THE ENFORCEMENT OF COLLUSION IN OLIGOPOLY

by

David Knudsen Levine

B.A., UCLA
(1977)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1981

Signature of Author. Signature redacted .....
Department of Economics, May 1, 1981

Certified by... Signature redacted .....
Thesis Supervisor

Accepted by. Signature redacted .....
Chairman, Department Committee on Graduate Students

ARCHIVES
MASSACHUSETTS INSTITUTE
OF TECHNOLOGY
JUL 1 1981
LIBRARIES

ABSTRACT

Title: THE ENFORCEMENT OF COLLUSION IN OLIGOPOLY

Author: David Knudsen Levine

Submitted to the Department of Economics on May 1, 1981 in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Economics.

The three essays of this thesis examine oligopolistic outcomes from the perspective of rational disequilibrium adjustment.

The first essay studies the "local almost perfect" adjustment path of an industry with a fixed number of identical output controlling firms. Firms can respond to opponent's output deviations instantly but not costlessly. The long run industry equilibrium output is shown to lie between the Cournot-Nash and monopoly level depending on market conditions.

The second essay is closely related to the first - it asks what long run outcomes will be in an industry with an arbitrary technology but no costs of responding to opponents. Subject to technical qualifications I show that in the long run complete collusion occurs.

The third and final essay gives a rigorous analysis of "local almost perfect" equilibrium. In a dynamic game with adjustment costs firms do not have global information but are assumed to extrapolate linearly from complete local information. In an environment in which such extrapolation works well games theoretic regression is shown not to be a problem: the adjustment path is essentially unique.

Thesis Supervisor: Peter Diamond

Title: Professor of Economics
Massachusetts Institute of Technology

ACKNOWLEDGEMENTS

I owe a great debt of gratitude to the members of my thesis committee: Professor Peter Diamond, who served as chairman; Professor Timothy Kehoe; and Professor Franklin M. Fisher. Without their advice and encouragement this thesis could not have been written

During the two years and six or more drafts through which parts of these essays have passed a great many people have contributed valuable advice and comments. I am grateful to: Randall Ellis, James Friedman, Drew Fudenberg, Judith Gelman, Jonathan Hamilton, Jack Hirshleifer, Eric Maskin, John Riley; Richard Schmalensee, Sanjay Srivastava, Peter Temin, Jean Tirole and Michael Veall. I am also grateful to participants of seminars at MIT, UCLA, the University of Rochester and the University of Pennsylvania.

I am further grateful to the Department of Economics, Massachusetts Institute of Technology who gave me financial support while this thesis was being written.

Finally I would like to thank Deborah Fretz, Joanne Klys and Alice Sanderson who helped prepare, type and edit the manuscript.

TABLE OF CONTENTS

CHAPTER I:
The Enforcement of Collusion in a Quadratic Symmetric Oligopoly.... 5

CHAPTER II:
Long Run Collusion in a Partially Myopic Industry..... 70

CHAPTER III:
Local Almost Perfect Equilibrium in a Game with Adjustment Costs. 112

APPENDIX:
Some Results on Dynamical Systems..... 164

CHAPTER I: THE ENFORCEMENT OF COLLUSION IN A QUADRATIC SYMMETRIC OLIGOPOLY

0. Introduction

How and to what extent do self-interested oligopolistic firms successfully collude? The story of oligopolistic interaction ought to be a simple one. With the same firms in a static market over time uncooperative rivals can be punished and helpful rivals rewarded. Only to the extent that retaliatory strategies have associated frictional costs should less than fully collusive behavior be expected.

Current literature is divided on the question of whether oligopolists successfully collude. One class of models shows that various ad hoc assumptions about firm behavior lead to collusion.[1] Rigorously specified super-game models[2] lead to less satisfactory conclusions: almost any industry outcome at all is consistent with optimal play by firms with rational expectations.[3]

In a recent paper [10] I argued that when the bounded rationality of firms is taken account of there is only one economically significant adjustment process.[4] I argued that firms cannot be expected to know how their opponents will behave in every contingency, including contingencies that never occur. I argued that instead they will use local information near the status quo to extrapolate future income paths. I then showed that essentially only one adjustment path is consistent with

firms extrapolating future income linearly. This I called the local almost perfect adjustment path.

This paper studies the long-run steady states of the local almost perfect adjustment path of a simple oligopoly. A fixed number of identical firms control their own output, produce at constant marginal cost and face linear demand.[5] Only market interactions are considered. Legal cartels and non-market interactions such as violence are ruled out. Firms are shown to follow a simple, sensible adjustment procedure. They choose to reward cooperative opponents and punish uncooperative ones. Because firms are assumed to adjust gradually these are credible threats. In this environment I attempt to relate long-run industry output to exogenously determined features of the market. My main conclusions are

- Output exceeds the monopoly level only if there are frictional costs of engaging in retaliatory policies.
- When enforcement costs are sufficiently high, output rises to the static Cournot-Nash level.
- Output rises with the number of firms and the discount rate and falls with market profitability.
- If firms exchange enough information to avoid counter-reacting to each other's retaliation, collusion is enhanced.

The paper has five sections. Section one describes a market in which firms can communicate threats. Section two derives the local almost perfect adjustment path for such a market. Section three describes the

nature and stability of the long-run steady states of the adjustment path. Section four examines a duopoly in which firms cannot communicate, but do observe each other's output. Section five reexamines some widely believed myths about oligopoly in light of the findings of this paper.

1. The Model

This section describes a simple model of oligopoly without entry in which firms communicate threats, but cannot enter into legally binding contracts. The first half the section describes the actions available to firms and the information structure by which threats are communicated. The second half describes firm income. Discussion of firm behavior is deferred to section two.

The Environment: There are N identical firms, entry is prohibited and each firm j controls its own output x^j. Information is exchanged costlessly by a fixed information structure. At time t all firms j announce that they will respond to autonomous output changes by other firms k at a rate R^j_k.[6] At time t+Δt all firms k announce their autonomous output changes of y^k.[7] At time t+2Δt each firm j computes its total output change as

$$\Delta x^j = y^j + \sum_{k\neq j} R^j_k y^k = \sum_k R^j_k y^k \qquad (1\text{-}1)$$

where $R^j_j \equiv 1$. Firms are assumed to observe one another's actual output. When Δt is infinitesimal relative to the distant rate j's opponents observe immediately whether or not he fulfilled his announced commitment given in (1-1). Thus lying is detected instantaneously and (1-1) may be taken to determine the amount by which j actually

changes his output. In the continuous time limit as Δt goes to zero firm j chooses a vector of response rates (also called reaction coefficients) R^j and an autonomous rate of change of output y^j. Firm j's actual output follows the equation

$$\dot{x}^j = \sum_k R_k^j y^k \qquad (1\text{-}2)$$

It is important that in this environment firms react only to opponents' autonomous output change. They do not counter-react to opponents' retaliation. The ability of firms to communicate is crucial: it is only the information generated by communication which enables firms to distinguish between voluntary and reactive movements by rival firms. The case in which firms cannot communicate is examined in section four.

In addition to changing output firm j may gradually alter its commitment R^j over time. This is given as

$$\dot{R}_k^j = S_k^j \qquad (1\text{-}3)$$

where S_k^j is the rate at which firm j alters R_k^j. Thus firm j chooses paths for the control variables y^j and S^j in an effort to control the state variables x and R.

Firm Income: The income of firm j is a function of the state variables

$$a^j(x,R) = \pi^j(x) - bC^j(R^j) \qquad (1\text{-}4)$$

where π^j is sales revenue minus production costs, $bC^j(R^j)$ are frictional reaction costs discussed below and b>0 is a small scalar constant. Adjustment costs--costs which depend on j's control variables--could also be included. For notational simplicity I prefer a model of partial adjustment as described in section two.

All of the identical firms produce at constant marginal cost and face linear demand[8] so that

$$\pi^j(x) = E(1-q)x^j \qquad (1\text{-}5)$$

where E is price minus marginal cost at zero industry output, q is industry output as a fraction of the competitive level

$$q \equiv (\beta/E) \sum_k x^k \qquad (1\text{-}6)$$

and β is the slope of the demand curve.

What are the frictional costs bC^j? This paper focuses on the long-run industry steady state and describes the adjustment towards this steady state. The adjustment is described as a smooth path capturing long-run trends--short-run output fluctuations around this trend do not appear. In reality there will be short-run output fluctuations as firms engage in experimentation to learn about the environment they face, and because of various short-run random shocks. It is these short-run frictions (which are not explicitly modelled) which give rise to reaction

costs. When firm j is committed to a policy of responding to k at a rate R_k^j he must hold costly inventories to meet short-term output movements by k. The larger $|R_k^j|$ the greater the inventories he must hold. For simplicity the functional form

$$bC^j(R^j) = bm \sum_{k \neq j} |R_k^j| \tag{1-7}$$

is assumed where $m > 0$ is a scalar constant.

Besides inventories there are other short-run frictions which make response costly--most notably it may be costly to observe opponents messages or output. Explicit models along these lines are in Green [4], Stigler [20] and especially Spence [19]. Since I am primarily interested in the implications rather than the sources of frictional costs the function (1-7) is given exogenously rather than derived from underlying economic data. Note that since b is small frictional costs are presumed small. This is no limitation on the generality of the results here since large costs are a special limiting case of small costs.

To summarize, firm j controls y^j and S^j. The state variables are x and R which move according to (1-2) and (1-3). Firm income is given by (1-4) and firms' objectives are the present value of future income computed using a common discount rate $0 < \rho < \infty$. The next section describes firm behavior.

2. Firm Behavior

This section describes firm behavior by applying the notion of a local almost perfect equilibrium developed in Levine [10]. This attempts to model the bounded rationality of firms by assuming that they compute present values of income streams by extrapolating existing rates of income growth linearly into the future. The first half of the section derives the local almost perfect adjustment paths of firms. The second half explicitly computes the equations of motion of the state variables when initial conditions are symmetric. The qualitative features of the adjustment equations are developed in section three.

Local Almost Perfect Adjustment: A strategy (or closed-loop strategy) for firm j is a function

$$(y^j, S^j) = F^j(x,R) = (f^j(x,R), g^j(x,R)) \qquad (2\text{-}1)$$

which assigns a vector of control variables to the vector of state variables. This already embodies a degree of bounded rationality since firm j's choice depends only on the current state variables and not the entire past history of the market. Suppose that all firms k play $\tilde{F}^k$. Then firm j receives

$$\tilde{A}^j(x,r) = \int_0^\infty a^j(x(t),R(t)) \exp(-\rho t)dt \qquad (2\text{-}2)$$

where $x(t)$ and $R(t)$ satisfy the system of differential equations

$$\left.\begin{aligned} \dot{x}^j &= \textstyle\sum_k R_k^j \tilde{f}^k(x,R) \\ \dot{R}^j &= \tilde{g}^j(x,R) \end{aligned}\right\} \quad j = 1,\ldots,N$$

$$x(0) = x \qquad R(0) = R \qquad (2\text{-}3)$$

which are derived by substituting (2-1) into the equations of motion for the state variables (1-2) and (1-3).

If firm j has rational expectations it knows how $\tilde{A}^j$ depends on the state variables and simply chooses the controls to maximize this present value. It is unreasonable to suppose that firm j can in fact compute $\tilde{A}^j$ for this requires it to know how the industry will behave in all contingencies. I shall suppose instead that at (x,R) firm j knows an estimate $\hat{A}^j(x,R)$ formed by extrapolating local information in a manner indicated below. Firm j also knows the derivatives $D\hat{A}^j(x,R)$ (of $\hat{A}^j$ with respect to the state variables) which are assumed to approximate $D\tilde{A}^j(x,R)$. Firm j then chooses its control variables so as to move the state variables in a direction which (approximately) increases the (rationally expected) present value

$$\tilde{f}^j = b\,k \left\{ \sum_k \frac{\partial \hat{A}^j}{\partial x^k} \frac{\partial \dot{x}^k}{\partial y^j} + \sum_k \sum_\ell \frac{\partial \hat{A}^j}{\partial R_\ell^k} \frac{\partial \dot{R}_\ell^k}{\partial y^j} \right\}$$

$$= b\,k \left\{ \sum_k \frac{\partial \hat{A}^j}{\partial x^k} R_j^k \right\}$$

$$\tilde{g}_k^j = b \left\{ \sum_k \frac{\partial \hat{A}^j}{\partial x^k} \frac{\partial \dot{x}^k}{\partial S_k^j} + \sum_k \sum_\ell \frac{\partial \hat{A}^j}{\partial R_\ell^k} \frac{\partial \dot{R}_\ell^k}{\partial S_k^j} \right\}$$

$$= b \frac{\partial \hat{A}^j}{\partial R_k^j} \qquad (2\text{-}4)$$

where the equations of motion (1-2) and (1-3) are used to compute $\partial \dot{R}_\ell^k / \partial y^j = 0$, $\partial \dot{x}^k / \partial S_k^j = 0$ and $\partial \dot{R}_\ell^k / \partial S_k^j = 0$ $\ell \neq j$; $k > 0$ is a constant adjustment coefficient, and $b > 0$ is the same small scalar constant that appears in (1-4), the import of which is described below.

What (2-4) describes is a partial adjustment model with bk and b exogenous adjustment rates for the two types of controls. In this model the controls are set to increase the (approximate) level of present value over time. The factor b shows that the controls are small--equivalently that the state variables are adjusted gradually.

Implicitly it is expensive to choose large values of the controls. Indeed, in another paper [10], I show that if there are quadratic costs of choosing the controls the adjustment process (2-4) is almost optimal provided that $D\hat{A}^j$ is uniformly close to $D\tilde{A}^j$ the true present

value derivative.[9] Naturally adjustment costs and partial adjustment are dual.

One important reason for a partial adjustment model is bounded rationality. If firm j wishes to globally and instantly set the optimal output level (for example) he must know his demand curve everywhere. If he is content to restrict himself to change output slowly, he need only learn a small segment of his demand curve each day to make an optimal choice subject to his self-imposed constraint. The point is that the faster a firm moves the more quickly it must learn. Bounded rationality and large costs of gathering information imply it will move rather slowly.

How do firms use local information to approximate the present value of income? If the future income stream is not too badly non-linear, simply extrapolating income linearly should be a good approximation--indeed, casual empiricism indicates this is a widely used technique. The remainder of this sub-section gives a heuristic derivation of the (approximately) unique adjustment process which is smooth enough to allow reasonably accurate linear extrapolation by firms. Formal statements and proofs of these results can be found in Levine [10].

The linearly extrapolated present value of income is

$$\bar{A}^j \equiv \int_0^\infty [a^j + \dot{a}^j t]\exp[-\rho t]dt$$

$$= \delta a^j + \delta^2 \dot{a}^j \qquad (2\text{-}5)$$

where $\delta=1/\rho$ is the discount factor. Expanding $a^j(t)$ in a Taylor series under the integral in (2-2) shows that the error in this approximation is no worse than $\delta^3 \sup|\ddot{a}^j|$. Similarly $D\bar{A}^j$ is within $\delta^3 \sup|D\ddot{a}^j|$ of $D\tilde{A}^j$. If all firms k play $\tilde{F}^k$ with $|\tilde{F}^k|, |D^p\tilde{F}^k| \leq b\gamma$ for sufficiently high order p it can be shown that $|\ddot{a}^j|, |D\ddot{a}^j| \leq H(m,E,\beta,\gamma)b^2$ where H is a fixed function. Thus, if the adjustment of all firms is slow enough, linear extrapolation will work well.

Suppose that b is small enough that firms are willing to accept an approximation error of order b^2. I will now derive estimates $\hat{A}^j$ for each firm which: (1) depend solely on local information extrapolated linearly; (2) given that all firms k set $\tilde{F}^k$ according to (2-4) the estimates $\hat{A}^j$ are within order b^2 of the actual present value $\tilde{A}^j$ and this is true also for $D\hat{A}^j$ and $D\tilde{A}^j$; and (3) the functions $\hat{A}^j$ are given in terms of the exogenous data describing the economic environment. I will not show here, although it is shown in Levine [10], that any other adjustment process in which firms make errors of no worse than order b^2 and which permit the possibility of linear extrapolation are approximately the same as the particular adjustment process derived below.

To find what the functions $\hat{A}^j$ must be, assume that they exist and that $\tilde{F}^j$ are the corresponding strategies from (2-4). Once the functions are computed it is then straightforward to verify that they have the desired

properties. From (2-5) the linearly extrapolated present value is[10]

$$\bar{A}^j = \delta a^j + \delta^2 \dot{a}^j$$

$$= \delta a^j + \delta^2 \left\{ \sum_\ell \frac{\partial a^j}{\partial x^\ell} \left(\sum_k R_k^\ell \tilde{f}^k \right) + \sum_\ell \frac{\partial a^j}{\partial R_\ell^j} \tilde{g}_\ell^k \right\}$$

$$= \delta a^j + \delta^2 \left\{ \sum_\ell \pi_\ell^j \left(\sum_k R_k^\ell \tilde{f}^k \right) - bm \sum_\ell \operatorname{sgn}(R_\ell^j) \tilde{g}_\ell^k \right\} \qquad (2\text{-}6)$$

where $\pi_\ell^j \equiv \partial \pi^j / \partial x^\ell$,

$$\operatorname{sgn}(z) = \begin{cases} +1 & z > 0 \\ 0 & z = 0 \\ -1 & z < 0 \end{cases} \qquad (2\text{-}7)$$

and $\dot{a}^j$ is computed using $\dot{x}$ and $\dot{R}$ from (2-3) and (1-4). By assumption this approximation is good to order b^2. Substitution of (2-4) into (2-6) shows

$$\bar{A}^j = \delta a^j + b\delta^2 k \sum_\ell \pi_\ell^j \sum_k R_k^\ell \sum_m R_k^m \frac{\partial \hat{A}^k}{\partial x^m} + O(b^2)$$

$$= \delta a^j + b\delta^2 k \sum_k \sum_m \left(\sum_\ell \pi_\ell^j R_k^\ell \right) R_k^m \frac{\partial \hat{A}^k}{\partial x^m} + O(b^2) \qquad (2\text{-}8)$$

where $O(b^2)$ denotes an error of order b^2. Thus

$$\frac{\partial \bar{A}^k}{\partial x^m} = \delta \pi_m^k + O(b) \qquad (2\text{-}9)$$

Recall that $D\hat{A}^k$ is assumed to differ from $D\tilde{A}^k$ by at most $O(b^2)$. Since $D\bar{A}^k$ also differs from $D\tilde{A}^k$ by at most $O(b^2)$, $D\hat{A}^k$ and $D\bar{A}^k$ differ by at most $O(b^2)$. Certainly, then

$$\frac{\partial \hat{A}^k}{\partial x^m} = \delta \pi^k_m + O(b) \tag{2-10}$$

Substituting (2-10) into (2-9) shows

$$\bar{A}^j = \delta a^j + b\delta^3 \not{R} \sum_k \sum_m (\sum_\ell \pi^j_\ell R^\ell_k) R^m_k \pi^k_m + O(b^2) \tag{2-11}$$

Thus, if $\hat{A}^j$ is <u>defined</u> by

$$\hat{A}^j \equiv \delta a^j + b\delta^3 \not{R} \sum_k \sum_m (\sum_\ell \pi^j_\ell R^\ell_k) R^m_k \pi^k_m \tag{2-12}$$

it differs from $\bar{A}^j$ and thus $\tilde{A}^j$ by at most $O(b^2)$ and can be shown to have the properties assumed in its computation.

It is worth reflecting on how (2-12) solves the infinite regress problem that plagues game theory. It says that if firms are willing to extrapolate linearly they might as well assume in addition that in the future all firms will behave myopically--that (2-10) will hold without the error term. The point is that while each firm's behavior depends on what it thinks its opponents thinks it thinks, etc. in a model passing through time, the higher order terms in this chain of double-think have consequences which are in the distant future. While the optimal path for j depends on what k thinks j thinks

as represented by the error term in (2-10) this has such a small impact on k's behavior, j may as well ignore it entirely. Thus, the infinite regress is replaced with a finite regress and at the final stage opponents assume everyone is myopic and a unique solution is determined.

My view is that when firms enter an economic environment they do so with the working hypothesis that they can extrapolate linearly following (2-12) without substantial loss. In the environment of this paper, when all firms behave this way, this conjecture is true and firms will never have reason to reject it. In this environment the only economically meaningful adjustment process is that given above.

Nature of the Solution: From (2-4) the partial adjustment equation and (2-12) giving the approximate present value

$$\dot{x}^j = \sum_k R_k^j \tilde{f}^k$$

$$= \sum_k R_k^j \left(b \not{k} \sum_\ell \frac{\partial \hat{A}^k}{\partial x^\ell} \frac{\partial \dot{x}^\ell}{\partial y^k} \right)$$

$$= b \not{k} \sum_k R_k^j \sum_\ell R_k^\ell \delta \pi_\ell^k + O(b^2)$$

$$= b \delta \not{k} \sum_k \sum_\ell R_k^j R_k^\ell \pi_\ell^k + O(b^2) \qquad (2\text{-}13)$$

Since the first term in (2-13) is O(b) while the second

term is $O(b^2)$ to a good approximation[11]

$$\dot{x}^j = b\delta \sum_k \sum_\ell R_k^j R_k^\ell \pi_\ell^k \qquad (2\text{-}14)$$

I am not arguing that firms will ignore the second term--to achieve the same degree of extrapolative accuracy as provided by linear extrapolation; they cannot. I am pointing out instead that the qualitative features of the dynamical system describing the evolution of x and R over time can be understood without reference to the second term in (2-13). In economic terms, I don't care whether a firm's market share is 10% or 12% even though the firm itself may care tremendously. In mathematical terms, including the second term when b is small enough merely perturbs the location of steady states slightly without affecting their stability.[12]

The qualitative analysis in this paper focuses on the case when initial conditions are symmetric.[13] Since the adjustment equations are symmetric the system will remain symmetric for all time. Thus the system is entirely described by just two variables which may be taken to be q, industry output as a fraction of the competitive level given in (1-6), and $r \equiv R_k^j$ $j \neq k$ the common reaction of any firm to any other firm. In this case equation (2-14) simplifies to

$$\dot{q} = (b\delta \beta N/E)[1+(N-1)r][\pi^j+(N-1)r\pi_k^j] \qquad (2\text{-}15)$$

where by symmetry and (1-5) describing the quadratic profit function

$$\pi_j^j = (E/N)[N-(1+N)q]$$

$$\pi_k^j = -(E/N)q \qquad j \neq k \tag{2-16}$$

Turning to $\dot{R}_k^j$ from (2-4) the partial adjustment euqation and (2-12) giving the approximate present value

$$\dot{R}_k^j = \tilde{g}_k^j = b\frac{\partial \hat{A}^j}{\partial R_k^j}$$

$$= b\{-\delta bm\ \mathrm{sgn}(R_k^j)$$

$$+ b\delta^3 \not{k}\,[\textstyle\sum_m R_k^m \pi_j^j \pi_m^k + \pi_j^k \sum_\ell \pi_\ell^j R_k^\ell]\}$$

$$= b^2\delta\{\delta^2 \not{k}\,[\pi_j^j \textstyle\sum_m R_k^m \pi_m^k + \pi_j^k \sum_m R_k^m \pi_m^j] - m\ \mathrm{sgn}(R_k^j)\} \tag{2-17}$$

Notice that since all terms in (2-17) are $O(b^2)$ none can be ignored even as an approximation. This is because frictional reaction costs are assumed small of the same order as the adjustment rate. When these costs are not small (2-17) is dominated by the term $-b\delta m\ \mathrm{sgn}(R_k^j)$ which is special case of (2-17)--the case m large. When frictional reaction costs dominate (2-17) simply says that reaction coefficients are set to minimize the cost of reacting. When reaction costs are small the second term in (2-17)

is important. It shows how a firm should optimally punish and reward opponents--how to change R_k^j so as to get opponents to alter output in a preferred direction. The crucial insight (ignoring frictional response costs) is that the only effect reaction coefficients have is on the future behavior of opponents. As long as firms care about the future at least enough to use linear extrapolation rules it is precisely the effect of reaction coefficients on opponents' output that firms pay attention to in determining how to adjust these coefficients. This should be contrasted to (2-13) where the effect of autonomous output on opponents' future choice is relatively less important than the direct impact from having opponents respond instantaneously according to precommitted coefficients R_k^j.

When initial conditions are symmetric, (2-17) simplifies to

$$\dot{r} = b^2\delta\{\delta^2 R[\pi_j^j(\pi_j^j+(N-1)r\pi_k^j + \pi_k^j(r\pi_j^j+(1+(N-2)r)\pi_k^j)] - m\ \mathrm{sgn}(r)\} \qquad (2\text{-}18)$$

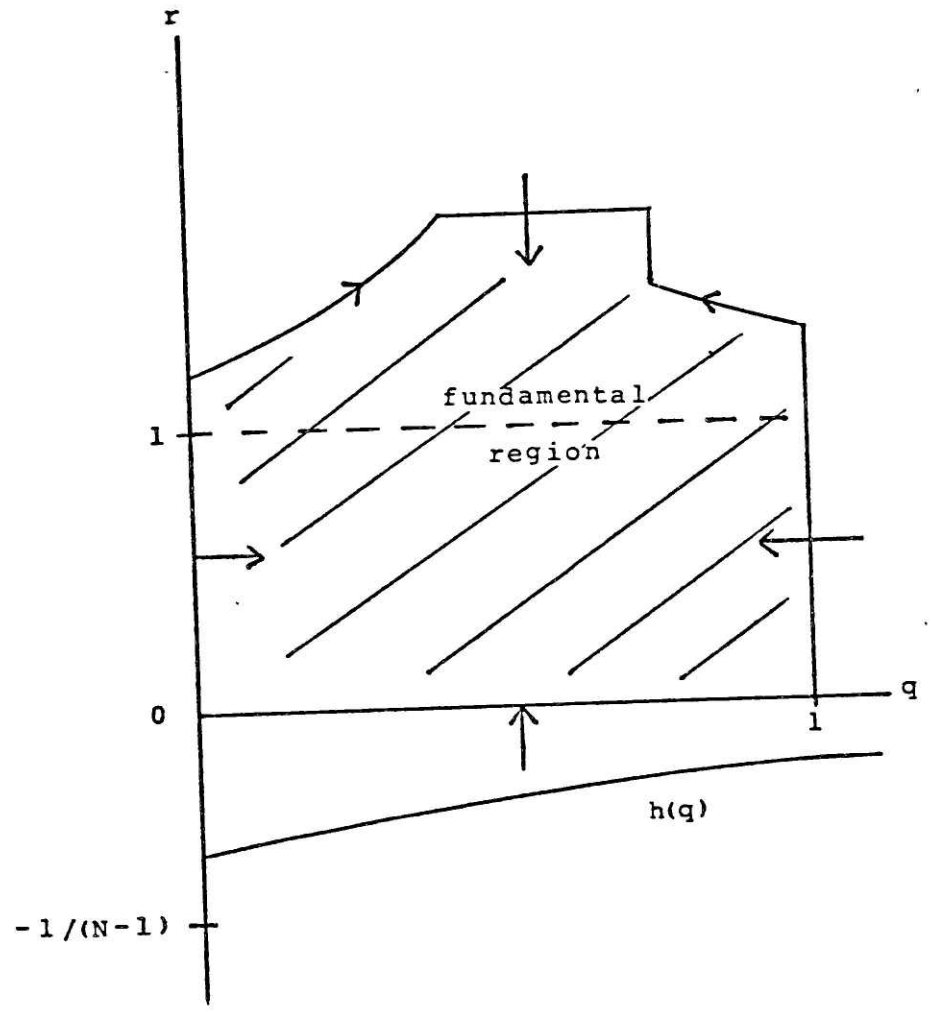
3. Oligopolistic Outcomes

This section studies the equations of motion derived in the previous section. It has three parts. The first part considers some global aspects of the dynamics. It shows that all stable steady states are contained in a compact positive invariant region which is reached (for all initial conditions with r not too greatly negative) in finite time. Part two considers steady state behavior in the short run. In the short run the response rate r is determined by initial conditions while industry output is at the "conjectural variational equilibrium" where firms conjecture that opponents will respond with the variation r. Part three analyzes the long run. In the long run r is determined endogenously: it is this feature which distinguishes this theory from previous oligopoly theories. In the long run, steady state output lies between the monopoly and Cournot-Nash output depending on the exogenous parameters of the market. The most significant result is that when there are no frictional costs of response output is at the monopoly level independent of other market parameters. Several comparative static exercises show how long-run output varies with market parameters when frictional costs are positive.

Global Aspects: Figure (3-1) sketches the fundamental region and the continuous strictly negative function $h(q)$ in q-r space. Note that the fundamental region contains the region $0 \leq r \leq 1$ and $0 \leq q \leq 1$. Appendix (A) derives the properties of this region. They are: (1) it is positive invariant--if the system starts in this region it never leaves it, and (2) if the system starts in the region $r \geq h(q)$ it reaches the fundamental region in finite time.

Intuitively what this analysis indicates is that we should expect to observe $r \geq 0$ but not too large and $0 \leq q \leq 1$. The latter property is sensible: since q is industry output as a fraction of the competitive output, and since frictional costs are a negligible component of total costs $q \leq 1$ is (approximately) the region of non-negative profits. The region $r < 0$ does not make economic sense: when r is negative each firm rewards its opponents for hurting it and punishes its opponents for cooperating. Finally if r is extremely large each firm gives its opponents such large rewards for cutting their output that industry output will actually be below the monopoly level--not a desirable situation from anyone's point of view.

This analysis has one drawback. If the system starts with $r < h(q)$ the fundamental region may never be reached. In fact there can be steady states with $r < 0$. Evidently the myopic behavior of firms can cause the system to get stuck in a region which is, in the long run, undesirable. It is shown in Appendix (B), however, that any steady



Figure(3-1): The Fundamental Region

state with $r < 0$ is unstable. I conjecture also that the set of initial conditions with $r < h(q)$ for which the system doesn't reach the fundamental region has measure zero so that small random perturbations would eventually cause the fundamental region to be reached.

