}^{(n)}$ *and* $T^{(n)}$ *denotes the last element in* $I^{(n)}$.
Moreover, conditions (1), (2), *and* (3) *are equivalent.* □

4. Pathwise stochastic integration

We now combine the results of Section 2 with the structure preserving feature of the skeleton technique (Theorem 3.2.1) to obtain a pathwise construction of stochastic integrals relative to continuous martingales. Throughout this section, we work under the probabilistic assumptions stated in Section 3.1.

4.1. An explicit construction

We first introduce the notion of predictable skeleton and then give the pathwise construction of stochastic integrals with respect to Z.

Definition 4.1.1. The triplet $(I^\nu, \boldsymbol{F}^\nu, \nu)$ is called an $\boldsymbol{F}^\nu$-*predictable* (*continuous-time*) *skeleton* if the following properties hold.

(a) $I^\nu = \{t(\nu,1), t(\nu,2), \ldots, t(\nu, N^\nu)\}$, $0 < t(\nu,1) < \cdots < t(\nu, N^\nu) \leq T$, $N^\nu < \infty$.

(b) $\boldsymbol{F}^\nu = (\mathcal{F}_t^\nu : 0 \leq t \leq T)$ is a skeleton filtration (see (b) and (c) of Definition 3.2.1).

(c) $\nu = (\nu_t : 0 \leq t \leq T)$ is an $\mathbb{R}^{d+1}$-valued stochastic process such that for each $0 < k \leq N^\nu$, $\nu_{t(\nu,k)} \in \mathcal{F}^\nu_{t(\nu,k-1)}$ $(\mathcal{F}^\nu_{t(\nu,0)} = \{\emptyset, \Omega\})$, and for $t \notin I^\nu$, $\nu_t = \nu_{t(\nu,k)}$ if $t \in (t(\nu,k-1), t(\nu,k)]$ for some $0 < k \leq N^\nu$ $(t(\nu,0) = 0)$.

Next recall that because of Theorem 3.2.1(1) there exists a continuous-time skeleton approximation $(I^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})_{n \geq 0}$ of $(\boldsymbol{F}, Z)$ such that for each $n \geq 0$, $(I^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})$ is a complete $(\boldsymbol{F}^{(n)}, P)$-martingale for which the results of Section 2 apply. (Note that as a result of the special construction, $T^{(n)}$, denoting the last element of $I^{(n)}$, is always slightly beyond T and converges to T as $n \to \infty$.) Thus, for each $Y^{(n)} \in \mathcal{F}^{(n)}_{T^{(n)}}$ there exists an $\boldsymbol{F}^{(n)}$-predictable skeleton $(I^{(n)}, \boldsymbol{F}^{(n)}, \phi^{(n)})$ with

$$\phi^{(n)}_{t(n,k)} \cdot \zeta^{(n)}_{t(n,k)} = \phi^{(n)}_{t(n,k+1)} \cdot \zeta^{(n)}_{t(n,k)} \quad P\text{-a.s.} \quad (t(n,k) \in I^{(n)}), \tag{4.1.1}$$

and such that

$$Y^{(n)} = (\phi^{(n)} \circ \zeta^{(n)})_{T^{(n)}} \quad P\text{-a.s.}, \tag{4.1.2}$$

or, equivalently (use 2.2.5)),

$$Y^{(n)} = \phi^{(n)}_{T^{(n)}} \cdot \zeta^{(n)}_{T^{(n)}} \quad P\text{-a.s.} \tag{4.1.3}$$

Here, $\phi^{(n)} \circ \zeta^{(n)}$ is viewed as continuous-time process and is defined for all $0 \leq t \leq T^{(n)}$ by

$$(\phi^{(n)} \circ \zeta^{(n)})_t = \phi^{(n)}_{t(n,1)} \cdot \zeta^{(n)}_0 + \sum_{k=1}^{N^{(n)}} \phi^{(n)}_{t(n,k)} \cdot (\zeta^{(n)}_{t(n,k) \wedge t} - \zeta^{(n)}_{t(n,k-1) \wedge t}) \quad P\text{-a.s.}$$

Applying the completeness property at each stage of the skeleton approximation of $(\boldsymbol{F}, Z)$ results in the following formal procedure for a pathwise construction of stochastic integrals with respect to Z. The feasibility of the construction is established in Section 4.2.

Pathwise construction of stochastic integrals.

Step 0. Choose $Y \in L^1(\Omega, \mathcal{F}, P)$ and consider the real-valued $(\boldsymbol{F}, P)$-martingale $(Y_t : 0 \leq t \leq T)$ with $Y_t = E_P[Y \mid \mathcal{F}_t]$ P-a.s.