The fact that the system could in principle remain forever in a region where firms make negative profits does point up a weakness in the preceding formulation. In most cases it is reasonable to suppose that firms know only the local effect of small changes near the status quo. The important exception is that firms know they can make zero profits by closing down. This can be integrated into the analysis here by simply assuming firms shut down whenever $\hat{A}^j$ the extrapolated present value becomes negative.

In the remainder of the section attention is restricted to the fundamental region.

The Short Run: Inspection of the equations of motion (2-15) and (2-18) shows that q adjusts at a speed proportional to b while $\dot{r}$ is proportional to b^2. Since b is assumed small this means q adjusts much more quickly than r. In the short run q moves rapidly towards the curve $\dot{q}=0$, while r is close to its starting value. Examination of (2-15) shows that when $r > -1/(N-1)$ $\dot{q}=0$ implies

$$[\pi^j_j + (N-1)r\pi^j_k] = 0 \qquad (3-1)$$

This is exactly the first order condition for choosing the optimal output level subject to the conjecture that opponents of j respond to output changes Δx^j by $r\Delta x^j$. Thus, in the short run, the steady state resembles the traditional conjectural variational equilibrium, as described for example by Seade [17].

From (3-1) and (2-16) giving the profit derivatives the curve $\dot{q}=0$ is seen to be

$$r = \frac{N - (1+N)q}{(N-1)q} \tag{3-2}$$

which is sketched in figure (3-2). The curve strictly decreases. This reflects the fact that the more j's opponents reward him for cutting output by responding with output cuts of their own, as reflected in large r, the lower j will choose to set his output. Note that (2-15) is only an approximation to (2-14)--incorporating the extra term of order b^2 would have the effect of shifting the curve (3-2) by an amount proportional to b.

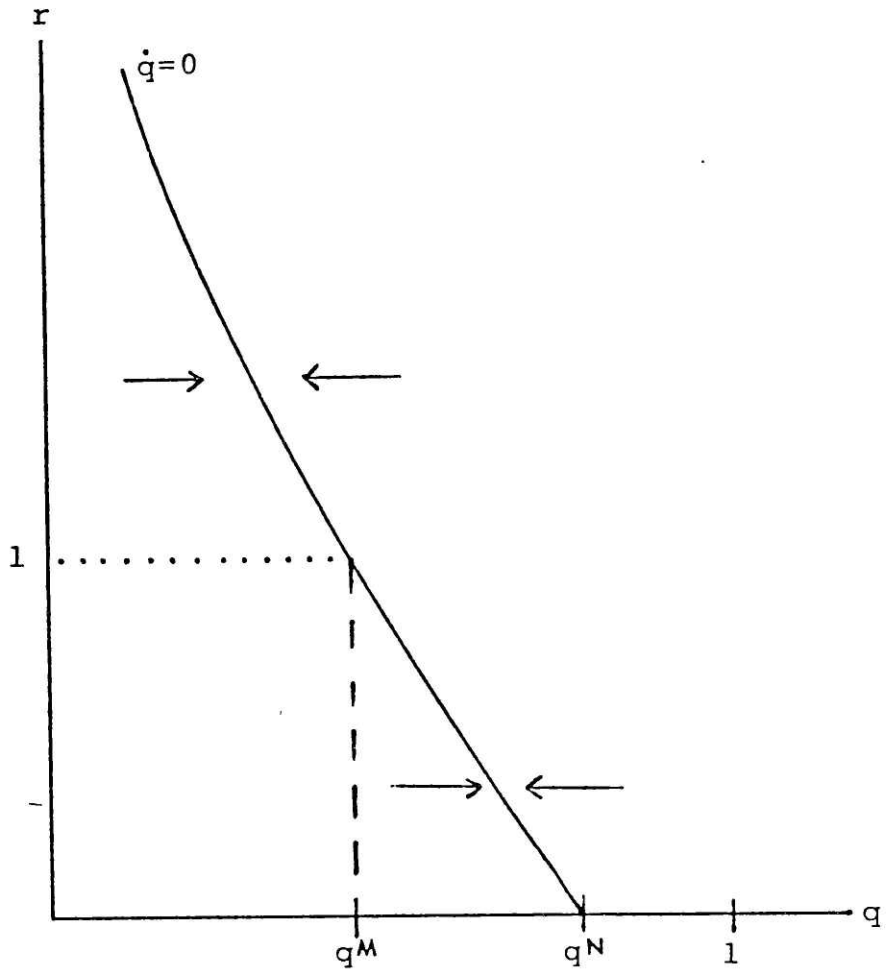
At the monopoly output

$$q^M \equiv (1/2) \tag{3-3}$$

the reaction r=1 while at the Cournot-Nash output

$$q^N \equiv \frac{N}{1 + N} \tag{3-4}$$

the reaction r=0. Finally, from (2-15) and (2-16)



Figure(3-3): The Short Run Steady State Curve

$$\frac{\partial \dot{q}}{\partial q} = -(b\delta \not{k} \beta)(1+(N-1)r)[(1+(N-1)r) + N] \qquad (3\text{-}5)$$

which is negative for $r > -1/(N-1)$. Thus in the short run q moves towards the curve $\dot{q}=0$, and the short-run steady states are globally stable.

The Long Run: The theory of oligopoly developed in the previous sections differs from orthodox theory because in the long run r is determined endogenously. In addition to satisfying $\dot{q}=0$, in the long run steady state $\dot{r}=0$ must be satisfied. Using the equation of motion for r (2-18) and the profit derivatives (2-16) shows that $\dot{r}=0$ along the curve

$$r = \frac{(mN^2/E^2\delta^2 \not{k}) - [(N^2+2N+2)q^2 - 2N(1+N)q + N^2]}{[(N^2+2N-2)q - N^2]q} \qquad (3\text{-}6)$$

Equating (3-6) with (3-2) and solving gives long-run steady state output as

$$q^S = \frac{3}{4} - \sqrt{\frac{1}{16} - M} \qquad (3\text{-}7)$$

$$q^U = \frac{3}{4} + \sqrt{\frac{1}{16} - M} \qquad (3\text{-}8)$$

where the constant

$$M \equiv \frac{(N-1)m}{2E^2\delta^2\ell} > 0 \qquad (3\text{-}9)$$

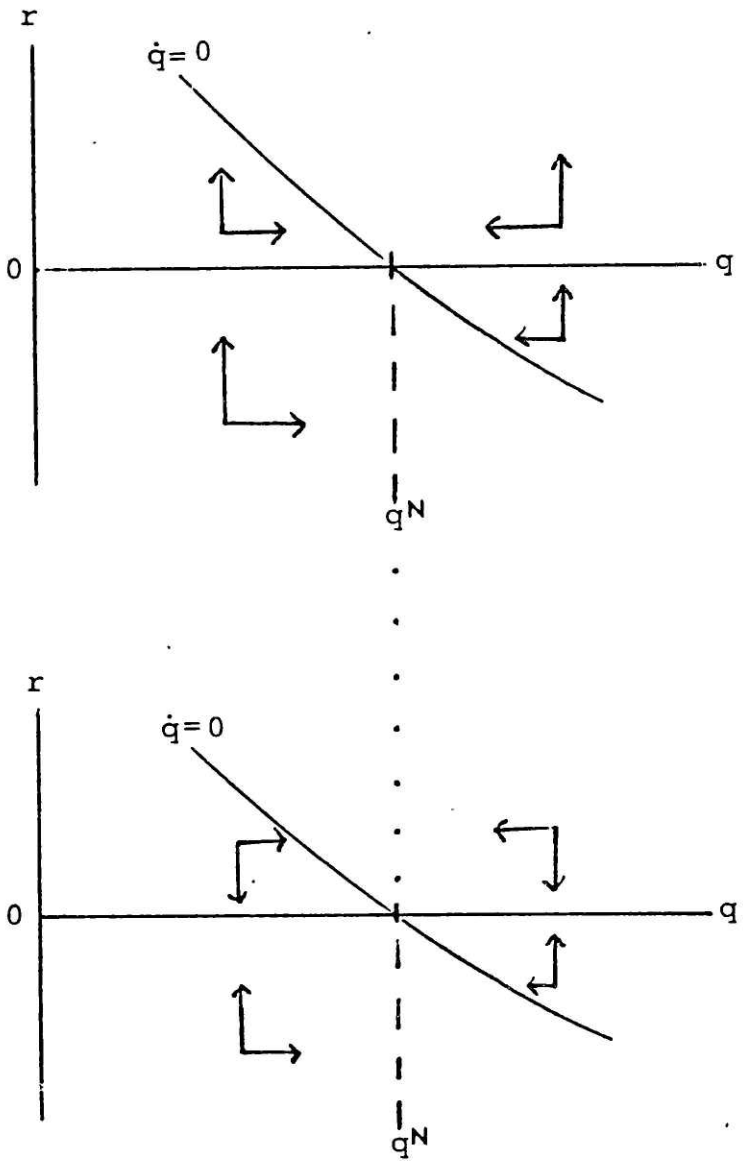
Since $r \geq 0$ is assumed, by (3-2) these steady states exist only when the corresponding output is no greater than q^N. In addition neither steady state exists if $M > (1/16)$. Appendix (C) shows that when they exist the steady state at q^S is stable and the steady state at q^U unstable.

There is one other possible steady state. There can be a steady state along r=0 since $\dot{r}=0$ is discontinuous there. Inspection of figure (3-2) shows that if this occurs it must be at the Cournot-Nash output q^N. As shown in figure (3-3) there will be a steady state at r=0, $q=q^N$ if and only if when evaluated at $q=q^N$ and $r > 0$ but small $\dot{r} < 0$. If this steady state exists, it is stable as figure (3-3) shows. A computation using (2-18) and (2-16) shows that for

$$M^N \equiv \frac{N-1}{2(1+N)^2} \qquad (3\text{-}10)$$

the condition for a steady state at q^N is $M > M^N$.

The next stage of analysis is to determine how the location of steady states depends on M, N fixed. After showing industry output as a fraction of the competitive level increases in M the issue of how M (and therefore market competitiveness) depends on market parameters is addressed. There are two cases in analyzing steady state



Figure(3-3): The Flow near q^N

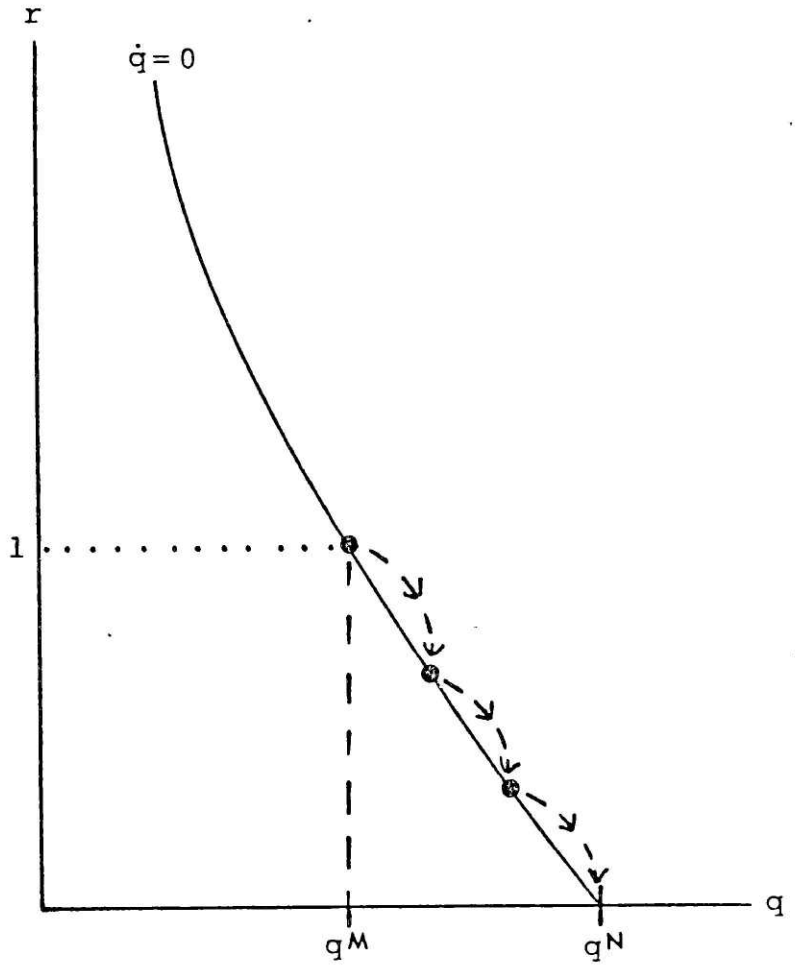
output N=2,3 and $N \geq 4$. The economic significance of the cutoff value N=4 is doubtful--it is probably an artifact of the choice of functional form.

Examination of (3-8) shows that when N=2,3 $q^U \geq q^N$ and thus there is no steady state at q^U. Examination of (3-7) shows q^S increases in M. When M=0, $q^S=q^M$. As M increases q^S increases until it reaches q^N when $M=M^N$. Thus when N=2,3 there is a unique steady state with $r \geq 0$, and it is stable. For $0 \leq M < M^N$ the steady state output is given by q^S. For $M \geq M^N$ steady state output is at the Cournot-Nash level. This situation is illustrated in figure (3-4).

When $N \geq 4$ the situation is more complex. Define the cutoff point

$$M^B = (1/16) \qquad (3\text{-}11)$$

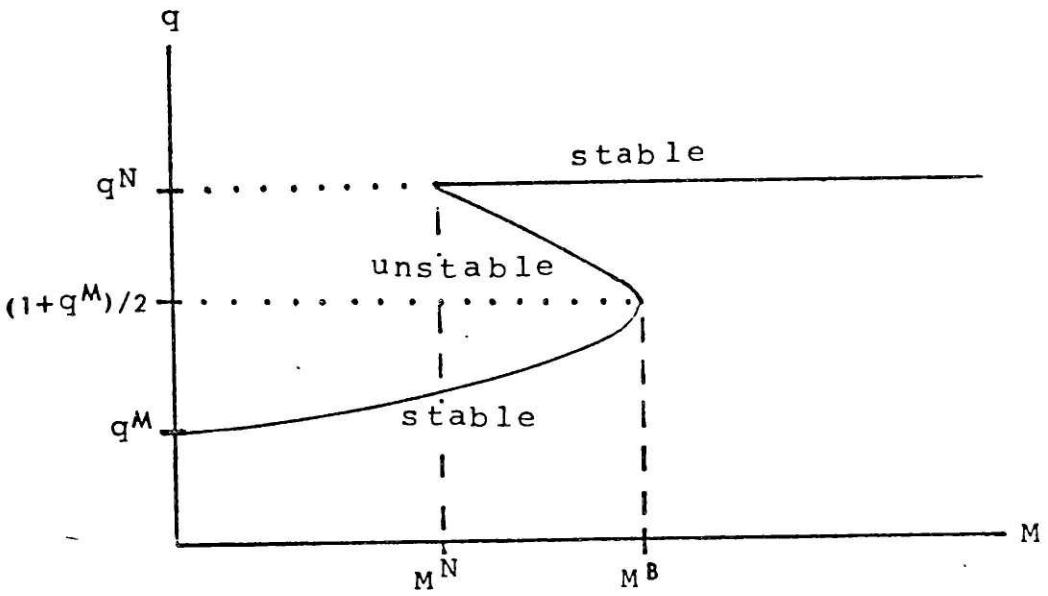
When M=0 there is a unique steady state (which is stable) at $q^S=q^M$ and r=1. As M increases to M^N there is still a unique steady state (stable) with output q^S. For $M^N < M < M^B$ there are three steady states with output q^S (stable), q^U (unstable) and q^N (stable). When $M > M^B$, q^S and q^U meet at $(3/4)=(q^M+1)/2$ and vanish leaving just one steady state (stable) at q^N. Figure (3-4) diagrams steady state output. As in the case N=2,3 output increases in M. However a discontinuity in steady state output can occur when $M=M^N$ or $M=M^B$ as the system jumps from one steady



Figure(3-4): The Case N=3,4$_S$
arrows denote movement in q^S for N fixed as M increases

state to another. In the language of catastrophe theory this is the double fold catastrophe.[12]

From (3-4) M and thus industry output as a fraction of the competitive level increases in N and m and decreases in E, δ and $\hbar$. That increases in N lead to more competition shouldn't be too surprising. In this model m represents the marginal cost of enforcing a collusive arrangement--of increasing R_k^j. Thus when m is large there is less collusion. There is also a public goods problem in allocating enforcement costs among firms. When N is large each firm has less incentive to bear its share of the burden and long-run output is greater. The fact that competitiveness declines in E is perhaps a bit surprising. This happens for two reasons. First, m/E, which is the cost of enforcing collusion divided by a variable describing the marginal profitability of increasing output, declines. Thus E serves to scale m: it isn't enforcement costs, but the ratio of marginal enforcement costs to marginal market profitability that matters. Second, from the equation of motion for q (2-15), when E increases, firms adjust output more quickly due to the increased marginal profits from doing so. Increasing $\hbar$ also increases the speed with which firms adjust. Why does more rapid adjustment of output enhance collusion? The benefit of setting high values of R_k^j and thus realizing relatively collusive arrangements lies in the effect this has on opponents' <u>future</u> output. The more quickly



Figure(3-5): Steady State Industry Output

N≥4 and held fixed

they adjust, the more quickly these benefits are realized and with discounting fixed, the more valuable they are. Naturally, increasing the discount factor δ has the same effect, placing more weight on the future benefits of collusion, less on the immediate costs. Thus raising E, R or δ all have the effect of reducing competitiveness.

4. Duopoly without Communication

One drawback of the preceding analysis is that it requires that firms distinguish between voluntary output changes by rivals and reactions by rivals to voluntary output changes. Without communication between firms this distinction is impossible. This section analyzes a duopoly in which the firms observe each other's output, but cannot communicate. It has four parts. Part one describes an environment in which each firm reacts after an infinitesimal lag to its rival's observed output change. Part two derives the local almost perfect adjustment path for firms. Part three computes steady state output. Part four compares this output with the steady state output in the model of section three.

The Environment: Consider first an infinitesimal reaction lag, Δt. Firms react to opponents' output change in the previous period

$$\Delta x^j(t+\Delta t) = y^j + \bar{R}^j_k \Delta x^k(t) \qquad j \neq k \tag{4-1}$$

where y^j is firm j's voluntary output change and $\bar{R}^j_k$ is its (structural) reaction coefficient. Define

$$\bar{R} \equiv \begin{bmatrix} 0 & \bar{R}_2^1 \\ \bar{R}_1^2 & 0 \end{bmatrix} \tag{4-2}$$

Let

$$\lambda(\bar{R}) \equiv \sqrt{|\bar{R}_2^1 \bar{R}_1^2|} \tag{4-3}$$

be the largest absolute eigenvalue of $\bar{R}$. Then for $\lambda < 1$ the discrete time dynamical system (4-1) converges to

$$\Delta x = Ry \tag{4-4}$$

where

$$R \equiv [I - \bar{R}]^{-1} \tag{4-5}$$

so that the (reduced form) reaction coefficients are

$$R_j^j = 1/(1-\lambda^2) \qquad R_k^j = \bar{R}_k^j/(1-\lambda^2) \tag{4-6}$$

and

$$\partial R_m^\ell / \partial \bar{R}_k^j = R_j^\ell R_m^k \tag{4-7}$$

Thus in the continuous time limit when $\lambda < 1$

$$\dot{x}^j = \textstyle\sum_k R_k^j y^k \tag{4-8}$$

while if $\lambda > 1$ the system explodes instantaneously. Since with even small adjustment costs an explosion is infinitely costly to all firms $\lambda \leq 1$ is assumed to act as a constraint on the system: firms will never choose to permit $\lambda > 1$.

Finally, letting S^j_k be the rate of change of $\bar{R}^j_k$ the rate of change of R^ℓ_m by (4-7) is

$$\dot{R}^\ell_m = R^\ell_j R^k_m S^j_k + R^\ell_k R^j_m S^k_j \tag{4-9}$$

Thus when $\lambda < 1$ the state equation for x remains unchanged from (1-2) while (4-9) replaces (1-3).

The frictional reaction cost function is taken to be

$$bC^j(R^j) = bm(|R^j_j - 1| + |R^j_k|) \tag{4-10}$$

which vanishes when no reaction takes place--when $\bar{R}^j_k = 0$. This is analogous to (1-7) except that it is no longer necessarily true that R^j_j is equal to one. The reason that frictional costs are taken to depend on the reduced form reaction coefficients is that it is these coefficients, rather than the structural coefficients, which magnify short-run fluctuations in autonomous output and thus determine the level of inventories that must be held.

The Local Almost Perfect Adjustment Equations:

Provided that the coefficients of section two are interpreted

as reduced form coefficients, the reasoning behind the derivation of (2-13) and (2-14) describing $\dot{x}^k$ remains unchanged and

$$y^k = \tilde{f}^k \cong b\delta \not{k} \sum_\ell \pi^k_\ell R^\ell_k \qquad (4\text{-}11)$$

is the approximate equation for autonomous output change. With symmetric initial conditions define

$$r = \bar{R}^j_k = R^j_k / R^j_j \qquad (4\text{-}12)$$

and observe that

$$R^j_j = 1/(1-r^2) \qquad (4\text{-}13)$$

Thus (4-11) reduces to

$$y^k = \frac{b\delta \not{k}}{1 - r^2} \quad [\pi^k_k + r\pi^k_\ell] \qquad (4\text{-}14)$$

Turning to S^j_k as in (2-17)

$$S_k^j = b\sum_\ell \sum_m \frac{\partial \hat{A}^j}{\partial R_m^\ell} \frac{\partial \dot{R}_m^\ell}{\partial S_k^j}$$

$$= b\left\{-\delta bm\left[\operatorname{sgn}(R_j^j - 1)\frac{\partial R_j^j}{\partial S_k^j} + \operatorname{sgn}(R_k^j)\frac{\partial \dot{R}_k^j}{\partial S_k^j}\right]\right.$$

$$\left. + \delta^2 \sum_h \pi_h^j \frac{\partial}{\partial S_k^j}\left(\sum_\ell R_\ell^h y^\ell\right)\right\} \qquad (4\text{-}15)$$

$$= b\delta\{\delta\sum_h \pi_h^j \sum_\ell [b\delta R_\ell^h \sum_m \pi_m^\ell R_j^m R_\ell^k + y^\ell R_j^h R_\ell^k]$$

$$- bm[R_j^j R_k^j + \operatorname{sgn}(R_k^j)(R_j^j)^2]\} \qquad (4\text{-}16)$$

where (4-8) and (4-9) are used and $\operatorname{sgn}(R_j^j - 1) = 1$ from (4-6) and $\lambda \leq 1$.

The Steady State: Reasoning as in appendix (A) shows that the region $r < 0$ is of no interest. When $r \geq 0$ (4-8) shows that

$$y^k = 0 \qquad S_k^j = 0 \qquad j,k = 1,2 \qquad j \neq k \qquad (4\text{-}17)$$

determine the position of the steady state. The condition $y^k = 0$ means from (2-16) giving the profit derivatives

$$r = \frac{2 - 3q}{q} \qquad (4\text{-}18)$$

which is (3-2) in the duopoly case. Using $y^k=0$ and (4-14) to cancel terms in (4-16) and equating the resulting expression to zero gives the second equation for the steady state

$$S_k^j = b^2\delta\left\{\frac{\delta^2 k}{(1-r^2)^3}\ [r\pi_j^j + \pi_k^j]^2 - \frac{m}{(1-r^2)^2}\ [1+r]\right\} = 0 \qquad (4\text{-}19)$$

Using (4-18) and (4-19) the steady state output is computed to be

$$q^{NC} = \frac{1}{4} + \sqrt{\frac{1}{16} + 2M} \qquad (4\text{-}20)$$

since the second root that occurs when (4-18) and (4-19) are solved has negative levels of output for $M > 0$. Here M is as in (3-9)

$$M \equiv \frac{m}{2E^2\delta^2 k} \qquad (4\text{-}21)$$

Comparison to the Previous Model: From section three equilibrium output when N=2 is

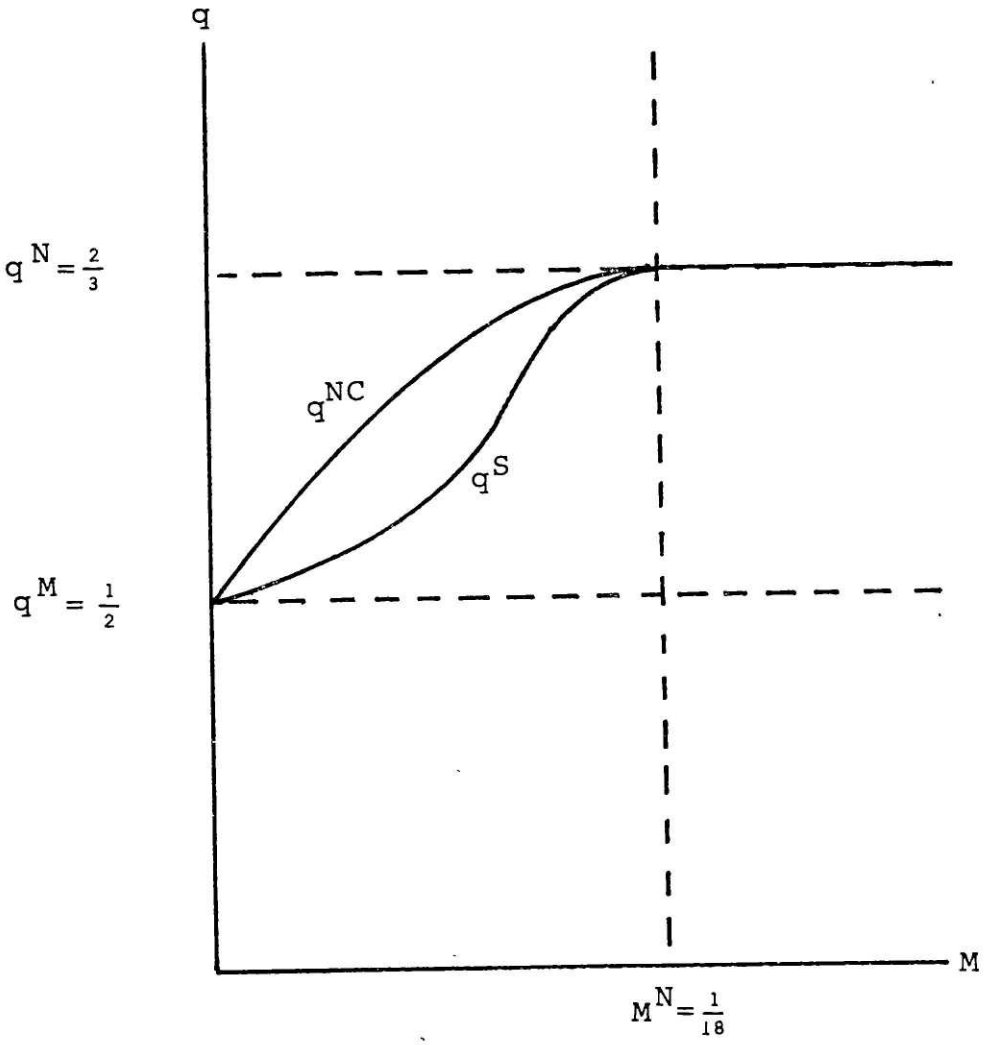
$$q^S = \frac{3}{4} - \sqrt{\frac{1}{16} - M} \qquad (4\text{-}22)$$

by (3-7). Equating (4-20) and (4-21) shows $q^{NC}=q^S$ if

and only if either M=0 in which case $q^{NC}=q^{S}=q^{M}$ or if $M=M^{N}$ in which case $q^{NC}=q^{S}=q^{N}$. Evaluating $\partial q^{NC}/\partial M = 4$ and $\partial q^{S}/\partial M = 2$ at M=0 shows that for $0 < M < M^{N}$ $q^{NC} > q^{S}$. Naturally, for $M \geq M^{N}$ output in both cases is at the Cournot-Nash level q^{N}. This situation is illustrated in figure (4-1).

In economic terms what is going on? Without communication the firms can't distinguish autonomous from total output movements. Thus they counter-react to each other's reactions. This has two effects. First, the incentives for choosing S_k^j are complicated by the fact that the rates of change of <u>all</u> the reduced form reaction coefficients are affected by S_k^j as (4-9) shows. In the previous case only $\dot{R}_k^j$ depends on S_k^j. Actually, this, while complicating (4-16) greatly, is of no economic significance: each firm can still control its rival's output by rewarding and punishing it, just as before.

The second consideration is that to achieve a given level of response as measured by R_k^j/R_k^k the sequence of counter-reactions forces R_j^j to be greater than one. The sequence of counter-reactions has the effect of magnifying reaction costs by enlarging the reduced form reactions required to enforce a particular output level along $\dot{q}=0$. Except when m=0 and there are no reaction costs to be magnified or r=0 so that no one reacts, firms have an incentive to react less strongly than in the previous section to avoid the increased marginal cost of response.



Figure(4-1): Output with and without Communication

Hence long-run output is greater without communication.

How significant is the difference between long-run output with and without communication? A calculation shows the largest case is M=(1/32) in which case $q^{NC}-q^{S}$ is 3% of competitive output. This is a modest effect relative to the gap between monopoly output q^{M} and Cournot-Nash output q^{N}, which is 17% of competitive output.

5. Facts and Myths about Oligopoly

The results of this paper contradict a number of widely believed myths about oligopoly. To conclude the paper I debunk several of these myths.

Infinite Response is Optimal: Frequently in seminars I have been told that by making sufficiently large threats against opponents a firm can get them to do anything it wants. Thus, when there are no frictional costs of response, an infinite response is optimal. Our results indicate that in the context of this paper the argument is false--the long-run steady state reaction coefficient (r) was computed equal to one. It is true in this paper that each firm controls the output of all its rivals when they follow the equilibrium adjustment procedure. The objective, however, is profit and not control of opponents output. A firm controls its rival's output only by losing control of its own output. It can drive opponents' output to zero, and perhaps even increase its market share while doing so, but to do so it must cut its own output. A large share of small profits doesn't amount to much. Of course if a firm could convince its opponents to leave the market without changing its own output it would do so--but it's hard to see how it could do so, and in the environment here it certainly can't.

A more sophisticated version of this myth holds that the way to convince opponents to leave the market is to first increase output a great deal, then begin the predatory policy outlined above. The initial increase in output insures that even when his opponents are driven from the market the predator will still have substantial sales. The reason such a policy does not work in the environment of this paper is that the potential predator's opponents are committed to retaliatory policies of their own. They will match the predator's initial output increase causing prices to drop precipitously. Although in the long run his rivals may be driven out of the market the short-run losses incurred by the predator more than offset these future gains.

If a firm can permanently commit itself to a policy of predatory threats, and if opponents adjust instantly to this policy, one firm may successfully dominate the market. In the environment here, as in real life, this possibility cannot occur.

Negative Response is Optimal: This myth might best be labelled the Cournot-Nash confusion. The assertion is that when an opponent increases output it is optimal to reduce output. In this paper, however, the converse is true--all stable steady states have non-negative response rates. Why is this?

Lowering output in response to an opponent's increase has two effects: it increases profits, and it encourages

the rival firm to increase output even further. At a steady state only the latter effect matters--rival firms aren't going to change output unless encouraged to do so. The Cournot-Nash confusion ignores the fact that retaliation affects the behavior of opponents. Only when it does not should opponents output increases be met with reductions.

Negotiation Matters: The institutional industrial organizational literature[15] frequently distinguishes between explicit collusion where firms negotiate output shares and implicit collusion where they do not. It argues that in the former case firms will always collude fully because it is "jointly optimal" to do so. Of course if firms can enter into legally binding contracts which can be costlessly enforced, this might be true, but is it true without legally binding contracts? The error does not lie in believing negotiation matters. Communication can (as we saw in section four) enhance collusion. It is also possible that explicit negotiations can reduce frictional costs of enforcing collusive arrangements. But does explicit collusion invalidate the results of this paper?

The key question is: how is collusion enforced? Even if an agreement is reached what keeps firms from cheating on it? Once an agreement is reached, firms will cheat and other firms retaliate--this process of learning and enforcement is what this paper is all about. A negotiated agreement could determine the starting point

for the dynamical system described here, but those equations of motion will determine the history of the industry from that point on. It is a mistake to think that talk alone will cure the problem of enforcing collusive arrangements.

Facts about Oligopoly: Learning is a time-consuming activity. Because it takes time to reoptimize, firms are implicitly committed to the status quo. This paper has examined how firms (almost) rationally choose these implicit commitments. The result is a simple sensible story of oligopoly: firms punish uncooperative opponents and reward cooperative ones. In the long-run steady state this implies that market competitiveness depends on the frictional costs of enforcing collusive arrangements.

Appendix A--The Fundamental Region

The objective is to verify that the fundamental region is as described in section three. This is done via a series of figures.

Figure (A-1) shows that $\dot{q} > 0$ along $q=0$ for $r > -1/(N-1)$; $\dot{q} < 0$ along $q=1$ and that $\dot{r} > 0$ for $r < 0$ and sufficiently small in absolute value. Let us verify these. For $q=0$ from (2-15) and (2-16) describing $\dot{q}$

$$\dot{q} = (b\delta \not{k} \beta)(1+(N-1)r)N \tag{A-1}$$

which is positive for $r > -1/(N-1)$. Along $q=1$

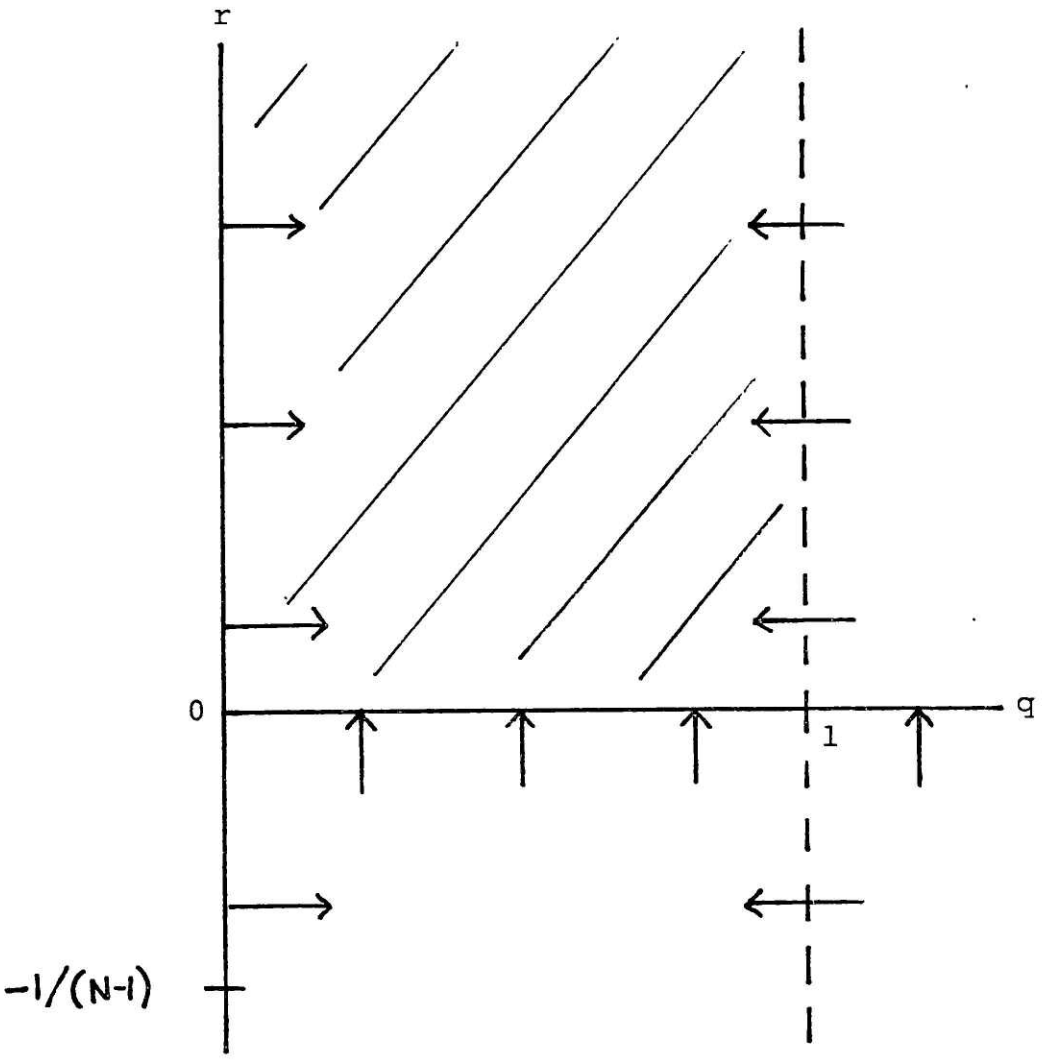
$$\dot{q} = -(b\delta \not{k} \beta)(1+(N-1)r)^2 \tag{A-2}$$

which is negative. Along $r=0$ from (2-18) describing $\dot{r}$

$$\dot{r} = b^2\delta\{\delta^2 \not{k} [(\pi_j^j)^2 + (\pi_k^j)^2] - m \text{ sgn } r\} \tag{A-3}$$

which is strictly positive when $r=-\varepsilon$ since by (2-16) π_j^j and π_k^j don't vanish simultaneously.

Now let us verify that any path starting with r not too negative reaches the shaded region $0 \leq q \leq 1 \quad r \geq 0$ in figure (A-1) in finite time. This requires two steps: (1) showing that $r < 0$ but small enough in absolute



Figure(A-1): Simple Global Features of the Flow

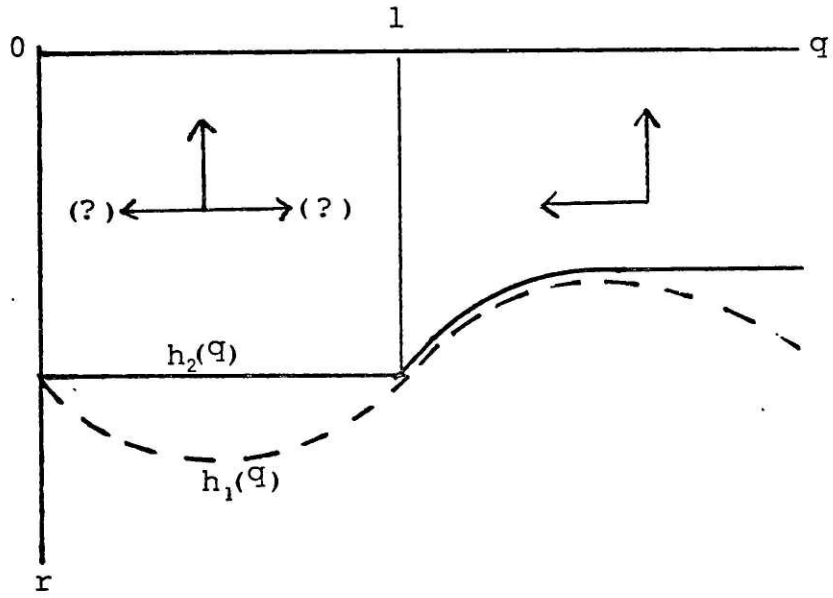
value implies $r \geq 0$ is reached in finite time and that (2) the shaded region is reached from any point in the region $q \geq 1$, $r \geq 0$ in finite time. The latter is easy. From (A-2) along $q=1$ $r \geq 0$ $\sup(\dot{q}) = \dot{q}^m < 0$; from (3-2) and (3-5) in the text this implies that for $q \geq 1$ $r \geq 0$ $\sup(\dot{q}) = \dot{q}^m$ also. Thus starting at q in this region it takes no longer than $(q-1)/|\dot{q}^m|$ to reach the shaded region.

Showing $r \geq 0$ reached in finite time from $r < 0$ and small is slightly more difficult. By (A-3) there is a function $h(q)$ such that for $0 > r > h(q)$ $\dot{r}(r,q) > 0$. Since $\dot{r}$ is continuous in q and r for $r < 0$ and since

$$\lim_{\substack{r\to 0 \\ r<0}} \dot{r}(r,q) > 0 \qquad \text{(A-4)}$$

again by (A-3), we may assume $h(q)$ is a continuous function and that $0 > r > h(q)$ implies $\dot{r}(r,q) > \varepsilon$ for some fixed $\varepsilon > 0$. Since $h(q)$ is continuous we can also assume $h(q)$ constant and greater than $-1/(N-1)$ on $0 \leq q \leq 1$ and strictly increasing for $q > 1$. This is illustrated in figure (A-2). Examining that figure and observing as above that for $q > 1$ $r > -1/(N-1)$ $\dot{q} < 0$ we see that the system once in the region $0 > r > h(q)$ can leave only if r becomes non-negative. But this takes no more time than $1/\varepsilon(N-1)$.

Finally we study the shaded region. First, we show that every path in the shaded region remains bounded. For if not along that path $r \to \infty$ and it must be that



Figure(A-2): The Case $r < 0$

observe that if $h_1(q)$ has $\dot{r} > \varepsilon$ for $0 > r > h_1(q)$
then $\dot{r} > \varepsilon$ for $0 > r > h_2(q)$ as well

$\dot{r}/\dot{q}$ is unbounded. But as $r \to \infty$ from the equations of motion (2-15) and (2-18) and the profit derivatives (2-16).

$$\frac{\dot{r}}{q} \to \left(\frac{b\delta^2}{k\beta N}\right) \frac{r\pi_k^j(N\pi_j^j + (N-2)\pi_k^j)}{r^2(N-1)^2\pi_k^j}$$

$$= \left(\frac{b\delta^2}{k\beta N(N-1)^2}\right) \frac{(N\pi_j^j + (N-2)\pi_k^j)}{r} \to 0 \qquad \text{(A-5)}$$

a contradiction. Next observe that

$$\frac{\partial \dot{r}}{\partial r} = (b^2\delta^3 k E^2/N^2)[(N^2+2N-2)q - N^2]q \qquad \text{(A-6)}$$

from (2-16) and (2-18). It is easy to check that for

$$q^r \equiv \frac{N^2}{N^2 + 2N - 2} > 0 \qquad \text{(A-7)}$$

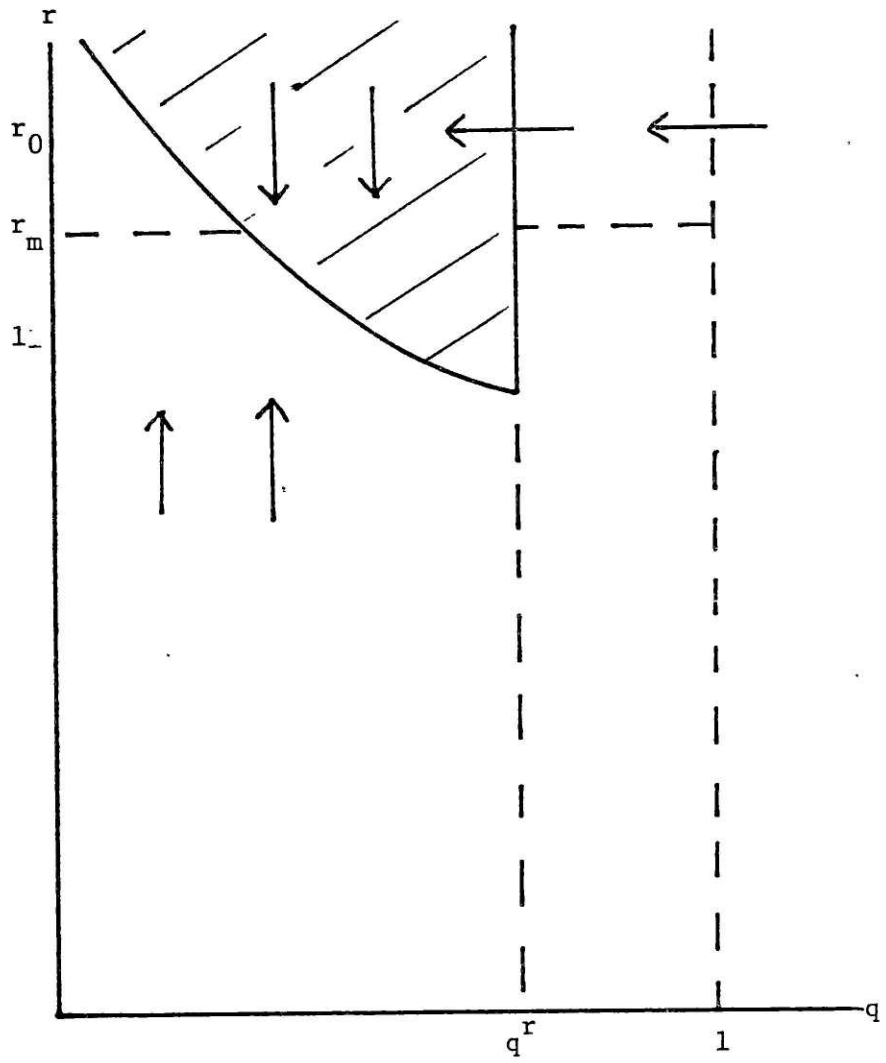
$0 < q < q^r$ implies $\partial\dot{r}/\partial r < 0$. This leads us to figure (A-3). The curve $\dot{r}=0$ from (3-6) is sketched. In the shaded region $\dot{r} < 0$. Note that the shaded region may reach the $q=0$ axis. This doesn't affect the analysis. As indicated in the figure for some $r_m > 1$ and $r \geq r_m$, $q^r \leq q \leq 1$ $\dot{q} < 0$ must hold--this can be verified from figure (3-2) in the text. Also for $0 \leq q < q^r$ and r below the $\dot{r}=0$ curve $\dot{r} > 0$. I next show how to construct the fundamental region.

Consider first the segment $r=r_m$ $q^r \leq q \leq 1$. Every path beginning here is bounded and $\dot{q} < 0$ whenever the path is above r_m and to the right of q^r. This implies that there is a continuous curve C_1 beginning at q=1, $r=r_m$ which meets the shaded region in figure (A-3) and which the flow does not cross from below. Let r_1^m be the r-coordinate where C_1 meets the shaded region. This is shown in figure (A-4).

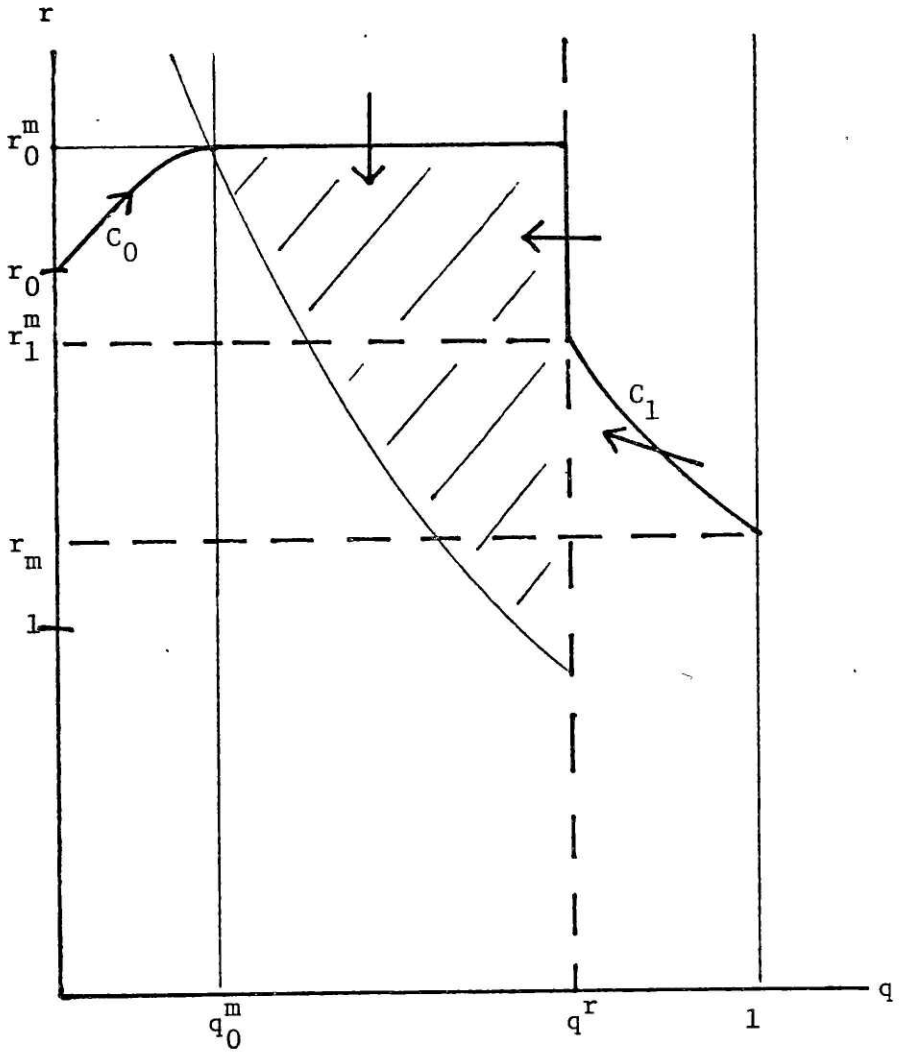
Next consider the path beginning at $r_0 \geq r_1^m$ shown in figure (A-4). Since $\dot{r} > 0$ and the path is bounded above, this curve meets the shaded region at some ordinate $r_0^m \geq r_0 \geq r_1^m$. Since this is an integral path it is not crossed by the flow. Comparing figures (A-3) and (A-4) we see that the piecewise continuous curve made up of C_0, the segment $r=r_0^m$, $q_1^m \leq q \leq q^r$, the segment $q=q^r$ $r_0^m \leq r \leq r_1^m$ and the curve C_1 connects q=0 and q=1 and is crossed only from above. This defines the upper boundary of the fundamental region shown in figure (3-1).

It remains to show that paths beginning above the fundamental region F reach it in finite time. To show this we observe that the ω-limit set of such a path is bounded, and therefore a compact non-empty set W. We now apply some results on planar systems from Hirsch and Smale [8] chapter 11.

We may as well assume $W \cap F = \phi$, otherwise the path reaches F in finite time since no limit point lies on the boundary of F. By the Poincaré-Bendixon theorem either W



Figure(A-3): Behavior of the Reaction Coefficient



Figure(A-4): Bounding the Fundamental Region

contains a steady state or it is a closed orbit with a steady state in its interior: either case implies that the region above F contains a steady state. However, every steady state has $r \leq 1 < r_m$ by the results of section three and this contradiction establishes the required result.

Appendix B--Instability with Negative Response

The objective is to show that any steady state with $r < 0$ is unstable.

The starting point of the analysis is the equation of motion for $\dot{q}$. From (2-15) and (2-16) this is

$$\dot{q} = (b\delta k\beta)(1+(N-1)r)(N-(1+N)q-(N-1)rq) \qquad \text{(B-1)}$$

Inspection shows $\dot{q}=0$ either when $r = -1/(N-1)$ or along the curve given in (3-2) as

$$r = \frac{N - (1+N)q}{(N-1)q} \qquad \text{(B-2)}$$

Since this curve strictly decreases and $r \to -(1+N)/(N-1)$ as $q \to \infty$, we may assume $0 > r > -(1+N)/(N-1)$. There are three cases $0 > r > -1/(N-1)$; $r = -1/(N-1)$ and $-1/(N-1) > r > -(1+N)/(N-1)$.

Case 1: $0 > r > -1/(N-1)$

Then any steady state lies along (B-2) implying $q^N < q < 1$. Using (2-16) and (2-18) to find the motion of r, and substituting in (B-2) shows the steady state is at

$$q^1 = \frac{3}{4} \pm \sqrt{\frac{1}{16} + M} \qquad \text{(B-3)}$$

where M is as in (3-9). Since $M > 0$ either

$$q > 1 \tag{B-4}$$

or

$$q < (1/2) = q^M < q^N \tag{B-5}$$

So in either case there is a contradiction. Thus in case (1) there can be no steady state.

Case 2: $r = -1/(N-1)$

Using the given value of r and the equation of motion for r from (2-16) and (2-18) shows that at a steady state

$$[N^2-2]q^2 - [2N^2-N-2]q + N(N-1) = -\bar{M} \tag{B-6}$$

where $\bar{M} > 0$ is a constant. Inspection of the polynomial in (B-6) shows that the steady state value of q must then lie in the interval

$$1 > q > \frac{N(N-1)}{N^2 - 2} > \frac{N^2}{N^2 + 2N - 2} = q^r \tag{B-7}$$

where q^r was defined in appendix (A) in (A-7). But the analysis of that section, and (A-6) in particular, showed that $q > q^r$ implies $\partial\dot{r}/\partial r > 0$. On the other hand from

section three equation (3-5) $\partial\dot{q}/\partial q = 0$. Thus $\partial\dot{r}/\partial r + \partial\dot{q}/\partial q > 0$ implying instability.

Case 3: $-1/(N-1) > r > (1+N)/(N-1)$

The steady state then lies along (B-2) implying $q > 1$. Equation (3-5) then shows $\partial\dot{q}/\partial q > 0$ while (A-6) shows since $q > q^r$ $\partial\dot{R}/\partial R > 0$. Thus $\partial\dot{q}/\partial q + \partial\dot{R}/\partial R > 0$ contradicting stability.

Appendix C--Stability with Positive Response

The objective is to verify that the steady state at q^S is stable, that at q^U is unstable. Observe that these steady states occur at the intersection of the curves $\dot{q}=0$ from (3-2)

$$r = \frac{N - (1+N)q}{(N-1)q} \qquad \text{(C-1)}$$

and $\dot{r}=0$ from (3-6)

$$r = \frac{(mN^2/E^2\delta^2\ell) - [(N^2+2N+2)q^2 - 2N(1+N)q + N^2]}{[(N^2+2N-2)q - N^2]q} \qquad \text{(C-2)}$$

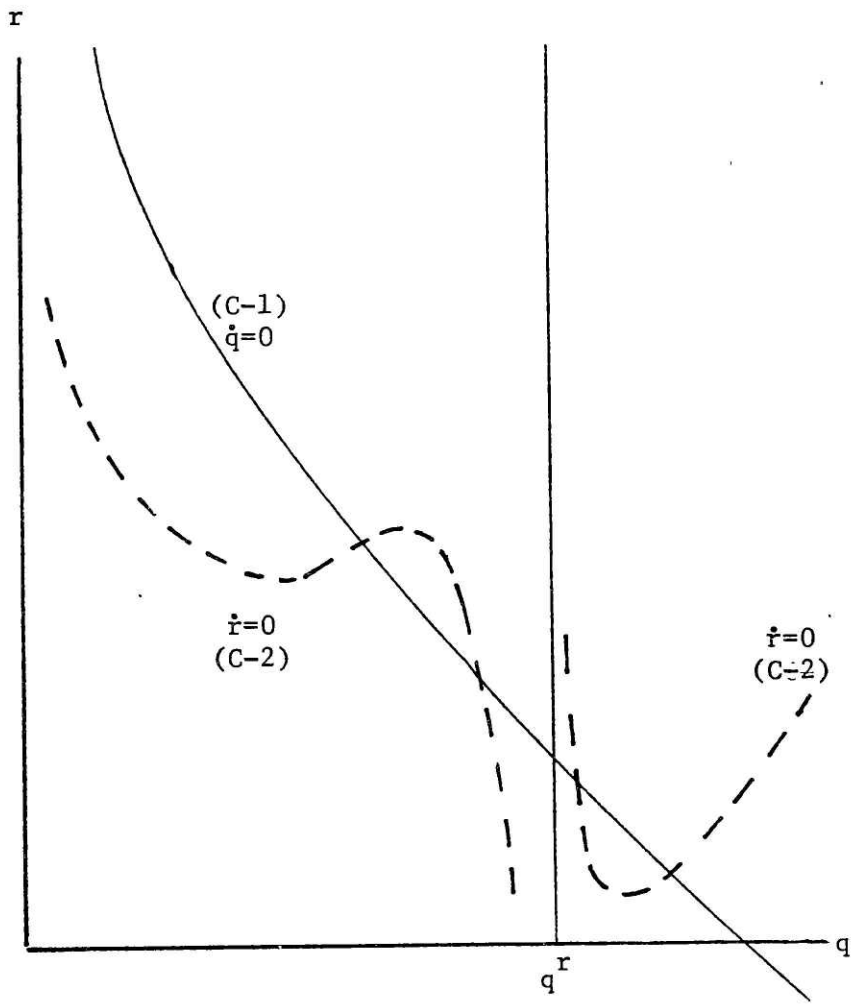
It is straightforward to verify that if these curves intersect at all and $q^S \neq q^U$ they intersect exactly twice at q^S and q^U. By definition q^S is the first intersection of these curves, q^U the second. From (3-5) at a steady state $\partial\dot{q}/\partial q < 0$ while $\partial\dot{r}/\partial r < 0$ if $q < q^r$ $\partial\dot{r}/\partial r > 0$ if $q > q^r$. Observe that the curve (C-2) has a pole at q^r and is continuous on $0 < q < q^r$ and on $q^r < q$. This information suffices, using some results from Levine [12] section (2) to determine the stability of the steady states: for $q < q^r$ a steady state is stable if and only if the curve (C-2) intersects the curve (C-1) from below; for

$q > q^r$ a steady state is unstable if the curve (C-2) intersects (C-1) from below, stable if from above and in addition $\partial\dot{q}/\partial q + \partial\dot{r}/\partial r < 0$. However, the later condition always holds for b small enough--$\partial\dot{q}/\partial q < 0$ and is of order b while $\partial\dot{r}/\partial r$ is only of order b^2.

Consider first $q < q^r$. As $q \to 0$ (C-1) approaches $(N/(N-1))(1/q)$ while (C-2) approaches $\{[(mN^2/E^2\delta^2\beta\mathcal{L}) - N^2]/N^2\}(1/q)$ from which it is seen as $q \to 0$ (C-1) lies above (C-2). Thus if $q^S < q^r$, since the first intersection of curves must be with (C-2) hitting (C-1) from below, q^S is stable. If $q^U < q^r$ it is at the second intersection which has (C-2) hitting (C-1) from above (since both curves are continuous on $0 < q < q^r$) and is unstable.

Taking the other case $q > q^r$ from (C-2) it is clear that as $q \to \infty$ (C-2) goes to $+\infty$, while from (C-1) the $\dot{q}=0$ curve becomes negative. Thus if $q^U > q^r$ it is at the second intersection with (C-2) hitting (C-1) from below and is unstable; if $q^S > q^r$ it is at the first intersection and is stable.

This line of reasoning is illustrated in figure (C-1).



Figure(C-1): Intersections of $\dot{q}=0$ and $\dot{r}=0$

References

[1] Robert Bishop, "Duopoly, Collusion or Warfare," AER, December 1960.

[2] James Friedman, "Non-cooperative equilibria for supergames," REStud, January 1971.

[3] _____, Oligopoly and the Theory of Games, North-Holland, 1977.

[4] Edward Green, "Non-cooperative Price Taking in Large Dynamic Markets," JET, April 1980.

[5] Joel Guttman, "Understanding Collective Action: Matching Behavior," AER, May 1978.

[6] Frank Hahn, "Exercises in Conjectural Equilibrium" Scandinavian Journal of Economics, 1977, 79(2), pp. 210-226.

[7] Morris Hirsch, Differential Topology, Springer-Verlag, 1976.

[8] Hirsch and Smale, Differential Equations, Dynamical Systems and Linear Algebra, Academic Press, 1974.