Step 1. Choose a continuous-time skeleton approximation $(I^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})_{n \geq 0}$ of $(\boldsymbol{F}, Z)$ such that the sequence $(I^{(n)}, \boldsymbol{F}^{(n)}, Y^{(n)})_{n \geq 0}$ defines a continuous-time skeleton approximation of $(\boldsymbol{F}, (Y_t : 0 \leq t \leq T))$ where for each $n \geq 0$, the process $Y^{(n)} = (Y^{(n)}_t : 0 \leq t \leq T^{(n)})$ is defined by $Y^{(n)}_t = E_P[Y \mid \mathcal{F}^{(n)}_t]$ P-a.s.

Step 2. For each $n \geq 0$, let $(I^{(n)}, \boldsymbol{F}^{(n)}, \phi^{(n)})$ denote an $\boldsymbol{F}^{(n)}$-predictable skeleton satisfying (4.1.1) and (4.1.2).

Step 3. For $0 \leq t \leq T$, and for P-almost all $\omega \in \Omega$, define

$$\int_0^t \phi_s(\omega) \cdot \mathrm{d}Z_s(\omega) \equiv \lim_{n \to \infty} (\phi^{(n)} \circ \zeta^{(n)})_t(\omega)$$

which will be called the *pathwise stochastic integral of ϕ with respect to Z.*

Note that in Step 2, The $\boldsymbol{F}^{(n)}$-predictable skeletons are obtained by solving equations along the sample paths of $\zeta^{(n)}$. These sample path operations are illustrated in Section 2.4 and explain the role of the 0th component of Z.

4.2. Feasibility of the construction

The discussion preceding the construction establishes the feasibility of Step 2. The proof of part (1) of the following theorem justifies Steps 1 and 3.

Theorem 4.2.1. (1) *The pathwise stochastic integral*

$$\int_0^t \phi_s(\omega) \cdot \mathrm{d}Z_s(\omega) \quad (0 \leq t \leq T)$$

is well defined; that is, $\phi^{(n)} \circ \zeta^{(n)}(\omega)$ converges uniformly on $[0, T]$ for P-almost all $\omega \in \Omega$ as $n \to \infty$.

(2) *If $X = (Z^1, Z^2, \ldots, Z^d) \in \mathcal{M}_0^2$ and $\psi \in \mathcal{L}^2(X)$ is given, then Itô's stochastic integral $\int \psi \, \mathrm{d}X$ and the pathwise stochastic integral $\int \phi_s(\cdot) \cdot \mathrm{d}Z_s(\cdot)$ (obtained by taking $Y = \int_0^T \psi \, \mathrm{d}X$ in Step 0 of the construction) are indistinguishable; that is,*

$$P\left[\{\omega \in \Omega: \left[\int_0^t \psi_s \, \mathrm{d}X_s\right](\omega) = \int_0^t \phi_s(\omega) \cdot \mathrm{d}Z_s(\omega); 0 \leq t \leq T\}\right] = 1.$$

Proof. Note that all sample paths of the $(\boldsymbol{F}, P)$-martingale $(Y_t = E_P[Y \mid \mathcal{F}_t]: 0 \leq t \leq T)$ can be assumed to be continuous. Indeed, assumption (3.1.1) implies the existence of a continuous modification of $(Y_t: 0 \leq t \leq T)$ (see, for example, Huang, 1985).

(1) To establish Step 1, we have to show that one can find sequences $(\boldsymbol{F}^{(n)})_{n \geq 0}$ of skeleton filtrations such that both $(I^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})_{n \geq 0}$ and $(I^{(n)}, \boldsymbol{F}^{(n)}, Y^{(n)})_{n \geq 0}$ are continuous-time skeleton approximations of $(\boldsymbol{F}, Z)$ and $(\boldsymbol{F}, (Y_t: 0 \leq t \leq T))$, respectively. It is easy to see that for any sequence $(b_n)_{n \geq 0}$ of positive real numbers with $b_n \to 0$ as $n \to \infty$, one can find skeleton filtrations $\boldsymbol{F}^{(n)}$ $(n \geq 0)$ such that for each $n \geq 0$, one has

$$P\left[\left\{\omega \in \Omega: \sup_{t \in D_n} |Z_t(\omega) - \zeta_t^{(n)}(\omega)| > b_n\right\}\right] \leq \frac{1}{2^n}$$

and also

$$P\left[\left\{\omega \in \Omega: \sup_{t \in D_n} |Y_t(\omega) - Y_t^{(n)}(\omega)| > b_n\right\}\right] \leq \frac{1}{2^n}$$

(where D_n denotes the nth dyadic partition of $[0, T]$). The rest of the proof is as in Theorem 4.3.1 of Willinger and Taqqu (1988).