[9] John Laitner, "Rational Duopoly Equilibria," QJE, December 1980.

[10] David Levine, "Local Almost Perfect Equilibria," MIT, 1981.

[11] _____, "Long Run Collusion in a Partially Myopic Industry," MIT, 1980.

[12] _____, "Some Results on Dynamical Systems," MIT, 1981.

[13] Marschak and Selten, General Equilibrium with Price Making Firms, Springer-Verlag, 1974.

[14] _____, "Restabilizing Responses, Inertia Supergames and Oligopolistic Equilibrium," QJE, February 1978.

[15] Roy Radner, "Collusive Behavior in Epsilon-Equilibria," JET, April 1980.

[16] R. M. Scherer, Industrial Market Structure and Economic Performance, Rand McNally, 1970.

[17] Jesus Seade, "On the Effects of Entry," Econometrica, March 1980.

[18] R. Selten, "A Re-examination of the Perfectness Concept," Int. J. Game Theory, 21 (1979), pp. 1-9.

[19] Michael Spence, "Tacit Coordination and Imperfect Information," Canadian Journal of Economics, August 1978.

[20] George Stigler, "A Theory of Oligopoly," SPE, February 1974.

[21] E. C. Zeeman, Catastrophe Theory, Addison-Wesley, 1977.

Notes

(1) See for example Laitner [9].

(2) Supergames are a concept due to Friedman [2].

(3) For a discussion of this see Friedman [3]. Marschak and Selten [13, 14] or Radner [15]. The relevant equilibrium concept is that of perfect equilibrium found in Selten [18]. Similar results hold for static conjectural equilibria discussed by Hahn [6] and Seade [17].

(4) The relevant environment is a continuous time model with discounting and adjustment costs.

(5) Non-identical firms and non-quadratic profit functions are examined in Levine [11].

(6) This differs from the formulation of Marschak and Selten [14] in that firms respond the same way to both increases and decreases in output by opponents. In the differentiable framework here there is no advantage to kinked response: the optimal response to punish opponents for cheating and the optimal response to reward them for colluding are the same.

(7) Reactions apply only to future changes in output and do not apply retroactively to past deviations by rival firms. This distinguishes the present model from the formulation in Guttman [5].

(8) More general technologies are examined in Levine [11]. If firms face a capacity constraint I assume that it is sufficiently large that it is not binding in competitive equilibrium.

(9) One insignificant difference between the two approaches is that when adjustment costs are explicitly introduced $\hat{A}^j$ must include an estimate of the present value of future adjustment costs. Fortunately Levine [10] shows that in the present case the only effect of this is to introduce some irrelevant constants into the adjustment equation.

(10) A mathematical technicality of no economic import is that $C^j(R^j)$ is not differentiable when $R^j_k=0$. This is ignored.

(11) With this simplification the model is formally and conceptually similar to that of Guttman [5]. I am grateful to Dr. Guttman for making available unpublished research conducted jointly with Michael Miller along lines similar to those here.

(12) See Hirsch and Smale [9] chapter 16.

(13) See Levine [11] for results with asymmetric initial conditions.

(14) For elementary catastrophe theory see Zeeman [21] especially essays one and ten.

(15) See Scherer [16], pp. 158-164 for example.

CHAPTER II: LONG RUN COLLUSION IN A PARTIALLY MYOPIC INDUSTRY

0. Introduction

Will partially myopic firms collude in the long run? In the first chapter we saw in a quadratic symmetric example the absence of frictional costs implies that they will. This chapter studies an unrestricted technology without frictional costs. I ask: how are long run steady states of the adjustment process related to pareto efficient outcomes of the static game? The answer has two parts:

- In "almost all" games steady states and outcomes satisfying the first order conditions for static pareto efficiency "almost" coincide.
- The general stability analysis is intractible. However, it is possible to show that with identical firms and symmetric initial conditions a steady state is stable if and only if it is (locally) efficient.

The paper has four sections. Section one reviews the model of chapter one. Section two studies steady states. Section three focuses on the stability of steady states. Section four discusses the implication of an approximation introduced in section one. The final section summarizes the conclusions of the paper.

1. Review

There are N firms, entry is prohibited and each firm j controls its own output x^j. The output vector x is presumed to lie in X, an open subset of $\mathbb{R}^N$. By assuming X is open behavior on the boundary is ignored.

Attention is limited to the case in which communication between firms can occur and there are no counter-reactions. Thus, from (1-2) of chapter one the movement of x is given by

$$\dot{x}^j = \sum_{k=1}^{N} R_k^j y^k \qquad (1\text{-}1)$$

where y^j are the autonomous output controls. From (1-3) of chapter one the reaction coefficients move according to

$$\dot{R}_k^j = S_k^j \qquad (1\text{-}2)$$

The profits of firm j are given by a smooth function $\pi^j : X \to \mathbb{R}$. Note that there are no frictional costs of reacting. Let $\pi_k^j = \partial \pi^j / \partial x^k$. To insure that firms are able to affect opponent's profits it is assumed that for $x \in X$ and $j \neq k$ $\pi_k^j(x) < 0$. Let $\pi^{\cdot}$ be the vector of profit functions, π_k be the column vector $(\pi_k^j)_{j=1}^N$, π^j the row vector $(\pi_k^j)_{k=1}^N$ and π be the matrix with rows π^j. Since X is open the static game with profit functions π^j may not have any efficient points. To rule out degeneracy at least some point $x \in X$ should satisfy the first order conditions for pareto efficiency.

As a second regularity condition it is assumed that for some $x \in \mathcal{X}$ $\det(\pi(x))=0$. In section two it is shown that this is in fact the first order condition for efficiency.

Firm j's intertemporal preferences are described by a discount rate ρ^j. The corresponding discount factor is $\delta^j = 1/\rho^j$.

The behavioral model is the same as in chapter one. As in equation (2-4) of chapter one the strategies $\tilde{f}^j = y^j$ and $\tilde{g}^j_k = S^j_k$ are given by

$$\tilde{f}^j = bk^j_j \sum_{k=1}^{N} \frac{\partial \hat{A}^j}{\partial x^k} R^k_j$$

$$\tilde{g}^j_k = bk^j_k \frac{\partial \hat{A}^j}{\partial R^j_k} \qquad (1\text{-}3)$$

where the bk^j_k are the partial adjustment coefficients. The approximate present value of income from (2-12) in chapter one is

$$\hat{A}^j = \delta^j \pi^j + (\delta^j)^2 b \sum_{\ell=1}^{N} \pi^j_\ell \sum_{k=1}^{N} R^\ell_k \, k^k_k \delta^k \sum_{m=1}^{N} R^m_k \pi^k_m \qquad (1\text{-}4)$$

Thus by (1-3) and (1-4)

$$\tilde{f}^j = b\, k^j_j \sum_{\ell=1}^{N} \frac{\partial \hat{A}^j}{\partial x^\ell} R^\ell_j = b\delta^j k^j_j \sum_{\ell=1}^{N} \pi^j_\ell R^\ell_j + O(b^2) \qquad (1\text{-}5)$$

which as in chapter one simplifies to

$$y^j = \tilde{f}^j \approx \sum_{\ell=1}^{N} \pi^j_\ell R^\ell_j \qquad (1\text{-}6)$$

where $b\delta^j k^j_j$ is normalized to one by choosing appropriate units for π^j and the term of $O(b^2)$ is dropped. Section four discusses the implications of this approximation in greater detail.

Finally from (1-3) and (1-4) the motion of the reaction coefficients is

$$\dot{R}^j_k = b\,k^j_k\,\partial\hat{A}^j/\partial R^j_k$$

$$= b(\delta^j)^2 k^j_k[\pi^j_j \sum_{m=1}^{N} R^m_k\,\pi^k_m + \pi^k_j \sum_{m=1}^{N} R^m_k\,\pi^j_m]$$

$$= b\eta^j_k[\pi^k_j \sum_{m=1}^{N} R^m_k\,\pi^j_m + \pi^j_j\,y^k] \qquad (1\text{-}7)$$

where the constant $\eta^j_k = (\delta^j)^2 k^j_k$ and use is made of the normalization rule $b\delta^j k^j_j = 1$.

2. Equilibrium

The remainder of the paper addresses the following question: what is the relationship between stable steady states of the dynamical system described in the first section and pareto efficient outcomes of the game? Mathematically this divides into two issues: how are steady states related to outcomes which satisfy the first order conditions for pareto efficiency (called FOPE), and how are the stability conditions related to the second order conditions for pareto efficiency? This section takes up the first issues and yields the following conclusion: in "almost all" games which satisfy the required regularity conditions steady states and FOPE coincide "amost exactly." The following section examines the relationship between stability and local efficiency.

A point x is supportable as a steady state (or is simply called a steady state) if there is some reaction matrix R such that $\dot{x}(x,R) = 0$. In this case the dynamical system is motionless for all time regardless of whether $y = 0$. There are two possible types of steady states. If $y = 0$ by (1-17) $\dot{x} = 0$. This is called an autonomous steady state to reflect the fact that the autonomous action control variables y are stationary. Steady states at which $y \neq 0$ are called non-autonomous. The first half of the section examines autonomous steady states and shows that they are all FOPE. As a partial converse in "almost all" games the FOPE form an N-1 dimensional set of which at worst an N-2 dimensional subset fail to be autonomous steady states. The second half of the section examines the possibility that some non-autonomous steady states might fail to be FOPE and demonstrates that in "almost

all" games such exceptional steady states are a set of isolated points. The section concludes by showing graphically the relationship between FOPE and steady states in the "generic" case.

Autonomous steady states are characterized by the conditions $y = 0$ $\dot{R} = 0$. To analyze the set of points x which are supportable as autonomous steady states R must be eliminated from these equations. The resulting condition will then be contrasted with the first order efficiency conditions to show that autonomous steady states are FOPE. Before proving a partial converse a digression will give a mathematical definition of "almost all" games.

Equating the expressions for y and $\dot{R}$ from (1- 6) and (1- 7) to zero gives the condition for an autonomous steady state

$$\sum_{p=1}^{N} \pi_p^j R_j^p = 0 \qquad j = 1,\ldots, N \qquad (2\text{-}1)$$

$$\pi_k^j \sum_{p=1}^{N} \pi_p^j R_k^p = 0 \qquad j = k = 1,\ldots, N \quad j \neq k \qquad (2\text{-}2)$$

Since the $\pi_k^j \neq 0$ (by assumption) they can be eliminated from (2-2) to yield the equivalent condition

$$\sum_{p=1}^{N} \pi_p^j R_k^p = 0 \qquad j, k = 1,\ldots, N \quad j \neq k \qquad (2\text{-}3)$$

It is instructive to combine the N^2 equations in (2-1) and (2-3) into the N vector equations

$$\pi R_k = 0 \qquad k = 1,\ldots,N \qquad (2\text{-}4)$$

where $\pi=\{\pi_k^j\}$ with j subscripting rows and k subscripting columns and $R_k=(R_k^j)_{j=1}^N$

If it is to be possible to solve (2-4) for N vectors satisfying $R_k^k = 1$ then π must be singular and admit in its null space a vector $\gamma = (\gamma^j)_{j=1}^N$ with $\gamma^j \neq 0$ for any j. I will call such a matrix regular singular. If π is regular singular setting $R_k^j = \gamma^j/\gamma^k$ solves (2-4) and satisfies the restriction $R_k^k = 1$. So it is necessary and sufficient for x to be an autonomous steady state that $\pi(x)$ be regular singular.

How does this compare with the first order condition for pareto efficiency? At a pareto efficient point for a non-zero vector of weights $\mu = (\mu^j)_{j=1}^N$ the weighted sum $\sum_{j=1}^N \mu^j \pi^j$ must be maximized. The first order conditions (which must be satisfied since $\mathcal{X}$ is an open set) are

$$\sum_{j=1}^N \mu^j \pi_k^j = 0 \qquad k = 1,\ldots, N \qquad (2\text{-}5)$$

which in matrix notation is

$$\mu'\pi = 0 \qquad (2\text{-}6)$$

or the condition that π be singular. This condition will be taken as the definition of a FOPE, the restriction that the weights μ^j have the same sign being viewed in this terminology as part of the second order conditions for efficiency. This definition is illustrated graphically at the end of the section. Note incidentally the difference between the weights γ corresponding to reactions and the weights μ corresponding to utility weights: the former satisfies

$\pi\gamma = 0$, the latter $\mu'\pi = 0$.

All autonomous steady states are FOPE. The converse is false since not all singular matrices are regular singular. To proceed further we must digress to give some technical definitions. By an L dimensional set is meant a finite collection of L or lower dimensional manifolds at least one of which has dimension L. For example in R^3 the set consisting of the $x^1 - x^2$ plane and the x^3 axis is a two-dimensional set. In appendix (A) it is shown that the set of singular NxN matrices is an $N^2 - 1$ dimensional set. Since singular matrices π satisfy the single restriction $\det(\pi) = 0$ and lie in an N^2 dimensional space this should not be too surprising. A singular matrix π which is not regular singular satisfies the additional restriction that every vector in its null space has $\gamma^j = 0$ for some j. As appendix (A) demonstrates these matrices lie in a $N^2 - 2$ dimensional set.

That most singular matrices are regular singular unfortunately does not imply that in an arbitrary game most FOPE are autonomous steady states. In a linear game $\pi(x)$ does not depend on x and it could well be that π is non-regular singular so that the entire game consists of FOPE which are not autonomous steady states. This example is clearly rather special and it would be useful to assert that this does not happen in "most" games.

It is now necessary to introduce some technical concepts from differential topology. References are Guillemin and Pollack [6], Milnor [15] and Hirsch [7]. The map π is said to meet a submanifold S

of a manifold M transversally iff for each x with $\pi(x) \varepsilon S$ the tangent to π at x and the tangent space to S at $\pi(x)$ span the tangent space to M at $\pi(x)$. It meets an L dimensional set S transversally in an N^2 dimensional space M if it meets each component manifold of S transversally. In this case the implicit function theorem shows that the set of x such that $\pi(x) \varepsilon S$ has the "right" dimension--that is, dimension $N - (N^2 - L)$. Thus if π meets both the singular and non-regular singular matrices transversally the FOPE are N - 1 dimensional and the points which are not autonomous steady states are confined to an N - 2 dimensional subset. Note that an L-dimensional set may be empty, although not in the case of FOPE which are assumed to exist.

As the example above shows not all games meet a given set transversally. "Most" games, however, do have this property. Let G be the set of all C^2 mappings $\mathcal{X} \dashrightarrow \mathbb{R}^N$ and let G^R be the mappings $\pi^{\cdot}$ which satisfy the regularity conditions $\pi^j_k(x) < 0$ for any $j \neq k$, $x \varepsilon \mathcal{X}$ and for some $x \varepsilon \mathcal{X}$ $\det(\pi(x)) = 0$. Then G is the set of all games and G^R the set of regular games. A topology can be introduced on G and G^R by specifying the neighborhoods of each game π. Let $K = K_i$ be a covering of $\mathcal{X}$ with compact sets such that no point lies in more than a finite number of these sets and let $\varepsilon = \varepsilon_i$ be positive real numbers. A neighborhood $U(\pi^{\cdot}, K, \varepsilon)$ of $\pi^{\cdot}$ corresponding to K and ε are all games $\hat{\pi}^{\cdot}$ satisfying

$$\sup \left\{ \left| \hat{\pi}^j(x) - \pi^j(x) \right|, \left| \hat{\pi}^j_k(x) - \pi^j_k(x) \right|, \left| \hat{\pi}^j_{K\ell}(x) - \pi^j_{K\ell}(x) \right| \right\} \leq \varepsilon_i$$

$$x \varepsilon K_i;\ j, k, \ell = 1, \ldots, N \qquad (2\text{-}7)$$

that is $\hat{\pi}^{\cdot}$ must be close to $\pi^{\cdot}$ in the assigned payoff and its first two derivatives. This is known as the Whitney C^2 topology.

A residual subset in G is the countable intersection of dense open sets. In $\mathbb{R}$ an example of a residual set is the set of irrational numbers--they are the intersection of the sets consisting of the real line with one rational point deleted. It can be shown that any residual subset of G is dense, and by almost all games in G is meant all games in some residual subset.

A theorem known as the jet transversality theorem asserts that almost all maps $\pi^{\cdot} \varepsilon G$ have derivative maps π that meet a given set transversally. This paper examines only regular games and requires a definition of residual in G^R rather than G. In appendix (B) it is shown that if D is dense in G then $D \cap G^R$ is dense in G^R. In light of this a subset of G^R is called residual iff it is the intersection of a residual set in G with G^R. This gives the desired conclusion that almost all games in G^R (and in particular all games in a dense subset) have derivative maps which meet a given set transversally.

The preceding discussion is conveniently summarized in two propositions.

Proposition (2-1): If S is an L dimensional set then for almost all games the set of $x \varepsilon \mathcal{X}$ which satisfy the restriction $\pi(x) \varepsilon S$ is $N - (N^2 - L)$ dimensional.

Proposition (2-2): All autonomous steady states are FOPE. In almost all games FOPE are N - 1 dimensional and are all autonomous steady states except possibly in an N - 2 dimensional set.

Exceptional Steady States: At an autonomous steady state $y = 0$, and the first order conditions for pareto efficiency are satisfied. It is possible, however that $\dot{x} = 0$ and $y \neq 0$ in which case the preceding analysis does not apply. Here it is shown that in almost all games there are at most an isolated set of points which are steady states but not FOPE. There are two steps in this undertaking: first R is eliminated from the equations $\dot{x} = 0$ $\dot{R} = 0$; then an application of Proposition (2-1) gives the desired genericity result.

The conditions for a (not necessarily autonomous) steady state are given by equating the expressions for $\dot{x}$ and $\dot{R}$ from (1-1) and (1- 7) to zero

$$\sum_{k=1}^{N} R_k^j y^k = 0 \qquad j = 1,\ldots, N \qquad (2\text{-}8)$$

$$[\pi_k^j (\sum_{p=1}^{N} \pi_p^j R_k^p) + \pi_j^j y^k] = 0 \qquad j \neq k \qquad (2\text{-}9)$$

To eliminate R from these equations use $\pi_k^j \neq 0$ $j \neq k$ to define

$$\lambda_k^j = \begin{cases} 1 & j = k \\ -\pi_j^j / \pi_k^j & j \neq k \end{cases} \qquad (2\text{-}10)$$

Then (2-9) is equivalent to

$$\sum_{p=1}^{N} \pi_p^j R_k^p = y^k \lambda_k^j \qquad j \neq k \qquad (2\text{-}11)$$

The equation (1- 6) defining y^k is

$$y^k = \sum_{p=1}^{N} \pi_p^k R_k^p \tag{2-12}$$

which can be expressed as

$$\sum_{p=1}^{N} \pi_p^k R_k^p = y^k \lambda_k^k \tag{2-13}$$

since $\lambda_k^k = 1$ by definition. The N^2 equations (2-11) and (2-13) can be combined into the N vector equations

$$\pi R_k = y^k \lambda_k \tag{2-14}$$

Throughout the remainder of this discussion it is assumed that the steady state is not a FOPE. In this case π is non-singular and (2-14) can be solved for the reaction coefficients.

$$R_k = y^k \pi^{-1} \lambda_k \tag{2-15}$$

The equation for R_k^k (which is one by definition) is

$$1 = R_k^k = y^k (\pi^{-1})^k \lambda_k \tag{2-16}$$

from which the autonomous controls are

$$y^k = 1/(\pi^{-1})^k \lambda_k \tag{2-17}$$

Observe that this implies $y^k \neq 0$ for any k, which is consistent with the earlier finding that the steady state must be non-autonomous since it is not a FOPE.

The N vector equations (2-15) can be rewritten using (2-17) as the N^2 scalar equations

$$R_k^j = (\pi^{-1})^j \lambda_k / [(\pi^{-1})^k \lambda_k] \tag{2-18}$$

So far the first steady state condition $\dot{x} = 0$ given in (2-8) has not been used. To eliminate R from the steady state conditions (2-8) and (2-9) substitute (2-18) into (2-8) to get the necessary conditions

$$\sum_{k=1}^{N} (\pi^{-1})^j \lambda_k / [(\pi^{-1})^k \lambda_k]^2 = 0 \qquad (2\text{-}19)$$

which since π^{-1} is non-singular is equivalent to

$$\sum_{k=1}^{N} \lambda_k^j / [(\pi^{-1})^k \lambda_k]^2 = 0 \qquad (2\text{-}20)$$

A useful implication of (2-20) is that $\lambda_k^j \neq 0$ and $\pi_j^j \neq 0$. If not since for $j \neq k$ $\lambda_k^j = -\pi_j^j / \pi_k^j$ it must be that $\pi_j^j = 0$ and so $\lambda_k^j = 0$ for every $k \neq j$. In this case since $\lambda_k^k = 1$ by definition

$$\sum_{k=1}^{N} \lambda_k^j / [(\pi^{-1})^k \lambda_k]^2 =$$

$$\lambda_k^k / [(\pi^{-1})^k \lambda_k]^2 > 0 \qquad (2\text{-}21)$$

contradicting the steady state condition (2-20).

Let H_E^N be the set of matrices π which satisfy (2-20). If H_E^N is an N^2 dimensional set then by Proposition (2-1) almost all games will have only an isolated set of points x at which $\pi(x) \epsilon H_E^N$. Define the surjective mapping $U: H_E^N \rightarrow H_U^N$ by

$$U_k^j = \text{sgn}\,(\pi_k^k) \cdot \lambda_k^j / [(\pi^{-1})^k \lambda_k]^2 \qquad (2\text{-}22)$$

Since $\pi_k^k \neq 0$ this map is smooth. To show that H_E^N is $N^2 - N$ dimensional

it suffices to show that H_U^N is $N^2 - N$ dimensional and diffeomorphic to H_E^N. Since U is smooth H_U^N and H_E^N are diffeomorphic provided U has a smooth inverse.

Let e be a vector of ones. By (2-20) if $U \varepsilon H_U^N$ $Ue = 0$. This shows that H_U^N is an $N^2 - N$ dimensional linear subspace.

To solve for π given U it is necessary only to solve for λ and π_j^j $j = 1,\ldots,N$. The π_k^j $j \neq k$ are then given by reversing the definition of λ in (2-10) as

$$\pi_k^j = -\pi_j^j/\lambda_k^j \qquad j \neq k \qquad (2\text{-}23)$$

which is smooth since $\lambda_k^j \neq 0$. Solving for λ_k^j $j \neq k$ is straightforward since by the definition of U in (2-22)

$$U_k^j/U_k^k = \lambda_k^j \qquad (2\text{-}24)$$

where $U_k^k \neq 0$ since π_k^k and λ_k^k are non-zero. To solve for π_j^j using λ and U observe from (2-23) and $\lambda_j^j = 1$ that π factors as

$$\pi = \beta d \qquad \beta_k^j = \begin{cases} 1 & j=k \\ -1/\lambda_k^j & j \neq k \end{cases} \qquad d_k^j = \begin{cases} \pi_j^j & j=k \\ 0 & j \neq k \end{cases} \qquad (2\text{-}23)$$

Since $\pi_j^j \neq 0$ d is non-singular, $\pi^{-1} = d^{-1}\beta^{-1}$ and

$$(\pi^{-1})^k = (1/\pi_j^j)(\beta^{-1})^k \qquad (2\text{-}26)$$

Substitute this expression into (2-22) the definition of U_j^j to find

$$U_j^j = \operatorname{sgn}(\pi_j^j)\lambda_j^j/[(1/\pi_j^j)(\beta^{-1})^j\lambda_j]^2 \qquad (2\text{-}27)$$

and since $\lambda_j^j = 1$ (2-27) solves as (2-28)

$$\pi^j_j = \text{sgn}\,(U^j_j)\sqrt{U^j_j}\;\left|\,(\beta^{-1})\,\lambda_j\,\right| \qquad (2\text{-}28)$$

which is smooth since it can never vanish.

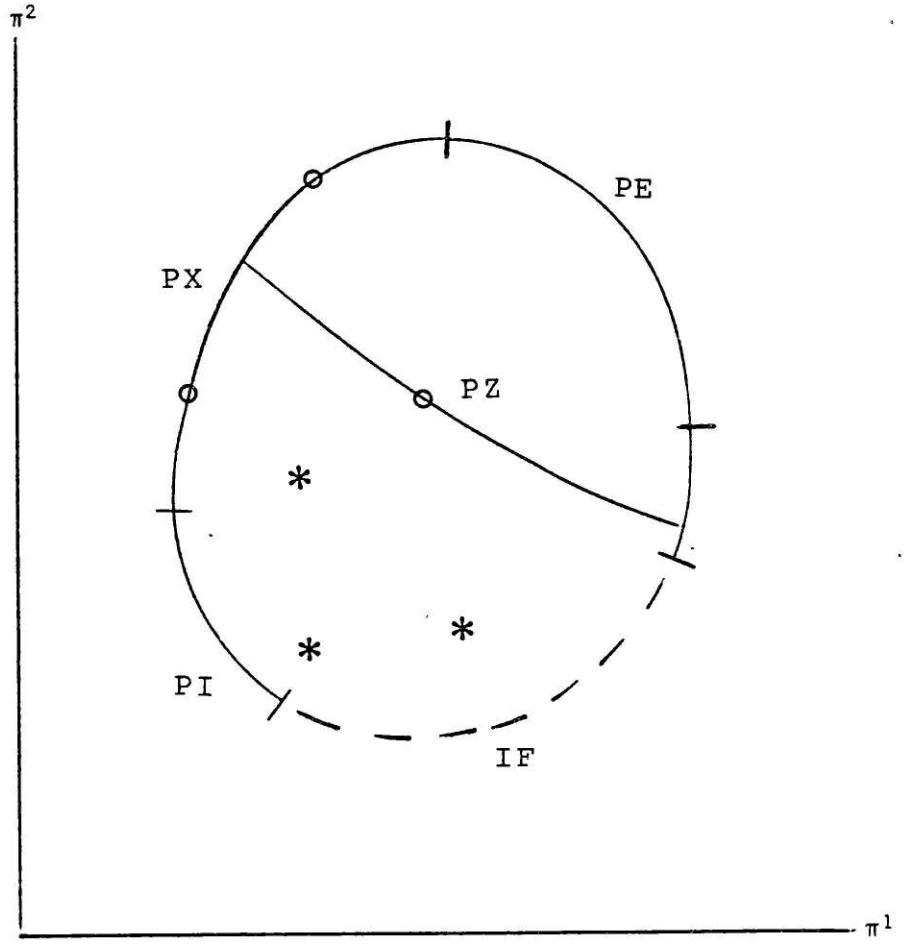
The preceding discussion can be summarized as

Proposition (2-3): In almost all games the set of steady states which are not FOPE are a set of isolated points.

Illustration of the Generic Case: Propositions (2-2) and (2-3) characterize the generic game. The FOPE form an N - 1 dimensional set which are all steady states except for an N - 2 dimensional subset. In addition there may be isolated steady states which are not FOPE.

The generic 2-agent game is illustrated in payoff space in figure (2-1). The shaded areas are all feasible payoff combinations in the game. Because the game is defined on an open set part of the boundary, labelled IF, may not be feasible. The remainder of the boundary are FOPE. Points along PE are pareto efficient. Points along PI do not pareto dominate any outcomes. Points along PX maximize $\pi^2 - \mu\pi^1$ for some $\mu > 0$. In addition to the boundary there may be interior FOPE along PZ which are saddle points of $\mu^1\pi^1 + \mu^2\pi^2$ for some $\mu \neq 0$.

Except for the isolated points labelled "0" all of the FOPE are steady states. The remaining steady states are the isolated non-autonomous steady states labelled "*". As shown the FOPE and steady states "almost" coincide.



Figure(2-1): The Generic Case

3. Stability

The previous section showed that pareto efficient points are a subset of the set of steady states. This section examines the extent to which the stable steady states coincide with the pareto efficient points.

The stability conditions in the general case are intractible and no satisfactory relationship between stability and efficiency is developed. The first half of the section limits attention to symmetric games with symmetric initial conditions. In these games steady states are stable if and only if they are (locally) pareto efficient. The second half of the section gives a sufficient condition for strong quasi-stability with non-unique steady states. As an application it is shown that symmetric pareto efficient outcomes in two firm games are strong quasi-stable even when non-symmetric perturbations are permitted.

Symmetric Games: In a symmetric game all firms are identical including in the initial conditions. Such games are easy to study since the game is characterized by two variables: the common output and the common reaction. First the steady state conditions are restated in terms of these variables. An examination of the stability conditions then shows that (local) pareto efficient points and stable steady states coincide.

Let $x^j = z$ and $R^j_k = r \quad j \neq k$ for each firm. In a symmetric game all firms are identical and these variables complete describe the game. The motion of z is given by the equations describing the motion of the x^j (1-1) and (1-6) as

$$\dot{z} = \dot{x}^j = \sum_{m=1}^{N} R_m^j \sum_{p=1}^{N} \pi_p^j R_j^p$$

$$= (\sum_{m\neq j} R_m^j + 1)(\sum_{p\neq j} \pi_p^j R_j^p + \pi_j^j)$$

$$= ((N - 1)\, r + 1)((N - 1) r\, \pi_k^j + \pi_j^j) \qquad (3\text{-}1)$$

The motion of r is given by the equation of motion for R_k^j $j \neq k$ (1- 7) as

$$\dot{r} = \dot{R}_k^j = b\eta_k^j\, [\pi_j^k (\sum_{p=1}^{N} \pi_p^j\, R_k^p) + \pi_j^j\, y^k]$$

$$= b\eta\, [\pi_j^k (\sum_{p=1}^{N} \pi_p^j\, R_k^p) + \pi_j^j\, (\sum_{p=1}^{N} \pi_p^j\, R_j^p)]$$

$$= b\eta\, [\pi_k^j (\pi_j^j r + \pi_k^j + \pi_k^j\, (N - 2) r)$$

$$+ \pi_j^j (\pi_j^j + (N - 1) r\, \pi_k^j)] \qquad (3\text{-}2)$$

where the motion y^k is from (1- 6) and η is the common value of the η_k^j.