(2) Next we establish feasibility of Step 3. We must show that the sequence $(\phi^{(n)} \circ \zeta^{(n)})_{n \geq 0}$ of discrete stochastic integrals converges uniformly on $[0, T]$ for P-almost all $\omega \in \Omega$ as $n \to \infty$. For each $n \geq 0$, we have

$$E_P[\, Y^{(n)}_{T^{(n)}} \,|\, \mathcal{F}^{(n)}_t] = Y^{(n)}_t \quad P\text{-a.s.} \quad (0 \leq t \leq T^{(n)});$$

and by the martingale property of $\phi^{(n)} \circ \zeta^{(n)}$ we also have

$$E_P[(\phi^{(n)} \circ \zeta^{(n)})_{T^{(n)}} \,|\, \mathcal{F}^{(n)}_t] = (\phi^{(n)} \circ \zeta^{(n)})_t \quad P\text{-a.s.} \quad (0 \leq t \leq T^{(n)}).$$

Because of Step 2 of the construction,

$$Y^{(n)}_{T^{(n)}} = (\phi^{(n)} \circ \zeta^{(n)})_{T^{(n)}} \quad P\text{-a.s.}$$

and, therefore,

$$Y^{(n)}_t = (\phi^{(n)} \circ \zeta^{(n)})_t \quad P\text{-a.s.} \quad (0 \leq t \leq T^{(n)}).$$

Pathwise convergence of $(\phi^{(n)} \circ \zeta^{(n)})_{n \geq 0}$ on $[0, T]$ for P-almost all $\omega \in \Omega$ now follows from the uniform convergence of $(Y^{(n)})_{n \geq 0}$ on $[0, T]$; the latter holds since $(I^{(n)}, F^{(n)}, Y^{(n)})_{n \geq 0}$ is a continuous-time skeleton approximation of $(F, (Y_t: 0 \leq t \leq T))$.

(3) In order to prove part (2) of the theorem, let $X = (Z^1, Z^2, \ldots, Z^d) \in \mathcal{M}^2_0$ and $\psi \in \mathcal{L}^2(X)$, and apply the pathwise construction with $Y = \int_0^T \psi_s \, dX_s$, Itô's stochastic integral of ψ with respect to X. As a result of part (1) of the theorem, we have

$$Y = \lim_{n \to \infty} (\phi^{(n)} \circ \zeta^{(n)})_{T^{(n)}} = \int_0^T \phi_s(\cdot) \cdot dZ_s(\cdot) \quad P\text{-a.s.},$$

that is, Y is the pathwise stochastic integral with respect to $Z = (1, X)^{\mathrm{T}}$. Moreover, since for each $t \in D = \{kT/2^n: k = 0, 1, \ldots, 2^n; n \geq 0\}$,

$$\int_0^t \phi_s(\omega) \cdot dZ_s(\omega)$$

$$= E_P\left[\int_0^T \phi_s(\cdot) \cdot dZ_s(\cdot) \,|\, \mathcal{F}_t\right](\omega) = E_P[\, Y \,|\, \mathcal{F}_t](\omega) \quad P\text{-a.s.}$$

and

$$\left[\int_0^t \psi_s \, dX_s\right](\omega) = E_P\left[\int_0^T \psi_s \, dX_s \,|\, \mathcal{F}_t\right](\omega) = E_P[\, Y \,|\, \mathcal{F}_t](\omega) \quad P\text{-a.s.}$$

and $(Y_t = E_P[\, Y \,|\, \mathcal{F}_t]: 0 \leq t \leq T)$ has continuous sample paths, we get

$$1 = P\left[\left\{\omega \in \Omega: \int_0^t \phi_s(\omega) \cdot dZ_s(\omega) = \left[\int_0^t \psi_s \, dX_s\right](\omega): t \in D\right\}\right]$$

$$= P\left[\left\{\omega \in \Omega: \int_0^t \phi_s(\omega) \cdot dZ_s(\omega) = \left[\int_0^t \psi_s \, dX_s\right](\omega): 0 \leq t \leq T\right\}\right]. \qquad \square$$

Although the pathwise stochastic integral agrees with Itô's integral (as a stochastic process, up to P-indistinguishability), we shall nevertheless distinguish the two integrals and use two slightly different notations for the same object. To explicitly indicate the pathwise nature of our construction, we write

$$\int \phi_s(\omega) \cdot \mathrm{d}Z_s(\omega) \tag{4.2.1}$$

and we use

$$\left[\int \psi_s \, \mathrm{d}X_s\right](\omega) \tag{4.2.2}$$

to refer to Itô's integral obtained through an L_2-isometry.