Equating (3-1) and (3-2) to zero and solving for z and r shows that there are three types of steady states

$$r = 1 \qquad \pi_j^j + (N - 1)\, \pi_k^j = 0 \qquad (3\text{-}3)$$

$$r = -1/(N - 1) \qquad \pi_j^j - \pi_k^j = 0 \qquad (3\text{-}4)$$

$$r = -1/(N - 1) \qquad (N - 1)\, \pi_j^j - \pi_k^j = 0 \qquad (3\text{-}5)$$

The steady state in (3-3) is autonomous and an extreme point of the weighted sum $\sum_{j=1}^{N} \pi^j$. The steady state in (3-4) is also autonomous and is an extreme point of the weighted sum $\pi^j - \pi^k$ $j \neq k$. It cannot be pareto efficient, since the weights do not all have the same sign. The steady state in (3-5) is not autonomous.

Necessary conditions for stability are

$$\partial \dot{z}/\partial z + \partial \dot{r}/\partial r \leq 0 \tag{3-6}$$

$$(\partial \dot{z}/\partial z)(\partial \dot{r}/\partial r) - (\partial \dot{z}/\partial r)(\partial \dot{r}/\partial z) \geq 0 \tag{3-7}$$

sufficient conditions are that (3-6) and (3-7) hold with strict inequality. Do the inefficient steady states in (3-4) and (3-5) satisfy the necessary conditions? Differentiating the expressions for $\dot{z}$ and $\dot{r}$ in (3-1) and (3-2) shows that

$$\partial \dot{z}/\partial z = ((N-1)r + 1) \cdot$$

$$((N-1)r\,(\pi^j_{jk} + \pi^j_{kk} + (N-2)\pi^j_{km}) + (\pi^j_{jj} + (N-1)\pi^j_{jk})) \tag{3-8}$$

$$\partial \dot{r}/\partial r = b\eta\,(\pi^j_k\,\pi^j_j + (\pi^j_k)^2(N-2) + \pi^j_j\,\pi^j_k\,(N-1))$$

$$= b\eta\,\pi^j_k(N\,\pi^j_j + (N-2)\,\pi^j_k) \tag{3-9}$$

when $r = -1/(N-1)$ as in (3-4) or (3-5) we see from (3-8) that $\partial \dot{z}/\partial z = 0$, and for stability from (3-6) it must be that $\partial \dot{r}/\partial r \leq 0$. In (3-7) $\pi^j_j = \pi^j_k$ and from (3-9)

$$\partial \dot{r}/\partial r = b\eta(\pi_k^j)^2\, 2(N - 1) > 0 \qquad (3\text{-}10)$$

implying instability. In (3-5) $\pi_j^j = \pi_k^j/(N - 1)$ and from (3-9)

$$\partial \dot{r}/\partial r = b\eta(\pi_k^j)^2\, (N/(N - 1) + N - 2) > 0 \qquad (3\text{-}11)$$

implying instability as well.

The steady state in (3-3) must be broken into two cases. Define

$$S \equiv \partial^2\pi^j/\partial z^2$$

$$= \pi_{jj}^j + (N - 1)\pi_{jk}^j + (N - 1)(\pi_{jk}^j + \pi_{kk}^j + (N - 2)\, \pi_{km}^j) \qquad (3\text{-}12)$$

A necessary condition for pareto efficiency is that each firm's profit be maximized subject to the symmetry constraint. The first order condition for this maximum is the condition in (3-3) $\pi_j^j + (N - 1)\pi_k^j = 0$. The second order necessary condition is $S \leq 0$. In Appendix (C) it is shown that in regular games $S < 0$ together with the first order condition are also sufficient for a symmetric outcome to be locally pareto efficient.

Examination of (3-3) shows $r = 1$ and combining this with (3-8) shows

$$\partial \dot{z}/\partial z = NS \qquad (3\text{-}13)$$

while $\pi_j^j = -(N - 1)\pi_k^j$ combined with (3-9) shows

$$\partial \dot{r}/\partial r = b\eta\, (\pi_k^j)^2\, [(N - 2) - N(N - 1)]$$

$$= -b\eta(\pi_k^j)^2[(N - 1)^2 + 1] < 0 \qquad (3\text{-}14)$$

So if $S < 0$ (3-6) holds. To determine when the other half of the sufficient condition (3-7) holds compute from (3-1), (3-2) and (3-3)

$$\dot{\partial z}/\partial r = (N - 1)((N - 1)\pi_k^j + \pi_j^j + N(N - 1)\pi_k^j)$$

$$= N(N - 1)\pi_k^j \qquad (3\text{-}15)$$

$$\dot{\partial r}/\partial z = b\eta[(\pi_j^j + \pi_k^j)S]$$

$$= - b\eta(N - 2)\pi_k^j \, S \qquad (3\text{-}16)$$

Using (3-13), (3-14), (3-15) and (3-16) to compute the expression in (3-7) yields

$$(\dot{\partial z}/\partial z)(\dot{\partial r}/\partial r) - (\dot{\partial z}/\partial r)(\dot{\partial r}/\partial z)$$

$$= - b\eta(\pi_k^j)^2 \, S[N[(N - 1)^2 + 1] - N(N - 1)(N - 2)]$$

$$= - b\eta \, N^2(\pi_k^j)^2 S \qquad (3\text{-}17)$$

If $S < 0$ the sufficient condition is satisfied and if $S > 0$ the necessary condition fails. Except for the unimportant case $S = 0$ a steady state is stable if and only if it is locally pareto efficient.

Strong Quasi-Stability: When steady states are not isolated it is possible to make arbitrarily small movements away from a steady state to another steady state. Stability is impossible. A weaker condition which I call strong quasi-stability requires only that a small movement away from a steady state leads to a nearby steady state. For

autonomous steady states which are N - 1 dimensional this is the relevant stability concept.

Define the stability matrix

$$A = \begin{bmatrix} \dot{\partial x}/\partial x & \dot{\partial x}/\partial R \\ \dot{\partial R}/\partial x & \dot{\partial R}/\partial R \end{bmatrix} \tag{3-18}$$

A necessary condition for strong quasi-stability is that the real parts of the eigenvalues of A be non-positive. If steady states form an N - 1 dimensional manifold N - 1 of the N^2 eigenvalues of A must vanish, the corresponding eigenvectors indicating directions on the steady state manifold. In Levine [11] section (3) it is shown that if (x^*, R^*) is a steady state at which $N^2 - N + 1$ of the eigenvalues of A have strictly negative real parts and if there is an open set surrounding (x^*, R^*) in which the set of steady states are an N - 1 dimensional manifold then (x^*, R^*) and all nearly steady states are strong quasi-stable.

The previous section showed that the points x^* which are supportable as steady states are usually N - 1 dimensional. This does not tell, however, the dimensionality of steady states in (x, R) space. Appendix (D) shows that if the matrix $\lambda(x^*)$ defined in (2-10) by

$$\lambda_k^j = \begin{cases} 1 & j = k \\ -\pi_j^j/\pi_k^j & j \neq k \end{cases} \tag{2-10}$$

does not admit a non-zero vector with non-negative components in its null space, and in addition for all k $[\pi \mid \lambda_k]$ has full rank, then there

is a unique R* such that (x*, R*) is a steady state and a neighborhood of this steady state in which steady states form an N - 1 dimensional manifold.

This discussion yields the following conclusion concerning quasi-stability.

Proposition (3-1): if (x*, R*) is an autonomous steady state and

- o $[\pi(x^*) \mid \lambda_k]$ has full rank for all k
- o $\lambda(x^*)\mu = 0 \Rightarrow \mu \not\geq 0$
- o $A(x^*, R^*)$ has $N^2 - N + 1$ eigenvalues with strictly negative real parts

then there is an open set $U \ni (x^*, R^*)$ in which all steady states

- o form an N - 1 dimensional manifold
- o are strongly quasi-stable.

Two Firm Symmetric Games: Symmetric pareto efficient points in symmetric games are stable with respect to symmetric shocks. As an application of Proposition (3-1) it is demonstrated here that in two firm games symmetric pareto efficient points are strong quasi-stable; that is with respect to asymmetric as well as symmetric shocks.

The first order condition for symmetric pareto efficiency was given in (3-3) as

$$\pi^j_j + (N - 1)\,\pi^j_k = 0 \qquad (3\text{-}3)$$

$$\pi^j_j + \pi^j_k = 0 \qquad (3\text{-}19)$$

where (3-19) follows from N = 2. The matrix π is

$$\pi = \begin{bmatrix} -\pi_k^j & \pi_k^j \\ \pi_k^j & -\pi_k^j \end{bmatrix} \tag{3-20}$$

which has rank one since $\pi_k^j \neq 0$ and is regular singular since it admits $e = (1,1)'$ in its null space.

The matrix λ is computed from (2-10)

$$\lambda_k^j = \begin{cases} 1 & j = k \\ -\pi_j^j/\pi_k^j & j \neq k \end{cases} \tag{3-21}$$

$$\lambda = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{3-22}$$

From inspection of (3-20) and (3-22) these matrices satisfy the hypotheses of proposition (3-1). To apply the proposition requires, in addition that three eigenvalues of A have strictly negative real parts.

In the two firm case the dynamic equations for agent one from (1-1) (1-6) and (1-7) are

$$\dot{x}^1 = [\pi_1^1 + R_1^2\pi_2^1 + R_2^1(\pi_2^2 + R_2^1\pi_1^2)] \tag{3-23}$$

$$\dot{R}_2^1 = b\eta[\pi_1^2\,(\pi_1^1 R_2^1 + \pi_2^1) + \pi_1^1(\pi_2^2 + \pi_1^2 R_2^1)] \tag{3-24}$$

Differentiating these equations, using symmetry and the equilibrium conditions in (3-19) shows that

$$\partial\dot{x}^1/\partial R_2^1 = [\pi_2^2 + R_2^1\,\pi_1^2 + R_2^1\,\pi_1^2] = b\pi_k^j$$

$$\partial \dot{x}^1/\partial R_1^2 = \pi_k^j$$

$$\partial \dot{R}_2^1/\partial R_2^1 = - b\eta \ (\pi_k^j)^2$$

$$\partial \dot{R}_2^1/\partial R_1^2 = 0 \qquad (3\text{-}25)$$

By symmetry the stability matrix A defined in (3-18) is

$$A = \begin{bmatrix} \partial \dot{x}^1/\partial x^1 & \partial \dot{x}^1/\partial x^2 & \pi_k^j & \dot{\pi}_k^j \\ \partial \dot{x}^1/\partial x^2 & \partial \dot{x}^1/\partial x^1 & \pi_k^j & \pi_k^j \\ \partial \dot{R}_2^1/\partial x^1 & \partial \dot{R}_2^1/\partial x^2 & - b\eta(\pi_k^j)^2 & 0 \\ \partial \dot{R}_2^1/\partial x^2 & \partial \dot{R}_2^1/\partial x^1 & 0 & - b\eta(\pi_k^j)^2 \end{bmatrix} \qquad (3\text{-}26)$$

Let $e_o = (0, 0, 1, -1)'$; then $Ae_o = -b\eta(\pi_k^j)^2 e_o$ so e_o is an eigenvector corresponding to a negative eigenvalue. Furthermore there are two other eigenvalues corresponding to symmetric departures from equilibrium and by the analysis of symmetric shocks these are strictly negative as well.

Proposition (3-1) now applies and the efficient steady state and nearby steady states are strong quasi-stable.

4. The Very Long Run

The system we have been studying was derived by approximating the true equation for y (1- 5) by (1- 6). Thus the true system is a (small) perturbation of the system we have studied. How does this perturbation affect our results? A good reference for the discussion that follows is Hirsch and Smale [8] chapter 16.

The first point is that a small perturbation can't have steady states far distant from the steady states of the unperturbed system. This means that the perturbed steady states are (approximately) a subset of the steady states we have studied. In effect, a perturbation can destroy steady states and it can introduce new steady states but the new ones must be close to the old ones.

A steady state is hyperbolic iff none of the real parts of the eigenvalues of the stability matrix vanish there. In a neighborhood of a hyperbolic steady state a perturbation creates no new steady states, it shifts the hyperbolic steady state only a small distance, and it preserves the stability properties of the hyperbolic steady state. In the symmetric case the perturbation is symmetric and the steady states are hyperbolic (generically) with respect to symmetric states. Thus small perturbations have no important effect on the results given in the symmetric case. Similarly, exceptional steady states are generically hyperbolic and thus won't be affected much by small perturbations.

The situation near the manifold of autonomous steady states is very different. Generically, perturbations of a dynamical system have

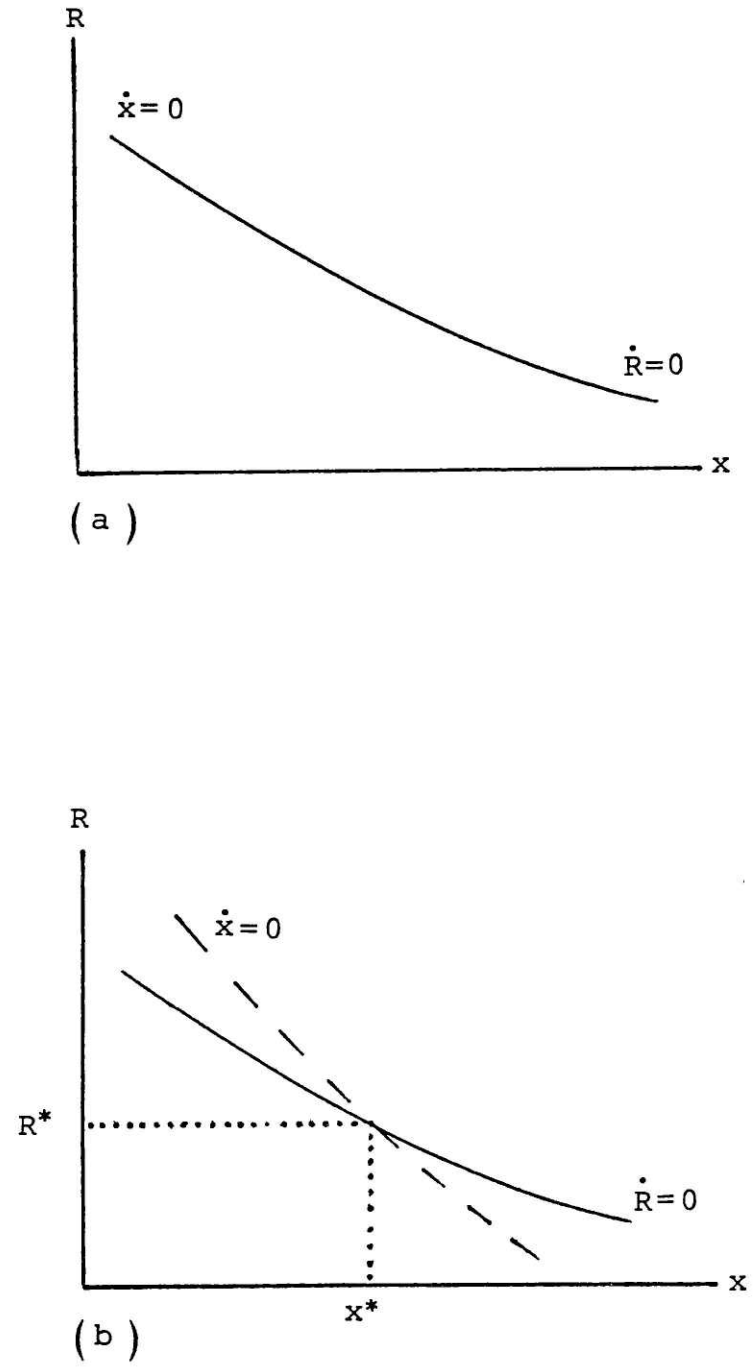
locally isolated steady states. It is to be expected that when the system is perturbed the steady state manifold will be replaced by a set of isolated steady states lying near the manifold.

This deserves a bit of explanation. Figure (4-1) illustrates what the system might look like before and after a perturbation. Initially the $\dot{x} = 0$ and $\dot{R} = 0$ curves coincide constituting a manifold of steady states. When (1- 6) is replaced by (1- 5) the $\dot{x} = 0$ curve shifts slightly by an amount proportional to b. Only the steady state (x^*, R^*) remains.

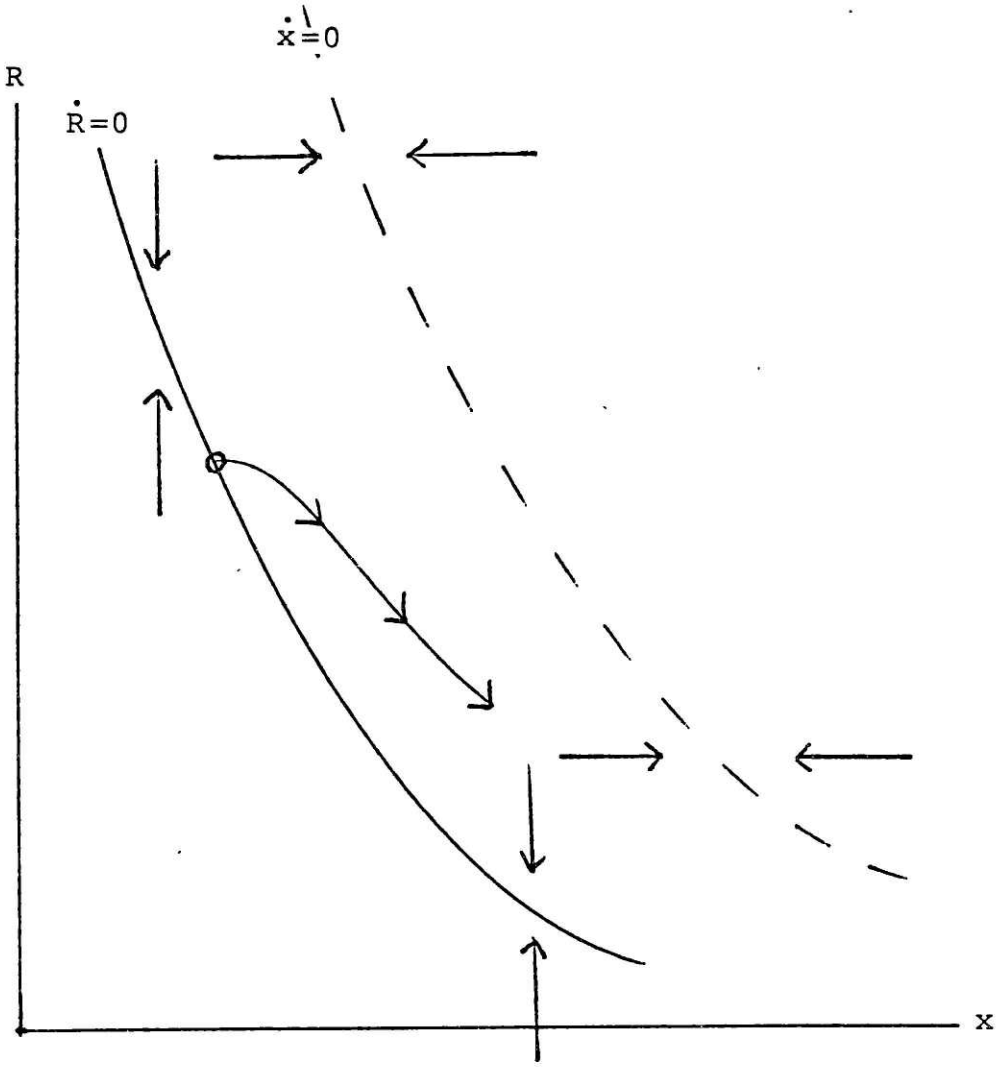
What happens along the old steady state manifold? Suppose the old manifold was stronge quasi-stable so both $\dot{x} = o$ and $\dot{R} = 0$ are attractors. Initially R doesn't move, while x moves towards the new $\dot{x} = 0$ curve. As x approaches $\dot{x} = 0$ R is no longer in equilibrium and begins to move along the old steady state manifold as illustrated in Figure (4-2).

If the old steady state manifold is unstable the situation is quite different: a small perturbation of the system will typically cause the system to move away from the old manifold entirely.

In the strong quasi-stable case, how rapid is the drift along the steady state manifold? From (1- 5) and (1- 7) $\dot{x} = bF(x, R)$ and $\dot{R} = b^2 G(x, R)$ where remember b is small. As the system moves along the old manifold x is approximately in equilibrium at $\dot{x} = 0$ since it equilibrates faster than $\dot{R}$. This means, since $\dot{x} = 0$ and $\dot{R} = 0$ lie apart by order b, that the distance of the system from $\dot{R} = 0$ is $\Delta x \cong bH(R)$. Thus, $\dot{R} = b^2 G_x \Delta x = b^3 G_x H(R)$ where G_x are the derivatives of G with respect to x.



Figure(4-1): Perturbation of the Steady State Manifold



Figure(4-2): Drift Along the Unperturbed Steady State Manifold

In the short run x equilibrates most rapidly at rate O(b) so R is determined by initial conditions and x by $\dot{x} = 0$. In the long run R equilibrates at rate $O(b^2)$ causing the system to move towards the unperturbed manifold of steady states. In the very long run, however, the perturbation causes the system to drift along the unperturbed steady state manifold at rate $O(b^3)$. This is similar to the notion of fast and slow manifolds introduced by Zeeman [13] chapter 3.

The exact nature of very long run steady states where output shares are determinate is an interesting question for future research.

5. Conclusion

When firms can engage in costless retaliatory policies we anticipate that they will reward opponents who make pareto improving output adjustments and punish those who selfishly try to increase output. This paper has shown that, subject to technical qualifications, this is true. The important qualifications:

- o In asymmetric games pareto efficient outcomes may be unstable steady states. A small subset of the pareto efficient outcomes may not be steady states at all.
- o In asymmetric games there may be stable steady states which satisfy the first order conditions for pareto efficiency, but not the second order conditions. A small number of steady states may not even satisfy the first order conditions.

APPENDIX (A)--Singularity and Regular Singularity

The objective is to prove two lemmas:

Lemma (A-1): m x n matrices of rank r are a set of codimension (m - r) x (n - r).

Lemma (A-2): regular singular n x n matrices are a set of codimension 2.

Proof of (A-1): From McCoy [14] section 15.5 a matrix A of rank r has an r x r non-singular submatrix A_{11}. Assume

$$A = \begin{matrix} r & n-r \\ \left[\begin{matrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{matrix}\right] & \begin{matrix} r \\ m-r \end{matrix} \end{matrix} \qquad \text{(A-1)}$$

Any matrix of rank r is obtained from a matrix of the form (A-1) by permuting rows and columns. Since only a finite set of such permutations is possible it suffices to prove the lemma for a matrix of the form (A-1). Following Guilleman and Pollack [6] chapter 1.4 problem 13 define

$$B = \begin{bmatrix} I & -A_{11}^{-1} A_{12} \\ 0 & I \end{bmatrix} \qquad \text{(A-2)}$$

Since B is nonsingular rank (AB) = rank (A). Also

$$AB = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1} A_{12} \end{bmatrix} \tag{A-3}$$

so that A has rank r if and only if

$$F(A) = A_{22} - A_{21} A_{11}^{-1} A_{12} = 0 \tag{A-4}$$

Since $\partial F/\partial A_{22} = I$ the transformation F has full rank. Thus since F(A) is $(m - r) \times (n - r)$ the lemma follows. Q.E.D.

Proof of (A-2): Let A be square, singular and non-regular. Any such matrix can be obtained from a matrix that admits a vector of the form $(0, X_2, \ldots, X_n)'$ in its null space by a finite permutation of columns, so assume A has this form. Set

$$A = [a_{11}, A_{12}] \tag{A-5}$$

where a_{11} is an n-vector. Then A has the required form if and only if A_{12} has rank (n - 2) or less. By the same reasoning used in lemma (A-1) this implies a codimension of 2. Q.E.D.

APPENDIX (B)--Genericity in Regular Games

Let G be the set of C^2 mappings $\pi^{\cdot}:\mathcal{X} \rightarrow |R^N$ in the Whitney C^2 topology where $\mathcal{X}$ is open in $|R^N$. Let G^R be the subset of $\pi^{\cdot} \in G$ which also satisfy $\pi^j_k(x) < 0 \quad j \neq k$ and $\det(\pi(x)) \neq 0$. The objective is to prove

Lemma (B-1): D residual in G implies D dense in G^R.

From general topological considerations it suffices to prove that $G^R \subset$ closure (interior G^R)). Define G^1 to be the subset of G which satisfies $\pi^j_k(x) < 0 \quad j \neq k$ and G^2 the subset which satisfies $\det(\pi(x)) \neq 0$. Obviously small perturbations of π satisfying $\pi^j_k(x) < 0$ will satisfy the restriction so G^1 is open. Thus it suffices to show $G^2 \subset$ closure (interior G^2)). Define G^3 to be the subset of G^2 such that for $\pi^{\cdot} \in G^3$ there is an x_1 and x_2 with $\det(\pi(x_1)) > 0$ and $\det(\pi(x_2)) < 0$. Obviously G^3 is open. To prove lemma (B-1) it is then necessary to only show that

Lemma (B-2): $G^2 \subset$ closure (G^3)

This requires a preliminary lemma.

Lemma (B-3): If A is singular there is a matrix B such that for $\lambda > 0$ $\det(A + \lambda B) > 0$.

Proof:

By the Jordan decomposition theorem $A = C^{-1}JC$ where J is upper triangular, has zeroes in the first $(N - r)$ diagonal positions and the non-zero eigenvalues of A in the remaining r diagonal positions. Assume without loss of generality that the product of the non-zero eigenvalues of A is positive. Let D be a diagonal matrix with ones in the first r positions, zeroes in the other $(N - r)$. Then for $\lambda > 0$ $\det (J + \lambda D) > 0$. Thus $B = C^{-1}DC$ satisfies the required property.

Q.E.D.

Now suppose $\pi^{\cdot} \in G^2 - G^3$. Ignoring the case $\det(\pi)$ vanishes identically (left as an exercise) suppose without loss of generality for some x_1 $\det(\pi(x_1)) < 0$. By assumption for some x_2 $\det(\pi(x_2)) = 0$. Let B be a ball centered on x_2. Because $\mathcal{X}$ is open in $\mathbb{R}^N$ we may assume $B \subset \mathcal{X}$ and $x_1 \notin$ closure (B). Let π^* be (by lemma B-3)) such that $\det(\pi(x_2) + \lambda\pi^*) > 0$ for $\lambda > 0$. Using techniques similar to those of chapter 2 section 2 of Hirsch [7] there is a function $\bar{\pi}^{\cdot}: \mathcal{X} \dashrightarrow \mathbb{R}^N$ which is C^{∞}, vanishes outside closure (B) and has $D\bar{\pi}^{\cdot}(x_2) = \pi^*$. Then $(\pi^{\cdot} + \lambda\,\bar{\pi}^{\cdot}) \in G^3$ for $\lambda > 0$ and approximates π^* arbitrarily well. This proves lemma (B-2) and thus (B-1).

APPENDIX (C)--Symmetric Efficiency

The objective is to show that in a symmetric game with $\pi_k^j < 0$ $j \neq k$ $\pi_j^j + (N - 1)\ \pi_k^j = 0$ and $S < 0$ imply local efficiency. It was already shown that this implies no small symmetric perturbation makes any firm better off. Suppose π has rows π^j as always. I will show that if z is an asymmetric N-vector then $\pi_k z < 0$ for some k thus implying any non-symmetric perturbation makes at least one firm worse off via the mean value theorem.

Lemma (C-1): If z is non-symmetric for some k $\pi_k z < 0$.

Proof: Suppose conversely z is asymmetric and $\pi z \geq 0$. Since $\pi_k^j \neq 0$ $j \neq k$ a check shows that π has rank N - 1. Since $\pi e = 0$ where e is symmetric (by assumption) it cannot be that $\pi z = 0$. Thus we may assume $\pi z \geq C$ where $C_1 = 1$ $C_k = 0$ $k > 1$. By a theorem on linear inequalities found for example as theorem 2.7 in Gale [3] the system $\pi z \geq C$ has a solution if and only if the system $y'\pi = 0$ $y'C = 1$ has no non-negative solution. Letting $e = (1, \ldots, 1)'$ we see that $e'\pi = 0$ and $e'C = 1$. Thus $\pi z \geq C$ has no solution, a contradiction.

Q.E.D.

APPENDIX (D)--The Manifold of Autonomous Steady States

The objective is to prove that when the hypotheses of proposition (3-1) are satisfied at (x^*, R^*) there is an open set $U \supset (x^*, R^*)$ in which steady states form a manifold of dimension $(N - 1)$. This is done by means of four lemmas.

Lemma (D-1): If $\lambda(x^*)$ does not admit a weak positive null vector then there is an open $U \supset x^*$ such that $x \in U$ and (x, R) a steady state implies $\pi(x)$ is singular.

Proof: Since $\lambda(x)$ is continuous choose U so that $x \in U$ implies $\lambda(x)$ has no weak positive null vector. If $\pi(x)$ is non-singular the steady state is exceptional and from (2-20) satisfies $\sum_{k=1}^{N} \lambda_k^j / [(\pi^{-1})^k \lambda_k]^2 = 0$ contradicting the fact λ doesn't admit weak positive null vectors.

Q.E.D.

Lemma (D-2): If $[\pi(x^*) \mid \lambda_k(x^*)]$ has full rank for all k then there is open $U \supset x^*$ such that $x \in U$ and $\pi(x)$ singular imply (x, R) can't be a non-autonomous steady state.