Part (2) of Theorem 4.2.1 shows how to define the pathwise stochastic integral when $\psi \in \mathscr{L}^2(X)$ is given. Namely, in Step 0 of the construction, take $Y = \int_0^T \psi_s \, \mathrm{d}X_s$, Itô's integral of ψ with respect to X and proceed as in the construction above. In fact, we conjecture that $((\phi^{(n)})^1, (\phi^{(n)})^2, \ldots, (\phi^{(n)})^d)$, the first d component processes of the $\boldsymbol{F}^{(n)}$-predictable, continuous time skeleton $\phi^{(n)} = ((\phi^{(n)})^0, (\phi^{(n)})^1, \ldots, (\phi^{(n)})^d)$ converge (in $\mathscr{L}^2(X)$) to the integrand process ψ of Itô's integral $\int \psi_s \, \mathrm{d}X_s$. Thus far, however, we have been unable to prove such a result.

The connection between Itô's approach and our pathwise construction leading to (4.2.1) can be illustrated as follows. The skeleton approach required us to assume the minimal filtration, whereas Itô's method works with an *arbitrary* filtration (with respect to which the integrator is adapted). Itô's work is based on L^2-theory and is therefore typically not concerned with sample path considerations. However, by relying on an L^2-theory one loses control over the fine structure of the filtration or, in the case of the minimal filtration, over the sample path behaviour of the underlying stochastic process. Moreover, if at the end, a reconstruction of the underlying fine structure is desired, then an L^2-approach requires the cumbersome task of looking for appropriate (almost-sure convergent) subsequences. This not only explains the main character of existing pathwise approaches (see Section 1.3) when compared to Itô's theory (finding the "right" subsequence), but it also shows that the underlying filtration will have to be taken into consideration when one wants to extract a sample path result from an L^2-theory. Note that our pathwise approach relies on the fine structure of the filtration alone and uses no L^2-theory at all.

Remarks. (1) Although Step 1 of the construction of pathwise integrals does not uniquely specify the continuous-time skeleton approximation $(I^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})_{n \geq 0}$ of $(\boldsymbol{F}, Z)$, the pathwise stochastic integral does not depend on the choice of approximating skeletons (as long as they are feasible in the sense of Step 1). This property follows directly from the proof of part (1) of Theorem 4.2.1.

(2) The $\boldsymbol{F}^{(n)}$-predictable skeletons $(I^{(n)}, \boldsymbol{F}^{(n)}, \phi^{(n)})$ $(n \geq 0)$ obtained in Step 2 of the construction have the property that $\phi^{(n)}_{t(n,1)} \cdot \zeta_0^{(n)} = (\phi^{(n)}_{t(n,1)})^0$ $(Z_0^k = 0$ P-a.s.; $k = 1, 2, \ldots, d)$ is independent of n and equals $E_P[Y]$ since

$$E_P[Y] = E_P[Y^{(n)}_{T^{(n)}}] = E_P[(\phi^{(n)} \cdot \zeta^{(n)})_{T^{(n)}}] = \phi^{(n)}_{t(n,1)} \cdot \zeta_0^{(n)}.$$

Here we used, in succession, the definition of $Y^{(n)}$, property (4.1.2), and properties of the martingale transforms. Observe, however, that Theorem 4.2.1 contains no statement concerning the convergence (in any sense) of the sequence $(\phi^{(n)})_{n\geq 0}$. Thus far, we have been unable to prove such a result but conjecture that $\phi^{(n)}$ converges to an $\boldsymbol{F}$-predictable process $\phi = (\phi_t\colon 0 \leq t \leq T)$ for $P \times \lambda$-almost all $(\omega, t) \in \Omega \times [0, T]$ (here, λ denotes Lebesgue measure on $[0, T]$). In particular note that $(\phi^{(n)})^k$, the kth component process of $\phi^{(n)}$ $(1 \leq k \leq d)$, can be viewed as a discrete approximation of $\mathrm{d}Y/\mathrm{d}Z^k = (\mathrm{d}Y_t/\mathrm{d}Z_t^k\colon 0 \leq t \leq T)$. To see this, consider the case $d = 1$ (for $d > 1$, similar arguments apply and rely on a modified construction of continuous-time skeleton approximations of $(\boldsymbol{F}, Z)$ (see Willinger and Taqqu, 1988, Sections 5.2 and 5.3). Fix $t = kT/2^m$ $(k = 0, 1, \ldots, 2^m;\ m \geq 0)$ and consider the behavior of $\phi_t^{(n)}$ for $n \geq m$. On any set $A_n \in \mathscr{P}_{t^{(n)}}^{(n)}$ with $P[A_n] > 0$ (where $t^{(n)}$ denotes the last element in $I^{(n)}$ preceding t), Corollary 2.3.1 characterizes $\phi_t^{(n)}$ as a solution to the system $y \cdot B_t^{(n)} = \nu_t^{(n)}$ of linear equations where the columns of the matrix $B_t^{(n)}$ and the right-hand side $\nu_t^{(n)}$ are given by the values of $\zeta_t^{(n)}$ and $Y_t^{(n)}$, respectively. More precisely, since $(I^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})$ results from an application of the special construction to $(\boldsymbol{F}, Z)$ (see Willinger and Taqqu, 1988, Section 4.4), $B_t^{(n)}$ and $\nu_t^{(n)}$ are given by

$$B_t^{(n)} = (E_P[Z_t \mid A_n \cap \{Z_t^1 > Z_0^1\}],\ E_P[Z_t \mid A_n \cap \{Z_t^1 \leq Z_0^1\}])$$

and

$$\nu_t^{(n)} = (E_P[\,Y_t | A_n \cap \{Z_t^1 > Z_0^1\}],\ E_P[\,Y_t | A_n \cap \{Z_t^1 \leq Z_0^1\}]).$$