Proof: Since $\pi(x)$ and $\lambda(x)$ are continuous choose U so that $x \in U$ implies $[\pi(x) \mid \lambda_k(x)]$ has full rank for all k. From (2-14) a necessary condition for (x, R) to be a steady state is

$$[\pi \mid -\lambda_k] \begin{bmatrix} R_k \\ y^k \end{bmatrix} = 0 \qquad \text{all } k \tag{D-1}$$

Since π is singular and $[\pi \mid \lambda_k]$ has full rank the only solution to (D-1) is $\pi R_k = 0$ and $y^k = 0$. Thus all $y^k = 0$ and if (x, R) is a steady state it is an autonomous one. Q.E.D.

Corollary D-3: If (x^*, R^*) are as in proposition (3-1) there is $U \supset (x^*, R^*)$ in which all steady states are autonomous.

Now define $w_k^j = \pi^j R_k$. From (1-1) and (1- 7) the equations of motion are

$$\dot{x}^j = \sum_{k=1}^{N} R_k^j w_k^k$$

$$\dot{R}_k^j = b\eta_k^j \left[\pi_j^k w_k^j + \pi_j^j w_k^k\right] \tag{D-2}$$

Let $\pi_{-k}^j = (\pi_1^j, \ldots, \pi_{k-1}^j, \pi_{k+1}^j, \ldots, \pi_N^j)$. The derivative of w_k^j with respect to all the state variables is the row vector

$$Dw_k^j = \underset{1}{[(D\pi^j R_k)' \mid 0 \mid \ldots} \underset{k}{\mid \pi_{-k}^j \mid \ldots \mid 0]} \tag{D-3}$$

where $D\pi^j$ is the matrix of second derivatives of π^j. Observe that $R_k^* = \gamma/\gamma_k$ so that this can be written as

$$Dw_k^j = [(D\pi^j \gamma)'/\gamma_k \mid 0 \mid \ldots \mid \pi_{-k}^j \mid \ldots \mid 0] \tag{D-4}$$

The matrix Dw is formed by stacking the Dw_k^j.

Lemma (D-4): If U contains only autonomous steady states and at steady states corank (Dw) $\leq$ (N-1) then autonomous steady states are a manifold in U.

Proof: Since π is continuous by choosing U small enough we may assume whenever π is singular the first row (say) is a linear combination of the other rows. Since all steady state in U are autonomous w = 0 is necessary and sufficient for a steady state there. Define $\bar{w}$ to be the sub-matrix of w formed by deleting the (N-1) rows corresponding to $w_2^1, w_3^1, \ldots, w_N^1$. I claim that $\bar{w} = 0$ implies w = 0 in U and $D\bar{w}$ has full rank, which will prove the lemma.

Suppose $\bar{w} = 0$. Then $w_1^1, w_1^2, \ldots, w_1^N = 0$ which reads $\pi R_1 = 0$. Since $R_1^1 = 1$ this implies π is singular. Does $w_k^1 = 0$?
This reads $\pi' R_k = 0$. But $\pi^j R_k = 0$ for $j \neq k$ and π^1 is a linear combination of the π^j. Thus $\pi' R_k = 0$. So w = 0.

Since corank (Dw) $\leq$ (N-1) to show $D\bar{w}$ has full rank it suffices to show that Dw_k^1 is a linear combination of the rows of $D\bar{w}$. Observe from (D-4) that

$$\gamma_k Dw_k^m - \gamma_1 Dw_1^m = [-\gamma_1 \pi_{-1}^m \mid 0 \mid \ldots \mid \gamma_k \pi_{-k}^m \mid 0] \qquad \text{(D-5)}$$

Thus

$$\gamma_k Dw_k^1 - \gamma_1 Dw_1^1 = [-\gamma_1 \pi_{-1}^1 \mid 0 \mid \ldots \mid \underset{k}{\gamma_k \pi_{-k}^1} \mid 0] \qquad \text{(D-6)}$$

Suppose $\pi^1 = \sum_{i=2}^{N} \mu_1 \pi^i$. Then from (D-5) and (D-6)

$$Dw_k^1 = (1/\gamma_k)[\gamma_1 Dw_1^1 + \sum_{i=2}^{N} (\gamma_k Dw_k^1 - \gamma_1 Dw_1^i)\mu_i] \qquad \text{(D-7)}$$

is the desired linear combination. Q.E.D.

Lemma (D-5): At an autonomous steady state if the stability matrix has corank (A) $\leq$ (N-1) then corank (Dw) $\leq$ (N-1).

Proof: From (D-2) the equations of motion can be written as

$$\dot{v} = Lw \qquad \text{(D-8)}$$

where v is the vector of state variables and L is an $N^2 x N^2$ matrix. Since w = 0 at the autonomous steady state A = LDw. Thus corank (A) $\leq$ (N-1) implies corank (Dw) $\leq$ (N-1). Q.E.D.

The proof of proposition (3-1) now follows from the fact that corollary (D-3) and lemma (D-5) imply the hypothesis of lemma (D-4).

References

[1] James Friedman, "Non-cooperative Equilibria for Supergames," RE Stud, January 1971.

[2] _____, Oligopoly and the Theory of Games, North-Holland, 1977.

[3] David Gale, Theory of Linear Economic Models, McGraw-Hill, 1960.

[4] Edward Green, "Non-cooperative Price Taking in Large Dynamic Markets," JET, April 1980.

[5] Joel Guttman, "Understanding Collective Action: Matching Behavior," AER, May 1978.

[6] Guillemin and Pollack, Differential Topology, Prentice-Hall, 1974.

[7] Morris Hirsch, Differential Topology, Prentice-Hall, 1974.

[8] Hirsch and Smale, Differential Equations, Dynamical Systems, and Linear Algebra, Academic Press, 1974.

[9] David Levine, "The Enforcement of Collusion in a Symmetric Quadratic Oligopoly," MIT, 1980.

[10] _____, "Local Almost Perfect Equilibria," MIT, 1981.

[11] _____, "Some Result on Dynamical Systems," MIT, 1981.

[12] Marschak and Selten, General Equilibrium with Price Making Firms, Springer-Verlag, 1974.

[13] _____, "Restabilizing Responses, Inertia Supergames, and Oligopolistic Equilibria," QJE, February 1978.

[14] Neal McCoy, Fundamentals of Abstract Algebra, Allyn and Bacon, 1973.

[15] John Milnor, Topology from the Differential Viewpoint, University Press of Virginia, 1976.

[16] Roy Radner, "Collusive Behavior in Epsilon Equilibria," JET, April 1980.

[17] R. Selten, "A Re-examination of the Perfectness Concept," Int. J. Game Theory, 21 (1979), pp. 1-9.

[18] E.C. Zeeman, Catastrophe Theory, Addison-Wesley, 1974.

CHAPTER III: LOCAL ALMOST PERFECT EQUILIBRIUM WITH LARGE ADJUSTMENT COSTS

0. INTRODUCTION

This paper studies continuous time games with closed-loop strategy spaces. A strategy selection such that every agent's strategy is optimal for all initial conditions is called a perfect equilibrium. This is an equilibrium of expectations - regardless of what happens now each agent expects that in the future everyone will behave optimally.[1]

There are two main difficulties with the perfectness concept. First, these equilibria are typically non-unique - in many cases entire continua of equilibria are known to exist.[2] Second, the informational requirements are absurd. Agents are supposed to know what will happen in all contingencies, including contingencies that never have and never will occur and which are far distant from the actual path of the system. This is particularly upsetting since opponents' behavior away from the realized path may not be uniquely determined by what they do along that path.

The objective of this paper is to point out that approximate perfect equilibria that satisfy reasonable informational assumptions exist and that these ε-perfect equilibria[3] are unique in the sense that they are approximately equal to one another.

A reasonable assumption about agents' information

is that they know all relevant information locally near states which have actually occured. They could gain this information by making small perturbations in their controls near the status quo.[4] The model of local information I propose is to assume that agents know low order derivatives of relevant functions at the status quo. Naturally this presupposes a relatively smooth environment.

For local data to be used in making intertemporal decisions it must first be extrapolated to global information. I model this by assuming that agents extrapolate income streams linearly into the future. Casual empiricism indicates that the use of such techniques is widespread. My view is this: agents entering an environment for the first time do so with the working hypothesis that linear extrapolation works relatively well. As long as their experience does not contradict this belief they will continue to use naive extrapolation to compute almost optimal paths. Hence ε-perfect equilibria sufficiently smooth that linear extrapolation works, if they exist, are the only economic equilibria likely to be observed.

Let me briefly indicate how the local adjustment procedure works. At each moment of time agents extrapolate income streams linearly and based on this make an (almost) optimal choice at that moment. By

continually reoptimizing in this manner at all future times it follows from Bellman's principle that the path chosen this way will be (almost) optimal. Notice that agents need not actually determine what they will do in the future, indeed if at some future time their decision were to be based on linear extrapolations from some far past date they would not do well at all.

One important smooth ε-perfect equilibrium, which I call the fundamental solution, is derived by basing linear extrapolation on the presumption that all agents behave myopically. The loss from doing this is of the same order of magnitude as that from complete linear extrapolation. As a result local almost perfect equilibria avoids the game theoretic infinite regress. It is true that each agent's behavior depends on what he thinks his opponents think he thinks, and so forth. In a model passing through time, however, the higher levels in this chain have consequences in the more distant future. With discounting these higher level conjectures may as well be abandoned. This leads to a finite recursion. At the final stage a solution is uniquely determined because everyone is presumed myopic. This accords well with reality, for although game theoretic paradoxes pose a curious and interesting intellectual challenge, game theoretic regressions don't seem to be an important part of observed economic behavior.

An important fact that follows from the recursive nature of the fundamental solution is that all ε-perfect equilibria smooth enough to permit accurate linear extrapolation and with ε small enough are close to the fundamental solution: the adjustment path is essentially unique.

The organization of the paper by sections is: (1) describes a simple dynamic model with adjustment costs; (2) describes myopic and linear extrapolation, the myopic and fundamental solutions and characterizes them in the main theorem; (3) contrasts the full optimum with the fundamental solution in a control problem with quadratic objective; (4) examines the fundamental solution in a simple duopoly problem; (5) states an extension of the earlier results; and (6) summarizes the salient conclusions.

1. THE MODEL

This section describes a simple dynamic economic environment with adjustment costs. Extensions are considered in section six.

There are N agents j=1,...,N. Agent j's vector of control variables is denoted $w^j \in \mathbb{R}^{m_j}$. The vector of state variables is $v \in \mathbb{R}^m$. The control and state spaces are assumed unbounded. The equation of motion is

$$\dot{v} = gw = \sum_{k=1}^{N} g_k w^k \tag{1-1}$$

where the g_k are fixed m x m_j matrices. You may wish to think of the scalar case with v^j firm j's capital stock, w^j its level of investment and $g^j \equiv [0,...,1,0...]$ with 1 in the jth position.

Agent j's objective is the present value of income. His discount rate is $0<\rho^j<\infty$ with corresponding discount factor $\delta^j \equiv 1/\rho^j$; his time horizon is infinite. The income of agent j is given by

$$a^j(v,w^j) \equiv \alpha^j(v) - \beta^j(w^j) \tag{1-2}$$

where a^j is net income, α^j is gross income and β^j are adjustment costs. It is assumed

$$\beta^j(w^j) = (1/b)(1/2)(w^j)^T w^j \tag{1-3}$$

where T denoted the transpose of a vector,

$0 < b << 1$ is a "small" scalar constant and adjustment costs are correspondingly "large". Since α^j may be non-linear, if agents move rapidly relative to the discount factor the income stream will be highly non-linear and linear extrapolation won't work well. To prevent this adjustment costs have been assumed large.

There are several reasons why it may be costly to adjust quickly. It may be physically difficult to change variables such as the capital stock or the physical location of the agent. From the perspective of this paper there is a more fundamental source of adjustment costs which needs to be made explicit: the cost of determining an optimal policy. Suppose I am selling apples and need to know how many I should pick today to sell tomorrow. I know I sold twenty-five apples today at one dollar each and can be fairly confident I can do the same tomorrow. If I contemplate changing output by five or less apples I must know my demand curve in the range fifteen to twenty-five apples in order to make an optimal decision subject to the self-imposed restriction that I'm not going to change output by more than five apples. This I can do fairly easily by inspecting prices and sales over the last several weeks, by recalling conversations I've had with customers, and the like. If, however, I wish to contemplate changing output by up to fifteen apples

I must know my demand curve in the range five to thirty-five apples. To form a reliable determination of demand over so broad a range in so little time as a day will require an appreciable expenditure of time and effort on my part: telephone marketing surveys, poring over sales/revenue records from many years, and the like. The point is that adjustment costs may stem from bounded rationality - the faster I move the more resources I must expend to determine how best to move.

Throughout this paper the following assumption is maintained

Assumption (A): α^j is five times continuously differentiable and for some $1 \leq K_0 < \infty$ $p=0,1,2,3,4,5$

$|D^p \alpha^j| \leq K_0$; $|g| \leq K_0$; $|\rho^j| \leq K_0$; $|\delta^j| \leq K_0$

Here D^p denotes pth order differentiation. The fact that all the bounds in assumption (A) are the same is a convention. The restrictive assumption is that α^j be smooth and that the derivatives remain bounded as $|v|$ grows large.

Definition (1-1): A (closed-loop) strategy for agent j is a bounded continuously differentiable function $f^j : \mathbb{R}^m \to \mathbb{R}^{m_j}$. This means that at v when playing f^j agent j chooses $w^j = f^j(v)$.

This strategy space is limited in the sense that agents don't choose their controls as a function of the entire past, merely as a function of the state variables. While this is restrictive with the present formulation of the equation of motion, in the more general state equation of section five the state space can be extended to include various informational variables (including time). The quite reasonable interpretation of definition (1-1) in this more general context is that agents' decisions are based on a finite set of variables which have a Markovian structure - i.e. evolve according to an ordinary differential equation.

Suppose f is a strategy selection. Since f is bounded and continuously differentiable there is a unique flow $\phi_t(f,v)$ defined for $-\infty<t<\infty$ and for all $v \in \mathbb{R}^m$ which satisfies

$$\begin{aligned} \phi_0(f,v) &= v \\ D_t\phi_t(f,v) &= gf(\phi_t(f,v)) \end{aligned} \tag{1-4}$$

and is the path of the state variables when all agents k play f^k and the system starts at v at time zero. By standard arguments it can be shown that

$$D_v\phi_t = \exp\left[g\int_0^t Df(\phi_s)ds\right] \tag{1-5}$$

or equivalently that $D_v\phi_t$ satisfies the variational equation

$$D_t(D_v\phi_t) = (gDf(\phi_t))(D_v\phi_t) \tag{1-6}$$

with initial condition $D_v\phi_0 = I$ the identity matrix.

Using this flow the present value of agent j's income starting at v until the horizon t when f is played is

$$A_t^j(f,v) \equiv \int_0^t a^j(\phi_s(f,v), f^j(\phi_s(f,v)))\exp(-\rho^j s)ds \tag{1-7}$$

Since f is bounded a^j is bounded and

$$A^j(f,v) \equiv \lim_{t\to\infty} A_t^j(f,v) \tag{1-8}$$

exists and is finite.

We shall need a smoothness criterion for strategies.

Definition (1-2): A strategy selection $\tilde{f}$ is called K-smooth iff for $K \geq K_0$, $p=0,1,2,3$ $|D^p\tilde{f}| \leq bK$

Thus f must be thrice continuously differentiable with bounded derivatives. The dependence of the bound on b makes good sense - if adjustment costs are large of order (1/b) it is to be expected that the adjustment rate will be small of order b. In section three it is

shown that f K-smooth with b<<1/K is precisely the condition required for linear extrapolation to provide good results.

Let f^{-j} be the strategy selection for all agents but agent j.

Definition (1-3): A strategy $\tilde{f}^j$ is called ε-optimal for j given $\tilde{f}^{-j}$ iff for all v

$$\tilde{A}^j(v) \equiv A^j(\tilde{f},v) \geq A^j(f,v) - \varepsilon$$

if, in other words, j is within ε of the optimum.

Definition (1-4): A strategy selection $\tilde{f}$ is called an ε-perfect equilibrium iff for all j $\tilde{f}^j$ is ε-optimal for $\tilde{f}^{-j}$.

This says that each agent ε-optimizes having rational expectations about everyone's behavior in all contingencies. While bounded rationality is consistent with ε-perfect equilibrium the converse isn't necessarily true: why if everyone has rational expectations do they sub-optimize? An alternative interpretation, and the one relevant to this paper, is that agents make small expectational errors and this causes them to undertake sub-optimal decisions relative to the full optimum.

A key technical fact is that in a smooth environment approximate optimization occurs if and only if Bellman's equation is approximately satisfied. To state this precisely some notation is required. If M(x) is a function the notation $M(x) = O^p(K)$ means that there is some positive real function $\bar{M}(\cdot)$ which <u>does not depend on the economic environment in any way</u> such that for $n = 0,1,\ldots,p$ $|D^n M(x)| \leq \bar{M}(K)$. The superscript T denotes the transpose of a matrix or vector, while $D\tilde{A}^j$ is the row vector of derivatives of $\tilde{A}^j$ with respect to the state variables. Then

<u>Proposition (1-1)</u>: $\tilde{f}$ K-smooth and $b<1/K$ imply

(A) $\tilde{A}^j = O^2(K)$ and therefore
$\bar{f}^j(v) \equiv bg_j^T D\tilde{A}^{jT}(v) = bO^1(K)$

(B) <u>Sufficiency</u>
$\tilde{f}^j = \bar{f}^j + (b\varepsilon)^{\frac{1}{2}} O(K)$ implies $\tilde{f}^j$ ε-optimal for j given $\tilde{f}^{-j}$

(C) <u>Necessity</u> if $b\varepsilon \leq 1$
$\tilde{f}^j$ ε-optimal for j given $\tilde{f}^{-j}$ implies
$\tilde{f}^j = \bar{f}^j + (b\varepsilon)^{\frac{1}{2}} O(K)$

Notice that there is no reason that the symbols O(K) in (B) and (C) should stand for the same function: the sole reason for introducing this notation is to avoid

having to continually recompute bounds.

To interpret proposition (1-1) it should be realized that $\bar{f}^j$ maximizes the approximate Hamiltonian that arises when $D\tilde{A}^j$ are used as costate variables in place of the present value derivatives along the optimal path. Note that the assumption of linear response and quadratic adjustment costs guarantees that $\bar{f}^j$ is unique. An implication of proposition (1-1) from the case $\varepsilon = 0$ is that $\tilde{f}^j = \bar{f}^j$ is necessary and sufficient for an optimum: this is simply Bellman's principle.

An ε-equilibrium requires that each agent ε-optimize over all closed-loop strategies. Although this rules out the use of strategies which depend on the entire past history of the game it is important to realize that this constraint is not binding: if a closed-loop strategy is ε-optimal (relative to other closed-loop strategies and for given initial conditions) then no strategy of any kind (provided the differential equation (1-1) has a unique solution) can yield a gain of more than ε over the closed-loop strategy.

The reason for this is fairly straightforward. The proof of proposition (1-1) actually shows that a smooth closed-loop strategy is ε-optimal relative to other closed-loop strategies if and only if it is ε-optimal relative to all smooth open-loop control paths originating at the same initial conditions - this

requires only notational changes in the proofs. Since any path for which (1-1) has a unique solution can be approximated arbitrarily closely by smooth paths it follows that a path ε-optimal with respect to smooth paths is ε-optimal with respect to all paths.

The significance of this is that while the restriction of agents to closed-loop strategies requires bounded rationality it does not entail any loss to agents - they do as well with closed-loop strategies as they would if all strategies were available.

A proof of proposition (1-1) can be found in the appendix.

2. LOCAL PERFECT EQUILIBRIA

This section investigates environments in which linear extrapolation works well. This means first that assumption (A) must be satisfied. If this fails - if the non-linearity of the exogenously given functions as measured by K_0 is large, and the adjustment costs as measured by (1/b) is small - agents will move rapidly over a non-linear income surface and the resulting income streams will be highly non-linear. As we will show, more than this is required. The agents' strategies must also be K-smooth with b small relative to K. In other words, even if the exogenous environment is relatively smooth, agents might choose to behave in such highly non-linear ways that linear extrapolation does poorly. In this section it is shown that in relatively smooth environments the loss from linear extrapolation is of order b^5. Thus I call K-smooth Kb^5-perfect equilibria K-linearly perfect equilibria to reflect the fact that they are smooth enough to permit relatively accurate linear extrapolation, and agents do no worse in order of magnitude loss than if they extrapolated linearly.

The goal of this section is to characterize linearly perfect equilibria. An adjustment path called the fundamental solution is explicitly computed and shown

to be linearly perfect, thus establishing existence. The fundamental solution is shown also to have an important dominance property that implies approximate uniqueness : every linearly perfect equilibrium is close to the fundamental solution.

The proofs of these facts are based on Taylor series representations simple in principle but complicated in detail. The first subsection is devoted to analyzing various bounds and approximations needed in the proof of the main theorem. Subsection two describes linear extrapolation and the fundamental solution, and states the main theorem. The final two subsections prove the main theorem.

To deal efficiently with high order derivatives the perspective that they are symmetric multi-linear maps is adopted. Thus if $M(x)$ is a function $D^pM(x)[y^1,\ldots,y^p]$ is the p-linear map $D^pM(x)$ applied to $y^1,\ldots,y^p$.

Expansions and Bounds

Let $\tilde{f}$ be K-smooth with $b<1/K$. The pth time derivative of income to j at v is denoted by $\tilde{a}^j_p(v)$ and can be computed recursively from

$$\tilde{a}^j_0(v) \equiv a^j(v,\tilde{f}^j(v))$$
$$\tilde{a}^j_p(v) = D\tilde{a}^j_{p-1}[g\tilde{f}(v)] \qquad (2-1)$$

The derivatives $D^p\tilde{a}_0^j$ needed to compute (2-1) recursively are

$$D\tilde{a}_0^j[y] = D\alpha^j[y] + (1/b)\tilde{f}^{jT}D\tilde{f}^j[y]$$

$$D^2\tilde{a}_0^j[y^1,y^2] = D^2\tilde{a}_0^j[y]$$
$$= D^2\alpha^j[y] + (1/b)\{D\tilde{f}^{jT}[y^1]D\tilde{f}^j[y^2]$$
$$+ \tilde{f}^{jT}D^2\tilde{f}^j[y]\}$$

$$D^3\tilde{a}_0^j[y^1,y^2,y^3] = D^3\tilde{a}_0^j[y]$$
$$= D^3\tilde{a}_0^j[y] + (1/b)\{\sum_{(i_1,i_2,i_3)=\text{all permutations of } (1,2,3)} D\tilde{f}^{jT}[y^{i_1}]D^2\tilde{f}^j[y^{i_2},y^{i_3}]\}$$
$$+ (1/b)\tilde{f}^{jT}D^3\tilde{f}^j[y] \qquad (2\text{-}2)$$

Using assumption (A) and K-smoothness with (2-1) and (2-2) it can be established that

$$\tilde{a}_p^j = b^pO^2(K) \quad p=0,1 \quad \tilde{a}_2^j = b^2O^1(K) \qquad (2\text{-}3)$$

and inspection shows that all parts of K-smoothness are required to bound $D\tilde{a}_2^j$.

Next consider

$$D\tilde{A}^j(v) = \int_0^\infty D\tilde{a}_0^j(\tilde{\phi}_s)\exp(-\rho^j s)ds \qquad (2\text{-}4)$$

Expand (2-4) under the integral sign in Taylor series

$$D\tilde{A}^j(v) = \int_0^\infty \{D\tilde{a}_0^j(v) + D\tilde{a}_1^j(\phi_s^*)\}\exp(-\rho^j s)ds$$
$$= \delta^j D\tilde{a}_0^j(v) + bO(K) \qquad (2\text{-}5)$$

where (2-3) is used to bound the remainder. Similarly

$$D^2\tilde{A}^j(v) = \delta^j D^2\tilde{a}_0^j(v) + bO(K) \qquad (2\text{-}6)$$

Finally, expand (2-4) into a first order Taylor series with remainder

$$D\tilde{A}^j(v) = \int_0^\infty \{D\tilde{a}_0^j(v) + D\tilde{a}_1^j(v) + (1/2)D\tilde{a}_2^j(\phi_s^*)\}\exp(-\rho^j s)ds$$
$$= \delta^j D\tilde{a}_0^j(v) + (\delta^j)^2 D\tilde{a}_1^j(v) + b^2 O(K) \qquad (2\text{-}7)$$

This final equation forms the basis of linear extrapolation techniques.

Linear Extrapolation

Suppose all agents play $\tilde{f}$ K-smooth with $b<1/K$. Agent j computes approximately the present value of income at v to be $\tilde{a}^j$ and chooses $\tilde{f}^j$ according to

$$\tilde{f}^j(v) = bg_j^T D\,\tilde{a}^{jT}(v) \qquad (2\text{-}8)$$

If $\tilde{a}^j = \tilde{A}^j$ so that no approximation is involved $\tilde{f}^j$ would be optimal by proposition (1-1). Also by proposition (1-1) necessary and sufficient conditions for $\tilde{f}^j$ to be ε-optimal are

$$|\tilde{f}^j(v) - \bar{f}^j(v)| \leq (b\varepsilon)^{\frac{1}{2}} O(K) \qquad (2\text{-}9)$$

(where however different functions $O(K)$ may be involved depending on whether necessity or sufficiency are involved.) What does this mean in terms of $\tilde{a}^j$?

$$|\tilde{f}^j(v) - \bar{f}^j(v)| = |b g_j^T [D\tilde{a}^{jT}(v) - D\tilde{A}^{jT}(v)]|$$
$$\leq bK_0 |D\tilde{a}^j(v) - D\tilde{A}^j(v)| \qquad (2\text{-}10)$$

Thus necessary and sufficient conditions for $\tilde{f}^j$ to be ε-optimal are

$$|D\tilde{a}^j(v) - D\tilde{A}^j(v)| \leq (\varepsilon/b)^{\frac{1}{2}} O(K) \qquad (2\text{-}11)$$

The objective is to consider methods of extrapolating future income from local information. A naive form of extrapolation is the purely myopic method

$$\tilde{a}^j(v) = \delta^j \tilde{a}_0^j(v) \qquad (2\text{-}12)$$

which assumes v won't change in the future. From the Taylor expansion of $\tilde{A}^j$ in (2-5) we see that $\tilde{a}^j = \tilde{A}^j + bO^1(K)$, where remember O^1 implies the first derivatives are close. Thus from (2-11) we are led to conclude (2-12) is $b^3 O(K)$-optimal for j. This leads us to define

Definition (2-1): $\tilde{f}^j$ is K-myopically optimal (for j given $\tilde{f}^{-j}$) iff $\tilde{f}^j$ is b^3K-optimal.

In other words, strategies that do as well in order of magnitude term as myopic extrapolation are called myopically optimal.

A more sophisticated extrapolation technique is to project the income stream linearly into the future, setting

$$\tilde{a}^j = \delta^j \tilde{a}_0^j + (\delta^j)^2 \tilde{a}_1^j \qquad (2\text{-}13)$$

Using the expansion (2-7) and (2-11) we see that linear extrapolation is $b^5O(K)$-optimal. This leads us to define

Definition (2-2): $\tilde{f}^j$ is K-linearly optimal (for j given $\tilde{f}^{-j}$) iff $\tilde{f}^j$ is b^5K-optimal.

Definition (2-3): $\tilde{f}$ is K-linearly perfect iff $\tilde{f}$ is K-smooth and $\tilde{f}^j$ is K-linearly optimal for j given $\tilde{f}^{-j}$.

Linear perfect equilibria are exactly those equilibria smooth enough to permit linear extrapolation, and in which agents do no worse than they would if they used linear extrapolation (in order of magnitude). As pointed out above, in an environment of the type under consideration these are the only economically interesting equilibria.

The fundamental myopic (or myopic) solution for

j is to set

$$\tilde{a}^j_m \equiv \delta^j \alpha^j(v) \tag{2-14}$$

which is the same as (2-12) except that adjustment costs are ignored. Since these are of order b anyway, we shouldn't expect any greater order of magnitude loss from (2-14) than from (2-12).

The fundamental linear (or fundamental) solution for j is to set

$$\tilde{a}^j_\ell \equiv \delta^j a^j(v, \tilde{f}^j_m) + (\delta^j)^2 D\alpha^j(v) g\tilde{f}_m(v) \tag{2-15}$$

which is derived by extrapolating linearly under the assumption that everyone is myopic (and also ignores an adjustment cost term of order b^2).

Theorem (2-1) [Main Theorem]:

(A) Smoothness: $\tilde{f}^j_m$ and $\tilde{f}^j_\ell$ are $O(K_0)$-smooth

(B) Existence: If $b<O(K_0)$ $\tilde{f}_\ell$ is $O(K_0)$-linearly perfect

(C) Dominance: If $\tilde{f}$ is K-smooth $K \geq O(K_0)$ $b<1/K$

(1) $\tilde{f}^j_m$ and $\tilde{f}^j_\ell$ are $O(K)$-myopically optimal

(2) if $\tilde{f}^{-j}$ is K-myopically optimal for -j
$\tilde{f}^j_\ell$ is $b^4 O(K)$-optimal for j

(3) if $\tilde{f}$ is K-linearly perfect for any $0 \leq \eta < 1$
$\tilde{f}^j_\ell$ is $b^{4+\eta} O(K)$-optimal for j

(D) Uniqueness: if $\tilde{f}$ is K-linearly perfect $K \geq O(K_0)$
for any $0 \leq \eta < 1$ $|\tilde{f} - \tilde{f}_\ell| = (b^{5+\eta})^{\frac{1}{2}} O(K)$

Note that the functions O(K) need not all be the same, and in particular in (C3) and (D) they may depend on the choice of η.

The main theorem uses the fundamental solution to categorize linearly perfect equilibria. For b small enough and K large enough part (B) shows linearly perfect equilibria exist and part (D) shows that they are essentially unique in the sense that they are all close to the joint fundamental solution. Part (C) is also of interest: it shows that as long as opponents are reasonably well-behaved the fundamental solution does reasonably well.