Without loss of generality, we can assume that $Z_t^1 \neq \text{const.}$ (P-a.s.) so that rank $(B_t^{(n)}) = \text{cardinality}\ (A' \in \mathscr{P}_t^{(n)}\colon A' \subseteq A_n) = 2$ (use Theorem 2.2.1(3)), and $y \cdot B_t^{(n)} = \nu_t^{(n)}$ has the unique $\mathscr{F}_{t^{(n)}}^{(n)}$-measurable solution

$$\begin{aligned}
\phi_t^{(n)} &= \nu_t^{(n)} \cdot (B_t^{(n)})^{-1} \\
&= (E_P[Z_t^1 | A_n \cap \{Z_t^1 \leq Z_0^1\}] - E_P[Z_t^1 | A_n \cap \{Z_t^1 > Z_0^1\}])^{-1} \\
&\quad \times (E_P[\,Y_t | A_n \cap \{Z_t^1 > Z_0^1\}] E_P[Z_t^1 | A_n \cap \{Z_t^1 \leq Z_0^1\}] \\
&\qquad - E_P[\,Y_t | A_n \cap \{Z_t^1 \leq Z_0^1\}] E_P[Z_t^1 | A_n \cap \{Z_t^1 > Z_0^1\}], \\
&\qquad E_P[\,Y_t | A_n \cap \{Z_t^1 \leq Z_0^1\}] - E_P[\,Y_t | A_n \cap \{Z_t^1 > Z_0^1\}]),
\end{aligned}$$

that is, $(\phi_t^{(n)})^1 = \Delta Y_t^{(n)} / \Delta \zeta_t^{(n)}$.

(3) Stochastic integrals cannot in general be defined in the Riemann-Stieltjes sense, namely as almost-sure limit

$$\lim_{n \to \infty} \sum_{t(k) \in D_n} \phi_{t(k)}(\omega) \cdot (Z_{t(k)}(\omega) - Z_{t(k-1)}(\omega)), \tag{4.2.3}$$

because typically, the sample paths $(Z_t(\omega)\colon 0 \leq t \leq T)$, $\omega \in \Omega$, of martingales are not rectifiable (e.g. almost all sample paths of Brownian motion are of unbounded variation and hence not rectifiable). However, our pathwise approach indicates that stochastic integrals are defined in an unconventional Riemann-Stieltjes sense, that

is, as almost-sure limit

$$\lim_{n\to\infty} \sum_{s(n,l)\in I^{(n)}} \phi^{(n)}_{s(n,l)}(\omega)\cdot(\zeta^{(n)}_{s(n,l)}(\omega)-\zeta^{(n)}_{s(n,l-1)}(\omega)). \tag{4.2.4}$$

Here, the set $I^{(n)}$ is a slightly expanded set D_n, $\zeta^{(n)}$ is an "averaged" version of Z, namely $\zeta^{(n)}_t = E_P[Z_t|\mathcal{F}^{(n)}_t]$, and $\phi^{(n)}$ is a certain predictable "control" process obtained in Step 2 of our construction. Thus, in addition to partitioning the time axis (see (4.2.3)), (4.2.4) also deals with a partition of the state space of the process. The following physical interpretation of the difference between (4.2.3) and (4.2.4) is illuminating. Let $Z_t(\omega)$ denote the position of a particle ω at time t. Then, (4.2.3) requires *exact* measurements of time and location of the particle at any instant. On the other hand, (4.2.4) reflects the physically more realistic situation where the position of a particle can only be determined with some error ("uncertainty principle"). For yet another version (random observation times, exact position in space), see F. Knight's pathwise approximation of Brownian motion (Knight, 1981; Itô and McKean, 1965).