The next two subsections prove the main theorem.

Outline of the Proof

This subsection states a number of lemmas and shows how they imply the main theorem. The next subsection demonstrates the truth of the lemmas.

Lemma (2-1): $\tilde{f}^j_m = bO^3(K_0)$ $\tilde{f}^j_\ell = \tilde{f}^j_m + b^2 O^3(K_0)$

Lemma (2-2): if $\tilde{f}$ K-smooth $b < 1/K$ then

(A) $|\tilde{f}^j - \tilde{f}^j_m| \leq b^2 K$ implies $\tilde{f}^j$ O(K)-myopically optimal

(B) $\tilde{f}^j$ K-myopically optimal implies $|\tilde{f}^j - \tilde{f}^j_m| \leq b^2 O(K)$

Lemma (2-3): if $\tilde{f}$ K-smooth $K \geq O(K_0)$ $b<1/K$ and $|D^p\tilde{f} - D^p f^{\sim}_m| \leq (b^{3+\eta})^{\frac{1}{2}}K$ where $0 \leq \eta \leq 1$ $p=0,1$ then

(A) $\tilde{f}^j_\ell$ is $b^{4+\eta}O(K)$-optimal

(B) if in addition $\tilde{f}$ is K-linearly perfect

$$|\tilde{f}^j - \tilde{f}^j_\ell| \leq (b^{5+\eta})^{\frac{1}{2}}O(K)$$

Lemma (2-4): M(x) twice continuously differentiable $|M| \leq Kb^\mu$ $|D^2M| \leq Lb$ imply $|DM| \leq (b^{\mu+1})^{\frac{1}{2}}\sqrt{8LK}$

Let's see how the main theorem follows from these lemmas. Part (A) follows from lemma (2-1) and the definition of K-smoothness. To prove part (B) we need only show that $\tilde{f}_\ell$ satisfies the hypothesis of lemma (2-3A) with $\eta = 1$. This is a direct consequence of lemma (2-1).

Part (C-1) is just lemma (2-2A). To get (C-2) observe by lemma (2-2B) $|\tilde{f}^{-j} - \tilde{f}^{-j}_m| \leq b^2O(K)$. Since $\tilde{f}^{-j}_m$ and $\tilde{f}^{-j}$ are K-smooth lemma (2-4) enables us to conclude that $|D\tilde{f}^{-j} - D\tilde{f}^{-j}_m| \leq b^{3/2}O(K)$. Then part (C-2) follows from lemma (2-3A).

Parts (C-3) and (D) can be proven simultaneously by induction on η. Take first $\eta=0$. Since $\tilde{f}$ linearly perfect implies $\tilde{f}^k$ myopically optimal (all k) we see as in part (C-2) $|D\tilde{f} - D\tilde{f}_m| \leq b^{3/2}O(K)$. By lemma (2-3A) this proves (C-3) for $\eta=0$, while by lemma (2-3B) it proves also (D) for $\eta=0$.

We now examine the implication of (D) for $\eta=0$. By lemma (2-4) it enables us to conclude $|D\tilde{f} - D\tilde{f}_\ell| \leq b^{7/4}O(K)$. By lemma (2-1) this in turn shows $|D\tilde{f} - D\tilde{f}_m| \leq b^{7/4}O(K)$. Using lemma (2-3) as before shows (C-3) and (D) hold for $\eta = 1/2$ (and therefore $0 \leq \eta \leq 1/2$). Continuing on in this way we get η arbitrarily close to one.

Proof of the Lemmas

Lemma (2-1): $\tilde{f}^j_m = bO^3(K_0)$ $\tilde{f}^j_\ell = \tilde{f}^j_m + b^2O^3(K_0)$

proof: From (3-14) defining $\tilde{a}^j_m$ and (3-8) giving $\tilde{f}^j_m$

$$\tilde{f}^j_m = b\delta^j g^T_j D\alpha^{jT} \tag{2-16}$$

By assumption (A) $|D^p\alpha^j| \leq K_0$ $0 \leq p \leq 5$ which with (2-16) shows $\tilde{f}^j_m = bO^3(K_0)$. From (2-15) defining $\tilde{a}^j_\ell$ and (2-16)

$$\tilde{a}^j_\ell = \tilde{a}^j_m - (1/2)b(\delta^j)^2 D\alpha^j g_j g^T_j D\alpha^{jT} + b(\delta^j)^2 D\alpha^j \textstyle\sum_k \delta^k g_k g^T_k D\alpha^{kT} \tag{2-17}$$

Observing from (2-8) that $\tilde{f}^j_\ell = bg^T_j D\tilde{a}^{jT}_\ell$ and using assumption (A) once again shows $\tilde{f}^j_\ell = \tilde{f}^j_m + b^2O^3(K_0)$.

Q.E.D.

Lemma (2-2): if $\tilde{f}$ is K-smooth $b<1/K$

(A) $|\tilde{f}^j-\tilde{f}^j_m| \leq b^2K$ implies $\tilde{f}^j$ $O(K)$-myopically optimal

(B) $\tilde{f}^j$ K-myopically optimal implies $|\tilde{f}^j-\tilde{f}^j_m|\leq b^2O(K)$

proof:

For any K-smooth strategy selection we have from the Taylor expansion (2-5)

$$\begin{aligned} D\tilde{A}^j &= \delta^j D\tilde{a}^j_0 + bO(K) \\ &= D\tilde{a}^j_m - (1/b)\delta^j(1/2)\tilde{f}^{jT}\tilde{f}^j + bO(K) \\ &= D\tilde{a}^j_m + bO(K) \end{aligned} \qquad (2\text{-}18)$$

From Proposition (1-1) giving $\bar{f}^j$ and (2-8) giving $\tilde{f}^j_m$ (2-18) reads

$$\bar{f}^j = \tilde{f}^j_m + b^2O(K) \qquad (2\text{-}19)$$

If $|\tilde{f}^j - \tilde{f}^j_m| \leq b^2K$ (2-19) implies $|\tilde{f}^j-\bar{f}^j| \leq b^2O(K)$ and by proposition (1-1B) this implies $\tilde{f}^j$ $b^3O(K)$-optimal against -j. Conversely by proposition (1-1C) $\tilde{f}^j$ b^3K-optimal implies $|\tilde{f}^j - \bar{f}^j| \leq b^2O(K)$. Using (2-19) and the triangle inequality then shows $|\tilde{f}^j_m-\bar{f}^j| \leq b^2O(K)$.

Q.E.D.

Lemma (2-3): if $\tilde{f}$ K-smooth $K\geq O(K_0)$ $b<1/K$ and $|D^p\tilde{f} - D^p\tilde{f}_m| \leq (b^{3+\eta})^{\frac{1}{2}}K$ $0\leq\eta\leq1$ $p=0,1$

(A) $\tilde{f}^j_\ell$ is $b^{4+\eta}O(K)$-optimal against -j

(B) if in addition $\tilde{f}$ is K-linearly perfect

$$|\tilde{f}^j - \tilde{f}^j_\ell| \leq (b^{5+\eta})^{\frac{1}{2}} O(K)$$

proof:

The idea of the proof is similar to that of lemma (2-2), using however the Taylor expansion (2-7) in place of (2-5). Throughout the proof K is assumed large enough that $\tilde{f}^j_\ell$ is K-smooth, where lemma (2-1) shows this is possible.

For any K-smooth $\tilde{f}$ from (2-7)

$$\begin{aligned} D\tilde{A}^j &= \delta^j D\tilde{a}^j_0 + (\delta^j)^2 D\tilde{a}^j_1 + b^2 O(K) \\ &= \delta^j [D\alpha^j - (1/b)\tilde{f}^{jT} D\tilde{f}^j] \\ &\quad + (\delta^j)^2 [D^2\alpha^j g\tilde{f} + D\alpha^j g D\tilde{f}] + b^2 O(K) \end{aligned} \tag{2-20}$$

so that from (2-8) and (2-15)

$$\begin{aligned} \bar{f}^j &= \tilde{f}^j_\ell - \delta^j (1/b)[\tilde{f}^{jT} D\tilde{f}^j - \tilde{f}^{jT}_m D\tilde{f}^j_m] \\ &\quad + (\delta^j)^2 [D^2\alpha^j g[\tilde{f} - \tilde{f}_m] + D\alpha^j g[D\tilde{f} - D\tilde{f}_m]] \\ &\quad + b^2 O(K) \\ &= \tilde{f}^j_\ell + |\tilde{f} - \tilde{f}_m| O(K) + |D\tilde{f} - D\tilde{f}_m| O(K) + b^2 O(K) \end{aligned} \tag{2-21}$$

If in addition $|D^p\tilde{f} - D^p\tilde{f}_m| \leq (b^{3+\eta})^{\frac{1}{2}} K \quad p=0,1$ then

$$\bar{f}^j = \tilde{f}^j_\ell + (b^{3+\eta})^{\frac{1}{2}} O(K) \tag{2-22}$$

Taking $\tilde{f}^j = \tilde{f}^j_\ell$ and observing that $\tilde{f}^j_\ell$ satisfies all the required hypothesis shows by proposition (1-1B) $\tilde{f}^j_\ell$ is $b^{4+\eta}O(K)$-optimal against $\tilde{f}^{-j}$. If in addition some other $\tilde{f}^j$ is K-linearly optimal against -j it follows from (2-22), proposition (1-1C) and the triangle inequality that $|\tilde{f}^j - \tilde{f}^j_\ell| \leq (b^{5+\eta})^{\frac{1}{2}}O(K)$.

Q.E.D.

Lemma (3-4): Let M(x) be twice continuously differentiable with $|M| \leq b^\mu K$ and $|D^2M| \leq bL$. Then $|DM| \leq (b^{\mu+1})^{\frac{1}{2}}\sqrt{8LK}$.

proof: Since $|DM(x)|$ is the largest directional derivative on the unit circle it suffices to consider the case where x is a scalar. Assume without loss of generality $DM(x) \geq 0$. The idea of the proof is similar to that in lemma (A-5). Since $|D^2M| \leq Lb$ the earliest DM reaches zero is at $x+DM(x)/Lb$ so that by the fundamental theorem of calculus

$$\begin{aligned} 2b^\mu K &\geq M(x + DM(x)/bL) - M(x) \\ &= \int_x^{x+DM(x)/bL} DM(z)dz \\ &\geq \int_x^{x+DM(x)/2bL} [DM(x)/2]dz \\ &= [DM(x)]^2/4bL \end{aligned} \qquad (2\text{-}23)$$

from which the lemma follows directly.

Q.E.D.

3. AN APPLICATION TO OPTIMAL CONTROL THEORY

The theory of the previous section can be applied to solve one-agent control problems. Because only one agent is involved the subscript j is dropped in this section. Let

$$\alpha(v) = (1/2)v^{T}Mv \quad M \text{ negative definite}$$
$$g = I \quad \text{the identity matrix} \tag{3-1}$$

In other words the control problem is

$$\text{maximize}_{w(\cdot)} \int_0^{\infty}(1/2)[v^{T}Mv-(1/b)w^{T}w]\exp[-\rho s]ds$$
$$\text{subject to } \dot{v} = w \tag{3-2}$$

First this is solved exactly, then compared to the fundamental solution.

The Hamiltonian is

$$H(w) = (1/2)[v^{T}Mv - (1/b)w^{T}w] + \lambda w \tag{3-3}$$

where $\lambda = DA$ along the optimal path are the costate variables. Maximizing the Hamiltonian in w gives

$$w = b\lambda \tag{3-4}$$

while from control theory

$$\dot{\lambda} = \rho\lambda - D_v H = \rho\lambda - Mv \tag{3-5}$$

Thus the first order condition is

$$\begin{bmatrix} \dot{w} \\ \dot{v} \end{bmatrix} = G \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} \rho I & -bM \\ I & 0 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \tag{3-6}$$

First the eigenvectors and eigenvalues of G must be computed. Let γ_i^1, γ_i^2 be the eigenvalues of G with corresponding eigenvectors x_i^1, x_i^2. Let μ_i be the eigenvalues of M and y_i the corresponding eigenvectors. A simple computation shows

$$\gamma_i^1 = \frac{\rho - \sqrt{\rho^2 - 4\mu_i b}}{2}$$

$$\gamma_i^2 = \frac{\rho + \sqrt{\rho^2 - 4\mu_i b}}{2}$$

$$x_i^j = \begin{bmatrix} x_{i1}^j \\ x_{i2}^j \end{bmatrix} = \begin{bmatrix} \gamma_i^j y_i \\ y_i \end{bmatrix} \tag{3-7}$$

Fix v, then w must be chosen so that the system (3-6) is stable; this means $[w^T, v^T]^T$ must be a linear combination of eigenvectors of G whose corresponding eigenvalues have negative real parts. From (3-7) these are the eigenvectors x_i^1.

Let $C = [y_1,\ldots,y_m]$. Then

$$v = \sum_{i=1}^{m} \eta_i y_i = C\eta \tag{3-8}$$

where η is an m-tuple of coefficients. From (4-8)

$$\eta = C^{-1}v \tag{3-9}$$

Finally

$$\begin{aligned} w_e &= \sum_{i=1}^{m} \eta_i x_{il}^1 \\ &= \sum_{i=1}^{m} \eta_i \gamma_i^1 y_i \\ &= C \text{ diag } (\gamma_i^1)\eta \\ &= C \text{ diag } (\gamma_i^1)\ C^{-1}v \end{aligned} \tag{3-10}$$

which gives the exact solution in closed-loop form.

Consider now the fundamental solution. The myopic present value is $\tilde{a}_m = (1/2)(1/\rho)v^T Mv$, and the myopic solution is

$$\tilde{f}_m = w_m = bg^T D \tilde{a}_m^T = (b/\rho)Mv \tag{3-11}$$

The fundamental present value is

$$\begin{aligned} \tilde{a}_\ell &= (1/\rho)[\alpha(v)-\beta(w_m)] + (1/\rho^2)[D\alpha(v)w_m] \\ &= (1/\rho)(1/2)[v^T Mv-(b/\rho^2)v^T M^2 v]+(1/\rho^2)(b/\rho)[v^T M^2 v] \\ &= (1/2)(1/\rho)\ v^T Mv + (1/2)(b/\rho^3)v^T M^2 v \end{aligned} \tag{3-12}$$

Thus the fundamental solution is

$$\tilde{f}_\ell = w_\ell = bg^T D a_m^T$$
$$= (b/\rho)[M + (b/\rho^2)M^2]v \qquad (3\text{-}13)$$

How are (3-10) the exact solution and (3-13) the fundamental solution related? From the main theorem w_ℓ and w_e differ by at most order b^3. Since (3-13) contains no terms of order b^3 it must be the second order Taylor's series approximation to (3-10) viewed as a function of b. Let's verify that this is so. The only expression in (3-10) that depends on b are the γ_i^1 which from (3-7) are

$$\gamma_i^1 = \frac{\rho}{2}\left[1 - \sqrt{1 - \frac{4\mu_i b}{\rho^2}}\right] \qquad (3\text{-}14)$$

Expanding (3-14) into a second order Taylor series around b = 0

$$\gamma_i^1 \simeq \frac{\mu_i b}{\rho} + \frac{\mu_i^2 b^2}{\rho^3} \qquad (3\text{-}15)$$

Consequently to second order

$$w_e \simeq (b/\rho)C\,[\mathrm{diag}(\mu_i)+(b/\rho^2)\mathrm{diag}(\mu_i^2)]\,C^{-1}v$$
$$= (b/\rho)[M+(b/\rho^2)C\,\mathrm{diag}(\mu_i)\,C^{-1}\,C\,\mathrm{diag}(\mu_i)C^{-1}]v$$
$$= (b/\rho)[M + (b/\rho^2)M^2]v \qquad (3\text{-}16)$$

which is exactly (3-13).

Two points should be raised from a purely analytic standpoint. First, the fundamental solution was easy to derive and could have been derived equally easily if α wasn't quadratic, while deriving the exact solution was tedious and was possible only because α was quadratic. Second, the expression for the fundamental solution (2-12) is relatively easy to interpret, while the exact solution is a rather mysterious function of eigenvalues and eigenvectors.

A final note is that the problem given here doesn't actually satisfy assumption (A) since α and $D\alpha$ are unbounded. This reflects merely the fact that the main theorem holds under quite general conditions - assumption (A) is intended largely to simplify the proofs.

4. APPLICATION TO A DUOPOLY PROBLEM

There are two firms, and v^j is firm j's output. Since each firm j chooses w^j, the rate of change of its own output, the matrix of responses of the state to control variables b = I the identity matrix. Each firm's discount rate $\delta^j \equiv 1$ and both firms have identical gross profit functions

$$\alpha^j = [\ 1 - v^1 - v^2\ ]v^j \qquad (4\text{-}1)$$

which corresponds to constant marginal cost, a linear demand curve with unitary slope and price minus marginal cost at zero industry output equal to one.

The myopic present value for firm j is $\tilde{a}^j_m = \alpha^j$ and the myopic strategy is $\tilde{f}^j_m = bD_j\alpha^j = 1-v^k-2v^j \quad k\neq j$. It is easy to see that if both firms behave myopically the symmetric steady state $\tilde{f}^1_m = \tilde{f}^2_m = 0 \quad v^1 = v^2 = \bar{v}_m$ occurs at the static Cournot-Nash output $\bar{v}_m = (1/3)$.

The fundamental solution is easily computed to be

$$\tilde{f}^j_\ell = b\{D_j\alpha^j + b[D_j\alpha^j D^2_{jj}\alpha^j + D^2_{jk}\alpha^j D_k\alpha^k + D_k\alpha^j D^2_{jk}\alpha^k]\}$$
$$j \neq k \qquad (4\text{-}2)$$

which can be solved for the symmetric steady state output

$$\bar{v}_f = (1/3)(1 + b/3) \qquad (4\text{-}3)$$

Thus when both firms play their fundamental strategies the industry is more competitive and both firms worse off than when they act myopically. This is reminiscent of the Stackelberg warfare discussed by Bishop [1]. Both firms realize that when they increase output their opponent's myopic response is to cut output, thus at the static Cournot-Nash output both firms will increase output slightly until the steady state in (4-3) is reached. This is an outstanding example of how, because of a public goods problem, increased rationality of agents can make all agents worse off (except consumers who aren't represented in the game).

Notice that fundamental steady state output in (4-3) is close to the static Cournot-Nash output. It is easy to see why: the condition for a myopic steady state is the first order condition for a static Nash equilibrium of the game with payoff functions α^j. Since the fundamental solution is a small perturbation of the myopic solution its steady states will typically lie nearby.

5. EXTENSIONS

The model of the previous sections had a very strong structure. While this greatly simplified the algebra in the proofs this structure is by no means necessary for our results. In this section several easy extensions are given.

What is important in the types of results here? Two assumptions are key: the exogenous environment must be such as to permit smooth adjustment by agents, and adjustment costs must be large enough that agents will choose to adjust slowly relative to the discount rate and the intrinsic non-linearity of the payoff structure. For example, bang-bang control problems will never fit into this framework - linear extrapolation won't work if large discontinuous changes will take place in the future. A sample of the type of problem that does fit the framework here is

$$\dot{v} = bh(v,w) + g(v)w \tag{5-1}$$

$$a^j(v,w) = \alpha^j(v,w) - \beta^j(v,w) \tag{5-2}$$

$$(1/2)(1/b)w^T\underline{B}^j w \leq \beta^j(v,w) \leq (1/2)(1/b)w^T\bar{B}^j w \tag{5-3}$$

where $\underline{B}^j$ and $\bar{B}^j$ are positive definite and β^j is strictly concave. The quadratic bounds on the adjustment costs illustrated in figure (5-1) insure that agents will choose small values of the

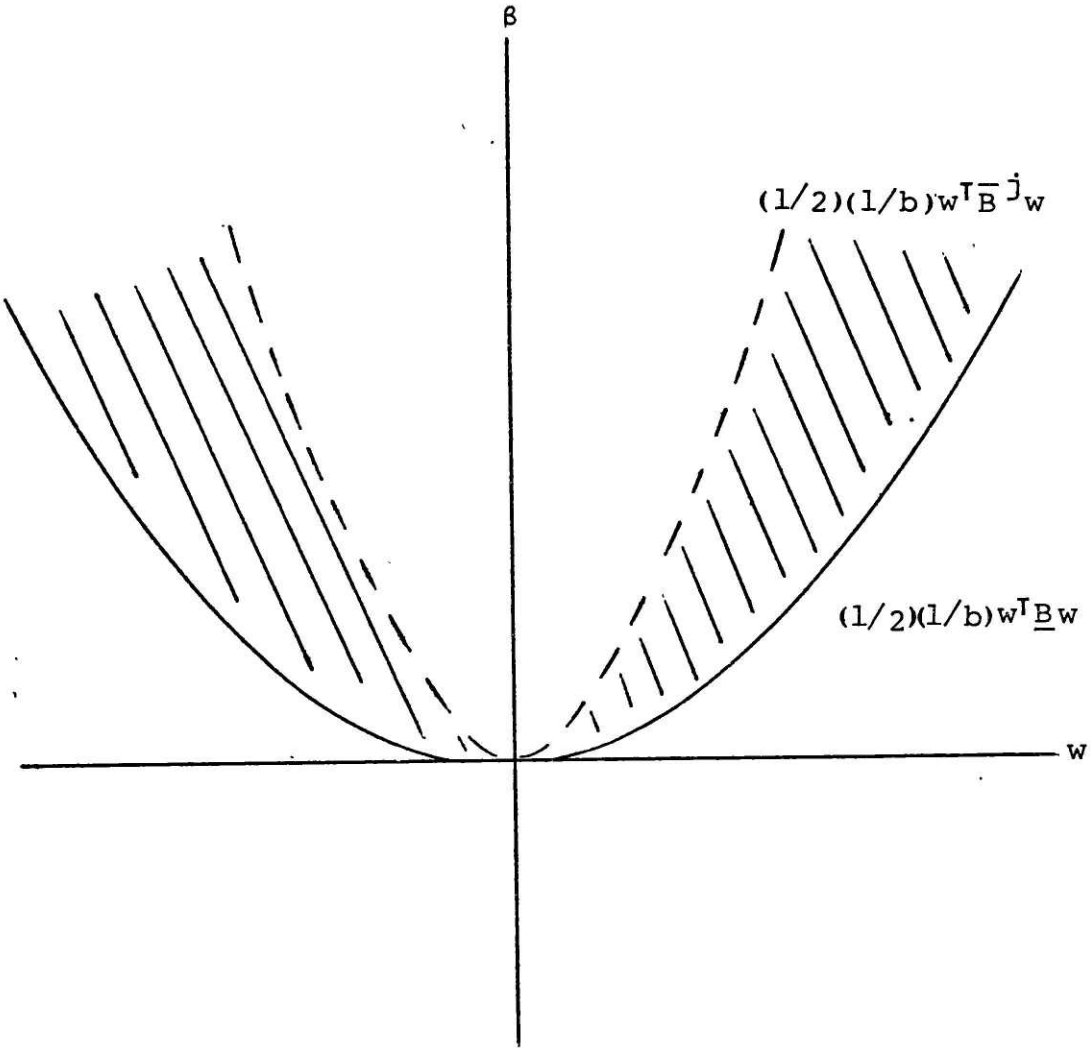


Figure (5-1): Bounded Adjustment Costs

controls; the adjustment equation (5-1) insures that when w is of order b $\dot{v}$ will be as well, so that extrapolated paths will be good approximations. Provided that the functions h,g,α and β are sufficiently differentiable with uniformly bounded derivatives all of the theorems of the previous sections go through unchanged. Note that in (5-3) adjustment costs can depend on all agents' controls. This complicates the expressions for the Hamiltonian and for $\bar{f}$ but otherwise has no significant impact on our results.

One important technical assumption was that the state space be unbounded. As long as (almost) optimal paths don't actually hit the boundary this makes no difference. If the paths do hit the boundary this creates a minor problem: payoff paths will be discontinuous when the boundary is reached. However, as long as the system is sufficiently far from the boundary the discontinuity will be far enough in the future that the discounted error from extrapolation will be small.

Behavior near the boundary is problematic. A far more serious problem, however, is that in economic models there are frequently important non-convexities on the boundary - start-up costs and the like - which are unrelated to the smooth structure in the interior. About these types of problems this framework has nothing

to say.

A second assumption that was primarily technical in nature was the assumption that various functions and their derivatives were bounded at infinity. It is not hard to see that what is really needed here is that income streams and their derivatives not grow faster than the discount rate. Although the algebra may be messy, there is no reason in principle why the results of this paper can't be extended to growth models.

The results can also be extended to discrete time models. This requires replacing assumptions on derivatives with assumptions on finite differences and using first differences as the basis of linear extrapolation. While algebraically messy this is conceptually straightforward.

6. CONCLUSION

What does it mean for b to be "small"? Apparently as b goes to zero adjustment costs become infinite and nobody moves - not a very interesting limit system. This, however, is an artifact of the units in which things are measured as the limit is approached. Two systems of measurement yield non-degenerate limits and give insight into the type of environment to which the results of this paper are applicable.

One way of rescaling the system is to change the units in which time is measured choosing new time units $\bar{t} = bt$. In these new time units the equation of motion (1-1) becomes

$$\dot{v} = (1/b)gw \tag{6-1}$$

and as b goes to zero both the myopic and fundamental solution approach

$$\begin{aligned} \dot{v} &= (1/b)g\tilde{f}_m \\ &= \textstyle\sum_k \delta^k g_k g_k^T D\alpha^k \end{aligned} \tag{6-2}$$

in which each agent moves the state variables in the direction which improves his own individual income. In the new time scale the discount rate is $\bar{\rho}^j = \rho^j/b$. As $b \to 0$ $\bar{\rho}^j \to \infty$ and with an infinite discount rate agents

behave completely myopically. If the discount rate is large (but not infinite) agents will still do reasonably well behaving myopically, but do even better using linear extrapolation or the fundamental solution.

An alternative method of rescaling the system is to change the units in which both income and the state variables are measured setting $\bar{a}^j = ba^j$ and $\bar{v} = v/b$. In these units with initial conditions v_0 the system approaches

$$\dot{v} = \sum_k \delta^k g_k g_k^T D\bar{\alpha}^k \qquad (6\text{-}3)$$

where the limit income function $\bar{\alpha}^k(\bar{v}) = D\alpha^k(v_0)[\bar{v}]$ is linear. In other words the myopic solution is exact if gross income is a linear function. If gross income is " almost " linear in the sense that $D\alpha^k$ changes slowly along the equilibrium adjustment path the myopic solution does reasonably well, but the fundamental solution does even better. In actual economic environments, of course, the gain to using a linear extrapolation rule may outweigh the informational and computational costs, and the use of these rules will be important.

The discussion of limiting systems is useful in understanding what it means for b to be "small". The non-linearity in gross income should be small over a

period of time during which substantial discounting occurs. From an economic perspective, however, this discussion misses the point. The objective of this paper has been to study a particular class of rules of thumb - forecasting techniques which extrapolate income linearly. Casual empiricism suggests that in many economic environments such rules are widely used and work relatively well. For this reason it is interesting to make assumptions which guarantee that linear extrapolation works. The central point of this paper is a simple one - in these environments there is a (nearly) unique adjustment path. It is characterized by the fact that along this path each agent can assume without significant loss that his opponents behave myopically. This is to say that in an economic environment in which linear extrapolation is a good rule of thumb game-theoretic regression is not an important aspect of behavior. This may explain why, although the paradoxes of game theory are an important intellectual question, they don't seem from an empirical standpoint to have much relevance to economics.

APPENDIX: ALMOST OPTIMAL CONTROL

In smooth systems Bellman's equation is necessary and sufficient for a smooth optimal path. This appendix demonstrates proposition (1-1): that in a smooth system a necessary and sufficient condition for a smooth almost optimal path is that Bellman's equation almost hold.

Throughout this section $\tilde{f}$ is assumed to be K-smooth with $b<1/K$. In outline the strategy of the proof is : (1) establish the smoothness of the state valuation functions $\tilde{A}^j$; (2) derive an expression for the loss from switching from $\tilde{f}^j$ to f^j in terms of $\bar{f}^j$ which maximizes the Hamiltonian, and (3) use (2) to find a necessary and a sufficient condition for an ε-optimal path.

To deal efficiently with high order derivatives it is convenient to view them as symmetric multi-linear operators. If M(x) is a function its pth derivative $D^pM(x)[y^1,\ldots,y^p]$ is a symmetric p-linear function of the variables y^i which have the same dimension as x. By definition the norm $|D^pM(x)|$ is the smallest number such that for all y

$$|D^pM(x)[y]| \leq |D^pM(x)| \prod_{i=1}^{p} |y^i|$$

Smoothness of the State Valuation Function

Lemma (A-1): $\tilde{A}^j = O(K)$

proof:

$$\begin{aligned} a^j(v,\tilde{f}) &= \alpha^j(v) - (1/2)(1/b)\tilde{f}^{jT}\tilde{f}^j \\ &= O(K_0) + O(bK^2/2) \\ &= O(K) \end{aligned} \tag{A-1}$$

using $|\alpha^j(v)| \leq K_0$ from assumption (A), $|\tilde{f}^j| \leq bK$ from K-smoothness and the maintained hypothesis $b<1/K$. Integrating and using $1/\rho^j = \delta^j \leq K_0$ from assumption (A) shows

$$\begin{aligned} \tilde{A}^j(v) &= \int_0^\infty a^j(\tilde{\phi}_s, \tilde{f}^j(\tilde{\phi}_s))\exp[-\rho^j s]ds \\ &= O(K) \int_0^\infty \exp[-\rho^j s]ds \\ &= O(K) \quad \delta^j = O(K) \end{aligned} \tag{A-2}$$

Q.E.D.

To bound $D^p\tilde{A}^j$ the derivatives of the flow $\tilde{\phi}_t$ must be bounded.