(4) Suppose that the filtration $\boldsymbol{F}$ is no longer minimal ($\boldsymbol{F}\supseteq\boldsymbol{F}^Z$) and does not satisfy the continuity assumption (3.1.1). Pathwise stochastic integration may still be possible but it requires the introduction of additional component processes, say $Z^{d+1},\dots,Z^{d+c}$ $(1\leqslant c<\infty)$ such that (i) $\boldsymbol{F}^{Z^*}=\boldsymbol{F}$ (where Z^* is the $\mathbb{R}^{d+c}$-valued process with component processes $Z^0, Z^1,\dots,Z^d,Z^{d+1},\dots,Z^{d+c}$) and (ii) the discontinuities in $\boldsymbol{F}$ are "explained" by Z^*. For an illustration of how to choose the "right" additional components, consider the following example which appears in Harrison and Pliska (1981) in a related context (see also Willinger and Taqqu, 1988). Take $d=1$ and consider the component processes

$$Z^0_t = 1, \quad 0\leqslant t\leqslant 1,$$
$$Z^1_t = \exp(W_t - \tfrac{1}{2}t), \quad 0\leqslant t\leqslant 1,$$

where $W=(W_t\colon 0\leqslant t\leqslant 1)$ denotes standard Brownian motion. Let $\sigma=(\sigma_t\colon 0\leqslant t\leqslant 1)$ denote a stochastic process independent of W which models the outcome of the toss of a fair coin for $\frac{1}{2}\leqslant t\leqslant 1$. For example, set

$$\sigma_t = \begin{cases} 2 & \text{if } 0\leqslant t<\frac{1}{2},\\ 1 & \text{if } \frac{1}{2}\leqslant t\leqslant 1 \text{ and head occurs},\\ 3 & \text{if } \frac{1}{2}\leqslant t\leqslant 1 \text{ and tail occurs}.\end{cases}$$

It is easy to see that Z is a $(\boldsymbol{F},P)$-martingale where $\boldsymbol{F}=\boldsymbol{F}^Z\vee\boldsymbol{F}^\sigma\supseteq\boldsymbol{F}^Z$; moreover, (3.1.1) does not hold since $\mathcal{F}_{1/2-}\neq\mathcal{F}_{1/2}$. Now choose the following additional component Z^2:

$$Z^2_t = \sigma_t - 1, \quad 0\leqslant t\leqslant 1.$$

Setting $Z=(Z^0,Z^1,Z^2)$, we obviously obtain an $(\boldsymbol{F},P)$-martingale with $\boldsymbol{F}^Z=\boldsymbol{F}$. Although Z has neither continuous sample paths nor does $\boldsymbol{F}$ satisfy the continuity assumptions (3.1.1), pathwise stochastic integration with respect Z is still possible since there exist continuous-time skeleton approximations of $(\boldsymbol{F},Z)$ (see Willinger and Taqqu, 1988).

5. Pathwise integration and the theory of continuous trading

In this section, we show that our pathwise approach to stochastic integration can be used to obtain a convergence theory for continuous security market models with exogenously given equilibrium prices. Throughout this section, it is assumed that the reader is familiar with the papers by Harrison and Kreps (1979) and Harrison and Pliska (1981) which introduce the modern theory of martingales and stochastic integrals to the analysis of stochastic models for trading securities in continuous time.

5.1. Towards a convergence theory for continuous security market models

The main problem with the existing theory of continuous security market models can be summarized as follows. The finite stochastic models for trading securities in which the underlying probability space is essentially discrete and finite and trading takes place at only finitely many points in time, are fully understood (Harrison and Pliska, 1981; Taqqu and Willinger, 1987). However, this understanding could not be applied to the study of many important features of continuous security market models (e.g. completeness, no-arbitrage) because of the absence of an appropriate approximation scheme and the typically non-constructive nature of the presently available stochastic integration theory. The properties of completeness and no-arbitrage are of theoretical interest as well since they contribute to a better understanding of the martingale representation theory (Jacod, 1979; Kopp, 1984).

For exogenously given equilibrium prices, we characterize convergence of "real-life" economies (i.e. finite security market models) to continuous market models in which securities can be traded continuously and the probability space is arbitrary. The characterization follows from our pathwise construction of stochastic integrals and provides a convergence theory for continuous security market models with *exogenously* given equilibrium prices that is—to our knowledge—the first of its kind and has been sought after as discussed in Kreps (1982) and Harrison and Pliska (1981). Such a convergence theory is important for justifying the study of idealized (i.e. continuous) markets such as the well-known Black-Scholes model (Black and Scholes, 1973). We also consider our results as a first step towards establishing a general convergence theory for continuous security markets with *endogenously* determined equilibrium prices (Kreps, 1982; Duffie and Huang, 1985).

5.2. An economic interpretation of the probabilistic setting

Consider the probabilistic setting introduced in Section 3.1 as a stochastic model of a *security market with continuous trading*; that is, interpret $Z_t^k(\omega)$ as the price of security k $(0 \leq k \leq d)$ at time t $(0 \leq t \leq T)$ if $\omega \in \Omega$ represents the state of nature, and $\mathcal{F}_t$ as the information available to an investor at time t $(0 \leq t \leq T)$. Then starting without knowledge ($\mathcal{F}_0$ is almost trivial), the investor ends up with all uncertainty resolved ($\mathcal{F}_T = \mathcal{F}$). Moreover, between times 0 and T, knowledge is based only on past and present values of the securities (assumption (3.1.2)) and is resolved

gradually (assumption (3.1.1)). Here, for the purpose of interpretation, we also assume that each component-process $Z^k = (Z_t^k: 0 \leq t \leq T), (k = 0, 1, \ldots, d)$ is strictly positive. Note that by taking $Z_t^0 \equiv 1$ $(0 \leq t \leq T)$, we consider an already discounted price process Z. (Harrison and Kreps, 1979, show that there is no loss of generality in assuming a riskless and constant bond price Z^0.) In the sequel, the continuous security market model corresponding to $(\Omega, \mathcal{F}, P)$ and the $(\boldsymbol{F}, P)$-martingale Z will be denoted by $(\boldsymbol{T}, \boldsymbol{F}, Z)$ with $\boldsymbol{T} = [0, T]$ representing the trading times.

Next consider a continuous-time skeleton $(\boldsymbol{T}^\zeta, \boldsymbol{F}^\zeta, \zeta)$ of $(\boldsymbol{F}, Z)$ where ζ is the projection (in the sense of a conditional expectation operator) of Z onto $\boldsymbol{F}^\zeta$. By restricting $(\boldsymbol{T}^\zeta, \boldsymbol{F}^\zeta, \zeta)$ to its finite index set $\boldsymbol{T}^\zeta$, we obtain a discrete-time stochastic process $((\mathcal{F}_t^\zeta: t \in \boldsymbol{T}^\zeta), (\zeta_t: t \in \boldsymbol{T}^\zeta))$ which defines a *finite security market model* in the sense of Taqqu and Willinger (1987). Identifying each element in a continuous-time skeleton approximation $(\boldsymbol{T}^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})_{n \geq 0}$ of $(\boldsymbol{F}, Z)$ with a finite security market model yields a *finite market approximation of* $(\boldsymbol{T}, \boldsymbol{F}, Z)$, again denoted by $(\boldsymbol{T}^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})_{n \geq 0}$, for convenience. Note that each element in a finite market approximation of $(\boldsymbol{T}, \boldsymbol{F}, Z)$ is defined on the same probability space as $(\boldsymbol{T}, \boldsymbol{F}, Z)$ but differs in the finite sets $\boldsymbol{T}^{(n)}$ of trading dates, the simple equilibrium price processes $\zeta^{(n)}$, and the finitely generated information structures $\boldsymbol{F}^{(n)}$. Since each $\zeta^{(n)}$ can be considered a stochastic process in continuous time or discrete time, we take in the sequel whichever setting is most convenient.

5.3. *Convergence results for the theory of continuous trading*

We first establish the existence of finite market approximations. The result below follows directly from the existence of continuous-time skeleton approximations of $(\boldsymbol{F}, Z)$ (see Willinger and Taqqu, 1988, Theorem 4.3.1) and is a simple translation of the skeleton terminology (Willinger and Taqqu, 1988) into security market language. It enables us to view the continuous model $(\boldsymbol{T}, \boldsymbol{F}, Z)$ as a pathwise limit of finite markets such that the underlying information structures converge, too.

Theorem 5.3.1 *Finite market approximations* $(\boldsymbol{T}^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})_{n \geq 0}$ *of the continuous security market model* $(\boldsymbol{T}, \boldsymbol{F}, Z)$ *always exist. They satisfy*

(1) *dense subset property of the sets* $\boldsymbol{T}^{(n)}$ *of finite trading dates*:

$$\bigcup_{n \geq 0} (\boldsymbol{T}^{(n)} \cap [0, T]) \text{ is a dense subset of } [0, T];$$

(2) *convergence of the finite market information* $\boldsymbol{F}^{(n)}$:

$$\mathcal{F}_t = \sigma\left(\bigcup_{k \geq 0} \mathcal{F}_t^{(k)}\right) \supseteq \cdots \supseteq \mathcal{F}_t^{(n+1)} \supseteq \mathcal{F}_t^{(n)} \supseteq \cdots \supseteq \mathcal{F}_t^0$$

(*up to P-null sets*) $(n \geq 0, 0 \leq t \leq T)$;

(3) *pathwise convergence of the finite market price processes* $\zeta^{(n)}$:

$$P\left[\left\{\omega \in \Omega: \lim_{n \to \infty} \sup_{0 \leq t \leq T} |Z_t(\omega) - \zeta_t^{(n)}(\omega)| = 0\right\}\right] = 1. \qquad \square$$

Next we concentrate on the martingale property of the process Z of equilibrium prices. Willinger and Taqqu (1988) show that there exist continuous-time skeleton approximations $(\boldsymbol{T}^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})_{n\geq 0}$ of the $(\boldsymbol{F}, P)$-martingale Z such that for each $n \geq 0$, $\zeta^{(n)}$ is a $(\boldsymbol{F}^{(n)}, P)$-martingale. However, in the corresponding finite market models $(\boldsymbol{T}^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})$, the martingale property of the process $\zeta^{(n)}$ is known to be equivalent to the absence of *arbitrage opportunities*. An arbitrage opportunity (also called a *free lunch*) is a riskless plan for making profit without investments and can be formally defined in finite security models by the *self-financing* condition (4.1.1). (For details, see Taqqu and Willinger, 1987). For the continuous model $(\boldsymbol{T}, \boldsymbol{F}, Z)$, the notions of "no-arbitrage" and "self-financing" can be defined and understood through the following convergence result.

Theorem 5.3.2. *There exist finite market approximations* $(\boldsymbol{T}^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})_{n\geq 0}$ *of the continuous security market model* $(\boldsymbol{T}, \boldsymbol{F}, Z)$ *such that for each* $n \geq 0$, *the finite market model* $(\boldsymbol{T}^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})$ *contains no arbitrage opportunities.* □

Finally, we present a convergence result for the continuous model $(\boldsymbol{T}, \boldsymbol{F}, Z)$ that is an immediate consequence of the pathwise construction of stochastic integrals (Section 4.1) and its feasibility (Section 4.2). In particular, observe that each $(\boldsymbol{T}^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})$ along the continuous-time skeleton approximation $(\boldsymbol{T}^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})_{n\geq 0}$ of the $(\boldsymbol{F}, P)$-martingale Z is assumed to be complete (Step 1). For the corresponding finite market model $(\boldsymbol{T}^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})$, *completeness* is an economically desirable property, since it enables one to price any given *contingent claim* unambiguously (see Taqqu and Willinger, 1987). Completeness of the continuous security market model $(\boldsymbol{T}, \boldsymbol{F}, Z)$ (i.e. the ability to write any $Y \in L^1(\Omega, \mathcal{F}, P)$ as pathwise stochastic integral) can then be explained via the following convergence result.

Theorem 5.3.3. *There exist finite market approximations* $(\boldsymbol{T}^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})_{n\geq 0}$ *of the continuous security market model* $(\boldsymbol{T}, \boldsymbol{F}, Z)$ *such that for each* $n \geq 0$, *the finite-market model* $(\boldsymbol{T}^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})$ *is complete (for* $(\Omega, \boldsymbol{F}^{(n)}, P)$*).* □

We conclude by recalling that because of Theorem 3.2.1, completeness of each finite market model $(\boldsymbol{T}^{(n)}, \boldsymbol{F}^{(n)}, \zeta^{(n)})$ can be characterized in terms of condition $(C'_{\zeta^{(n)}})$ $(n \geq 0)$. This condition dictates almost completely the dynamic nature of the price process $\zeta^{(n)}$ (i.e. the flow of information and the possible changes of the values of $\zeta^{(n)}$) and relates the fine structure of the filtration $\boldsymbol{F}^{(n)}$ to the number of (nonredundant) securities needed for completeness of $\zeta^{(n)}$ (Harrison and Pliska, 1981; Taqqu and Willinger, 1987). Thus, Theorems 5.3.3 and 3.2.1 state that this relationship can be maintained along finite market approximations of $(\boldsymbol{T}, \boldsymbol{F}, Z)$ and identify the fine structure of the filtration $\boldsymbol{F}$ as the key factor for completeness. The relationship is made explicit in Willinger and Taqqu (1988, Corollaries 5.2.2 and 5.3.2) and provides an intuitive and rigorous explanation for why $d+1$ securities suffice to uniquely

price each contingent claim: In each finite market model along the approximation, $d+1$ securities are needed to complete the market. This result makes obvious the startling observation of Black and Scholes (1973) that two securities suffice to complete their model; it also explains why the generalized Black-Scholes model is complete with $d+1$ securities (see Harrison and Pliska, 1981; Duffie and Huang, 1985). Moreover, the ability to explicitly deal with the fine structure of $\boldsymbol{F}$ through condition $(C'_{\xi^{(n)}})$ along finite market approximations enables us to complete *incomplete* market models by finding "suitable" additional securities which yield a new price process Z such that Theorem 5.3.3 applies (see Willinger and Taqqu, 1988, Section 5.4). Finally we mention that such an explicit description of the fine structure of the filtration of a complete process (in terms of condition $(C'_{\xi^{(n)}})$ along a continuous-time skeleton approximation of $(\boldsymbol{F}, Z)$) is new to the theory of continuous trading as well as to the martingale representation theory and solves an open problem stated in Kopp (1984, p. 169).

Acknowledgement

This work is based in part on the first author's dissertation under the direction of the second author. We acknowledge Harry Kesten for pointing out some technical problems in an early version of Willinger (1987) and for helpful discussions.

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