Lemma (A-2): $|D_v\tilde{\phi}_t| \leq \exp[K^2 bt]$

proof: By (1-4)

$$D_v\tilde{\phi}_t = \exp\left[g \int_0^t D\tilde{f}(\tilde{\phi}_s)ds\right] \tag{A-3}$$

and the lemma follows from $|g| \leq K_0$ $|D\tilde{f}| \leq K$

Q.E.D.

Lemma (A-3): $|D_v^2 \tilde{\phi}_t| \leq O(K)bt \exp[K^2bt]$

proof: From (A-3)

$$D_v^2\tilde{\phi}_t[y^1,y^2] = \{(g/2)\int_0^t\{D^2\tilde{f}[y^1,D_v\tilde{\phi}_s[y^2]] + D^2\tilde{f}[y^2,D_v\tilde{\phi}_s[y^1]]\}ds\} \exp[g\int_0^t D\tilde{f}(\tilde{\phi}_s)ds] \qquad (A\text{-}4)$$

Using lemma (A-2) and the additional bound $|D^2\tilde{f}| \leq bK$ yields the desired result.

Q.E.D.

Lemma (A-4): $D\tilde{A}^j = O^2(K)$

proof: Since

$$\tilde{A}^j(v) = \int_0^\infty a^j(\tilde{\phi}_s, \tilde{f}^j(\tilde{\phi}_s))\exp[-\rho^j s]ds \qquad (A\text{-}5)$$

$$D\tilde{A}^j = \int_0^\infty\{D\alpha^j - D\beta^j D\tilde{f}^j\}D_v\tilde{\phi}_s\exp[-\rho^j s]ds \qquad (A\text{-}6)$$

$$D^2\tilde{A}^j[y^1,y^2] = \int_0^\infty(\{D^2\alpha^j[y] - D^2\beta^j[D\tilde{f}^j[y^1],D\tilde{f}^j[y^2]] - (1/2)D\beta^j[y^1]D\tilde{f}^j[y^2] - (1/2)D\beta^j[y^2]D\tilde{f}^j[y^1]\}D_v\phi_s + \{D\alpha^j - D\beta^j D\tilde{f}^j\}D_v\phi_s[y])\exp[-\rho^j s]ds \qquad (A\text{-}7)$$

the bounds in lemmas (A-2) and (A-3) with (A) and K-smoothness now yield the desired result.

Q.E.D.

This proves part (A) of proposition (1-1).

The Hamiltonian

For agent j the Hamiltonian corresponding to $\tilde{f}$ is defined as

$$\tilde{H}^j(v,w^j) \equiv a^j(v,w^j) + D\tilde{A}^j(v)g\tilde{f} \qquad \text{(A-8)}$$

which exists by lemma (A-4) and the maintained hypothesis that $\tilde{f}$ is K-smooth with $b<1/K$. This differs slightly from the usual definition of the Hamiltonian since $\tilde{A}^j$ is evaluated along $\tilde{\phi}$ which is not necessarily an optimal path for agent j.

Consider the problem of maximizing $\tilde{H}^j(v,w^j)$ over w^j. From (A-8), (1-1) and (1-2)

$$\tilde{H}^j(v,w^j) = \alpha^j(v) - (1/b)(1/2)\left[w^j\right]^T w^j + D\tilde{A}^j(v)\left\{\sum_{k\neq j} g_k \tilde{f}^k(v) + g_j w^j\right\} \qquad \text{(A-9)}$$

which is quadratic in w^j. The unique value of w^j denoted $\bar{f}^j(v)$ which satisfies the first order condition $D_2\,\tilde{H}^j(v,w^j) = 0$ is the unique maximizing choice of w^j. The first order condition is

$$-(1/b)\,w^j + g_j^T\,D\tilde{A}^{jT}(v) = 0 \qquad \text{(A-10)}$$

so that

$$\bar{f}^j(v) = bg_j^T\,D\tilde{A}^{jT}(v) \qquad \text{(A-11)}$$

Also since $\tilde{H}^j$ is quadratic in w^j if $\bar{H}^j(v) \equiv \tilde{H}^j(v, \bar{f}^j(v))$ is the value of the maximized Hamiltonian

$$\tilde{H}^j(v,w^j) = \bar{H}^j(v) - (1/b)(1/2)(w^j - \bar{f}^j(v))^T \quad (w^j - \bar{f}^j(v)) \tag{A-12}$$

The objective of this section is to use (A-12) to evaluate the gain to j from switching to f^j from $\tilde{f}^j$

$$\tilde{\Delta}^j(f^j,v) \equiv A^j(f^j,\tilde{f}^{-j},v) - \tilde{A}^j(v) \tag{A-13}$$

To this end define

$$\tilde{Q}_t^j(f^j,v) = A_t^j(f^j,\tilde{f}^{-j},v) + e^{-\rho^j t}\tilde{A}^j(\phi_t(f^j,\tilde{f}^{-j},v)) \tag{A-14}$$

to be the present value of playing f^j to time t then switching to $\tilde{f}^j$. Since by lemma (A-1) $\tilde{A}^j$ is bounded

$$\begin{aligned}\tilde{\Delta}^j(f^j,v) &= \tilde{Q}_\infty^j(f^j,v) - \tilde{Q}_0^j(f^j,v) \\ &= \int_0^\infty D_t Q_s^j(f^j,v)ds\end{aligned} \tag{A-15}$$

where the last line follows from the fundamental theorem of calculus. Using (A-14) it is seen that

$$D_t Q_s^j(f^j,v) = [\tilde{H}^j(\phi_s,f^j(\phi_s)) - \rho\tilde{A}^j(\phi_s)]e^{-\rho^j s} \tag{A-16}$$

where $\phi_s = \phi_s(f^j,\tilde{f}^{-j},v)$. Furthermore, by the usual properties of present values

$$\rho\tilde{A}^j(v) = \tilde{H}^j(v,\tilde{f}^j(v)) \qquad \text{(A-17)}$$

Equations (A-15), (A-16) and (A-17) combine to show

$$\tilde{\Delta}^j(f^j,v) = \int_0^\infty [\tilde{H}^j(\phi_s,f^j(\phi_s)) - \tilde{H}^j(\phi_s,\tilde{f}^j(\phi_s))]e^{-\rho^j s}ds \qquad \text{(A-18)}$$

To evaluate (A-18) observe that

$$\tilde{H}^j(v,w^j) - \tilde{H}(v,\tilde{w}^j) = [\tilde{H}^j(v,w^j) - \bar{H}^j(v)] - [\tilde{H}^j(v,\tilde{w}^j) - \bar{H}^j(v)] \qquad \text{(A-19)}$$

so that using (A-12) (A-18) becomes

$$\tilde{\Delta}^j(f^j,v) = (1/b)(1/2)\{\int_0^\infty (\tilde{f}^j-\bar{f}^j)^T(\tilde{f}^j-\bar{f}^j)e^{-\rho^j s}ds - \int_0^\infty (f^j-\bar{f}^j)^T(f^j-\bar{f}^j)e^{-\rho^j s}ds\} \qquad \text{(A-20)}$$

evaluated along ϕ_s. Some simple implications of (A-15) are that $\tilde{f}^j$ is optimal if and only if $\tilde{f}^j=\bar{f}^j$ which is Bellman's equation, and that if $\tilde{f}^j\neq\bar{f}^j$, $\bar{f}^j$ yields strictly greater present value than $\tilde{f}^j$.

Characterization of ε-Optima

Suppose as always $\tilde{f}$ is K-smooth with $b<1/K$. Set

$$\tilde{\gamma}^j \equiv \sup_v |\tilde{f}^j(v) - \bar{f}^j(v)| \qquad \text{(A-21)}$$

This section completes the proof of proposition (1-1) by showing that bounds on $\tilde{\gamma}^j$ of order $(b\varepsilon)^{\frac{1}{2}}O(K)$ are necessary and sufficient for an ε-optimum for j given $\tilde{f}^{-j}$.

Sufficiency is easy. Suppose $\tilde{\gamma}^j \leq (2/K_0)^{\frac{1}{2}}(b\varepsilon)^{\frac{1}{2}}$. Then the first term in (A-20) is

$$(1/b)(1/2)\int_0^\infty (\tilde{f}^j-\bar{f}^j)^T(\tilde{f}^j-\bar{f}^j)\exp(-\rho^j s)ds$$
$$\leq (1/b)(1/2)\delta^j(\tilde{\gamma}^j)^2$$
$$\leq (1/b)(1/2)K_0[(2/K_0)b\varepsilon] \leq \varepsilon \qquad \text{(A-22)}$$

and from (A-20) we see this implies that $\tilde{f}^j$ is indeed ε-optimal for j. This proves (B) of proposition (1-1).

Consider now the converse, part (C). Suppose $\tilde{f}^j$ is ε-optimal. Then by definition the gain from playing $\bar{f}^j$ $\tilde{\Delta}^j(\bar{f}^j(v)) \leq \varepsilon$ for all v. From (A-20) we see that this means

$$(1/b)(1/2)\int_0^\infty (\tilde{f}^j-\bar{f}^j)^T(\tilde{f}^j-\bar{f}^j)\exp(-\rho^j s)ds \leq \varepsilon \qquad \text{(A-23)}$$

when the integral is evaluated along $\bar{\phi}_s$. However, along $\bar{\phi}_s$ $D_t\tilde{f}^j$ and $D_t\bar{f}^j$ are absolutely bounded. Thus if $\tilde{\gamma}^j$ is too large $|\tilde{f}^j-\bar{f}^j|$ can't decrease fast enough for (A-23) to be satisfied. To formally conclude the proof of proposition (1-1) we prove

Lemma (A-5): (A-23) implies $\tilde{\gamma}^j \leq (b\varepsilon)^{\frac{1}{2}}O(K)$ provided

$b\varepsilon \leq 1$.

proof:

By part (A) $\bar{f}^j = bO^1(K)$ while by K-smoothness $\tilde{f}^j = bO^1(K)$. Thus

$$\begin{aligned} & |D_t[\tilde{f}^j(\bar{\phi}_t) - \bar{f}^j(\bar{\phi}_t)]| \\ & \leq |D\tilde{f}^j(\bar{\phi}_t) - D\bar{f}^j(\bar{\phi}_t)||D_t\bar{\phi}_t| \\ & \leq bO(K)|g_j\bar{f}^j + \Sigma_{k\neq j}\, g_k\tilde{f}^k| \\ & \leq b^2O(K) \end{aligned} \tag{A-24}$$

Let $d \equiv |\tilde{f}^j(v) - \bar{f}^j(v)|$. By (A-24) $|\tilde{f}^j - \bar{f}^j|$ declines no faster than linearly to zero at time $d/b^2O(K)$, so

$$\begin{aligned} & (1/b)(1/2)\int_0^\infty (\tilde{f}^j-\bar{f}^j)^T(\tilde{f}^j-\bar{f}^j)\exp(-\rho^j s)ds \\ & \geq (1/b)(1/2)\int_0^{d/2b^2O(K)} (d/2)^2\exp(-\rho^j s)ds \\ & = (1/b)(1/2)(d^2/4)\{1-\exp[-\rho^j d/2b^2O(K)]\} \\ & \geq (1/b)\,[O(K)d]^2\{1-\exp[-O(K)d]\} \\ & = (1/b)\,[M(O(K)d)]^2 \end{aligned} \tag{A-25}$$

where $M(x) = x\sqrt{1-\exp(-x)}$. Using the hypotheses (A-23) and (A-25) shows

$$d \leq O(K)\,[M^{-1}(b\varepsilon)]^{\frac{1}{2}} \tag{A-26}$$

and it is easy to verify that for $0 \leq b\varepsilon \leq 1$

$$M^{-1}(b\varepsilon) \leq K_1 b\varepsilon \qquad \text{(A-27)}$$

where K_1 is a fixed constant. This is the desired conclusion.

Q.E.D.

REFERENCES

[1] Robert Bishop, "Duopoly, Collusion or Warfare?" AER, December 1960

[2] David Levine, "Learning and Local Perfect Equilibrium", MIT, June 1980

[3] _____, "The Enforcement of Collusion in a Quadratic, Symmetric Oligopoly", MIT, July 1980

[4] _____, "Long Run Collusion in a Partially Myopic Industry", MIT, July 1980

[5] Roy Radner, "Collusive Behavior in Epsilon-Equilibria", JET, April 1980

[6] A. Rubinstein, "Equilibrium in Supergames with the Overtaking Criterion", JET, 21, pp.1-9

[7] R. Selten, "A Re-examination of the Perfectness Concept", International Journal of Game Theory, 4, pp.25-55

NOTES

(1) The concept of perfect equilibrium is due to Selten and discussed in Selten [7].

(2) This is pointed out in Rubinstein [4].

(3) Epsilon equilibria are due to Radner and are discussed in [3].

(4) Learning by small perturbations is examined formally in Levine [2].

APPENDIX: SOME RESULTS ON DYNAMICAL SYSTEMS

This paper discusses general dynamical systems. It draws together some existing results and proves a few simple extensions. Section one discusses the notion of the index of an equilibrium from a differential perspective. Section two gives a sufficient condition for stability in 2-dimensional dynamical systems. Section three studies stability in systems with locally non-unique equilibria -- it defines strong quasi-stability and gives a sufficient eigenvalue condition. Section four contains an eigenvalue approximation theorem for matrices. General terminology on dynamical systems can be found in Hirsch and Smale [4].

I. DEGREE, INDEX, AND STABILITY

A basic tool for analyzing global aspects of dynamical systems is the index of an equilibrium. This section reviews the basic concepts and theorems. References are Hirsch (3) ch. 5, Guilleman and Pollack (2) ch. 3 and Milnor (5) ch. 6. To conclude the section a simple application to a system with three equilibria is given.

Notation: By M and N are denoted oriented m and n dimensional manifolds with generic points $x \in M$ and $y \in N$. The derivative of the map $g: M \to N$ at x is denoted Dg_x. If $m = n$ and Dg_x is non-singular deg Dg_x can be defined. Choose local coordinates at x and $y = g(x)$ with a positive orientation. Let A be the matrix representation of Dg_x in these coordinates. Then deg Dg_x = sgn det A.

Degree of a Map: Two maps $g,h: M \to N$ are homotopic if there is a continuous function $F: M \times I \to N$ with $g = F(\cdot,0)$ and $h = F(\cdot,1)$. This says that each map is a continuous deformation of the other. Suppose $g: M \to N$ is C^0; $m = n$; M and N are compact without boundary and N is connected.

Theorem (1-1): There is a $h \in C^\infty$ homotopic to g. Let $y \in N$ be a regular value of any such h so that for $x \in h^{-1}(y)$ Dh_x is non-singular. Then

$$\deg g \equiv \sum_{x \in h^{-1}(y)} \deg Dh_x$$

does not depend on h or y.

Index of an Equilibrium: Now suppose $f: M \to \mathbb{R}^m$ is a C^0 vector field on M. If x is an isolated zero of f $\text{ind}_x f = \deg g$ where $g: S_x \to S^{m-1}$ is defined by $g(z) = f(z)/|f(z)|$; S_x is a sphere in local coordinates around x on which f does not vanish; and S^{m-1} is the $m - 1$ dimensional unit sphere. The next theorem gives somewhat easier techniques for computing the index:

Theorem (1-2):

(a) $\text{ind}_x f = (-1)^m \text{ind}_x(-f)$

(b) if Df_x is non-singular $\text{ind}_x f = \deg Df_x$

We next consider a vector field f on a compact manifold which has isolated zeroes x_i and points inward on the boundary.

Theorem (1-3):

$$\sum_i \text{ind}_{x_i} f = (-1)^m \chi(M)$$

where $\chi(M)$ is called the Euler characteristic and does not depend on f.

Index and Stability: If the zero x of f is hyperbolic $\text{ind}_x f = \text{sgn det } A$ where A is the stability matrix of the flow to which f is tangent. If x is a stable equilibrium then the eigenvalues of A have negative real parts and $\text{sgn det } A = (-1)^m$. If x is stable but not hyperbolic choose a local coordinate system in which the vector field f is the inward unitary normal field on S_x the unit sphere around x. This means that the map g used in the definition of $\text{ind}_x f$ is the same map as that induced by the uniform linear contraction. Since the latter field is stable and hyperbolic both fields have index $(-1)^m$ proving

Proposition (1-4): A locally stable equilibrium has index $(-1)^m$.

If the manifold M admits an inward pointing vector field with a unique stable zero (as is true if it is convex, star shaped or otherwise contractible to a point) by theorem (1-3) and proposition (1-4)

Proposition (1-5): If M admits an inward vector field with a unique stable equilibrium and f is any vector field with isolated zeroes

$$\sum_i \text{ind}_{x_i} f = (-1)^m$$

An Application: Suppose M admits an inward vector field with a unique stable equilibrium so that by proposition (1-5) $\chi(M) = 1$. Suppose f has three equilibria of which two are known to be stable. Then these have index $(-1)^m$ so $2(-1)^m + \text{ind}_{x_3} f = (-1)^m$ where $\text{ind}_{x_3} f = (-1)^{m+1}$ is the index of the third equilibrium of unknown stability. However this implies x_3 cannot be stable since if it was $\text{ind}_{x_3} f = (-1)^m \neq (-1)^{m+1}$.

We conclude

Proposition (1-6): If $\chi(M) = 1$ and f is inward pointing with two stable equilibria and one other equilibrium the other equilibrium is locally unstable.

2. SOME CONDITIONS FOR STABILITY IN PLANAR SYSTEMS

We examine an isolated equilibrium $z = (z^1, z^2)$ of a planar system $\dot{x} = f(x)$ occurring at the intersection of two locally unique curves $x^2 = a^1(x^1)$ and $x^2 = a^2(x^1)$ which solve $f^1(x^1, a^1(x^1)) = 0$ and $f^2(x^1, a^2(x^1)) = 0$.

Proposition (2-1): If $Da_z^1 < 0$ $\partial\dot{x}^1/\partial x^1$, $\partial\dot{x}^2/\partial x^2 < 0$ and a^2 intersects a^1 from below than z is stable (see figure 2-1).

The proof is in two parts. First we show that the proposition holds when $Da_z^2 > Da_z^1$. Then we prove the special case $Da_z^2 = Da_z^1$.

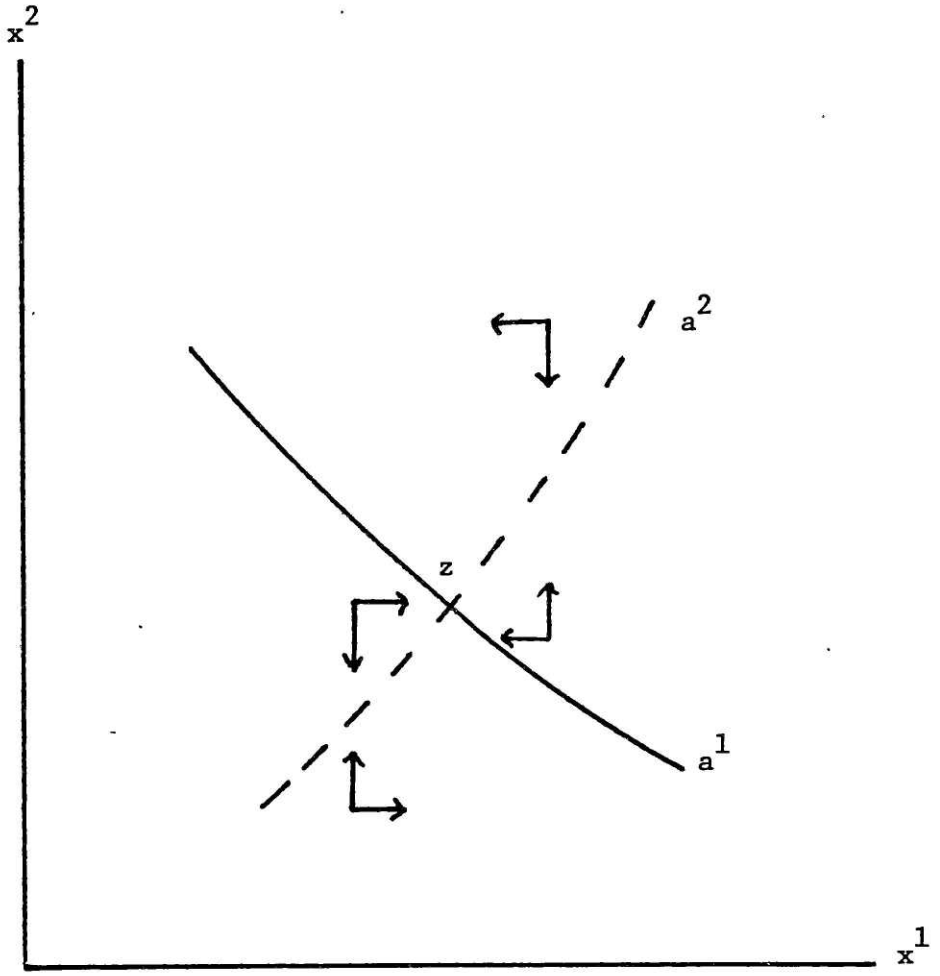
<u>Proof</u> ($Da_z^2 > Da_z^1$): It suffices to show

$$\partial\dot{x}^1/\partial x^1 \;\; \partial\dot{x}^2/\partial x^2 - \partial\dot{x}^1/\partial x^2 \;\partial\dot{x}^2/\partial x^1 > 0 \qquad (2\text{-}1)$$

To establish this note that

$$-(\partial\dot{x}^2/\partial x^1)/(\partial\dot{x}^2/\partial x^2) = Da_z^2$$

$$> Da_z^1 = -(\partial\dot{x}^1/\partial x^1)/(\partial\dot{x}^1/\partial x^2) \qquad (2\text{-}2)$$



Figure(2-1): The Sufficient Condition

Taken together $\partial\dot{x}^1/\partial x^1 < 0$ and $-(\partial\dot{x}^1/\partial x^1)/(\partial\dot{x}^1/\partial x^2) < 0$ imply $\partial\dot{x}^1/\partial x^2 < 0$. Cross multiplying in (2-2) yields

$$-(\partial\dot{x}^1/\partial x^2)(\partial\dot{x}^2/\partial x^1) > -(\partial\dot{x}^1/\partial x^1)(\partial\dot{x}^2/\partial x^2) \qquad (2\text{-}3)$$

which is seen to imply (2-1). QED

Proof ($Da_z^2 = Da_z^1$): The phase portrait is shown in figure (2-2). It is easily seen that inside any ball around z a rectangle can be drawn parallel to the axes and with a vertex in each of the regions bounded by the curves a^2 and a^1. Inspection shows the flow is strictly into this rectangle, so the equilibrium is stable. QED

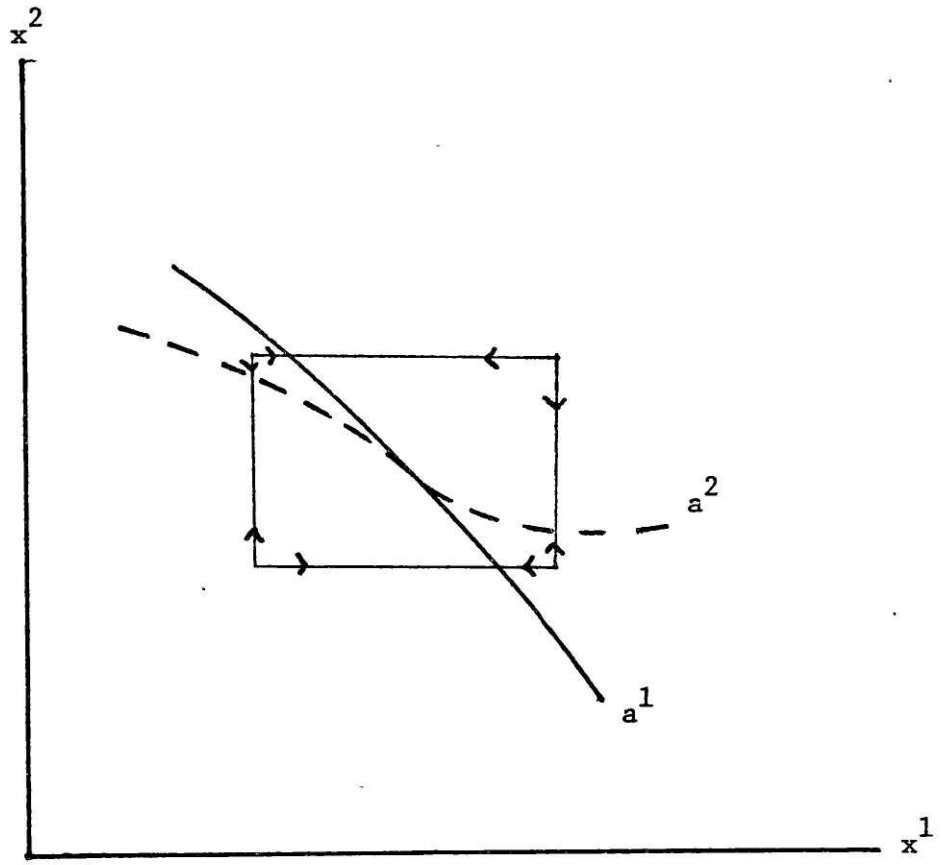
Proposition (2-2): If $Da_z^1 < 0$ $\partial\dot{x}^1/\partial x^1 < 0$ $\partial\dot{x}^2/\partial x^2 > 0$ $\partial\dot{x}^1/\partial x^1 + \partial\dot{x}^2/\partial x^2 < 0$ and a^2 intersects a^1 from above, then z is stable (see figure (2-3)).

Proof: The case $Da_z^2 = Da_z^1$ is left as an exercise. When $Da_z^2 < Da_z^1$

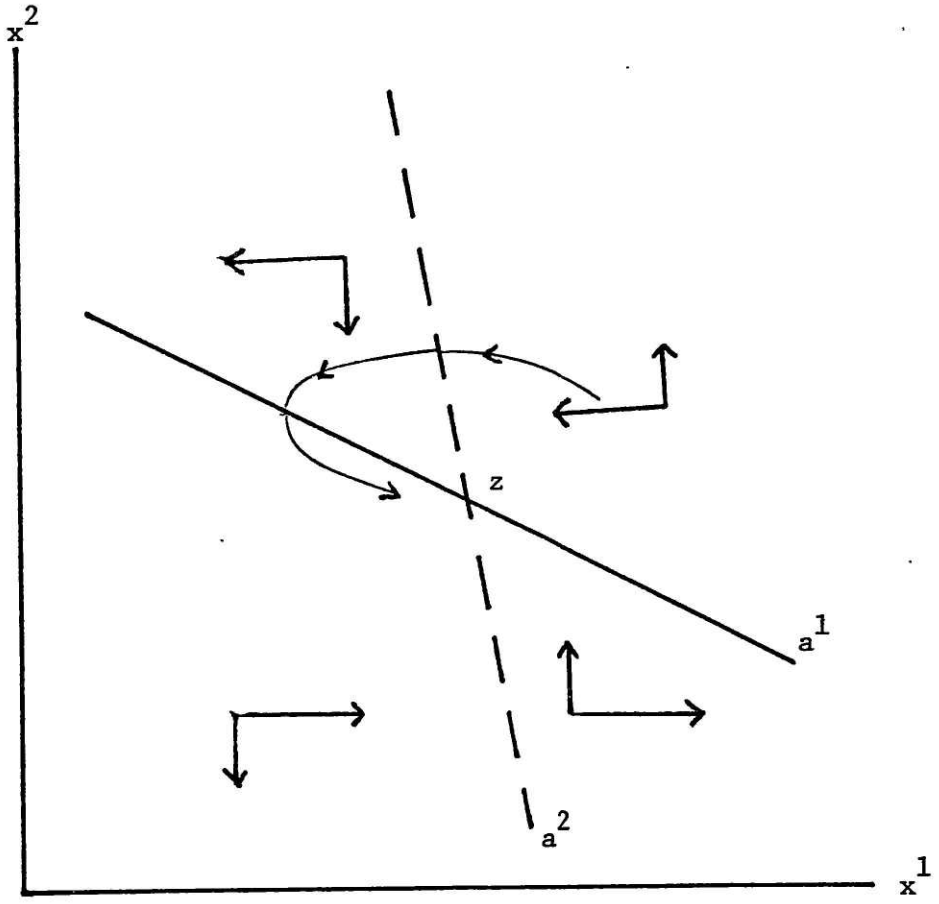
$$-(\partial\dot{x}^2/\partial x^1)/(\partial\dot{x}^2/\partial x^2) = Da_z^2$$

$$< Da_z^1 = -(\partial\dot{x}^1/\partial x^1)/(\partial\dot{x}^1/\partial x^2) \qquad (2\text{-}4)$$

Since $Da_z^1 < 0$ and $\partial\dot{x}^1/\partial x^1 < 0$ $\partial\dot{x}^1/\partial x^2 < 0$. Thus (2-4)



Figure(2-2): Stability with Tangency



Figure(2-3): Stability with a^2 a Repellor

implies

$$-(\partial\dot{x}^2/\partial x^1)(\partial\dot{x}^1/\partial x^2) > -(\partial\dot{x}^1/\partial x^1)(\partial\dot{x}^2/\partial x^2) \qquad (2\text{-}5)$$

which implies (2-1) as required. QED

Finally, reversing the proofs of propositions (2-1) and (2-2) shows

Proposition(2-3): If $Da_z^1<0$ $\partial\dot{x}^1/\partial x^1$, $\partial\dot{x}^2/\partial x^2<0$. and a^2 intersects a^1 from below then z is unstable.

Proposition (2-4): If $Da_z^1<0$ $\partial\dot{x}^1/\partial x^1<0$, $\partial\dot{x}^2/\partial x^2>0$ and a^2 intersects a^1 from below then z is unstable.

3. STRONG QUASI-STABILITY

Ordinary stability cannot hold at equilibria which are locally non-unique since carefully chosen perturbations will lead to a different nearby equilibrium. This section defines and gives a sufficient condition on the eigenvalues of the stability matrix for strong quasi-stability. Unlike ordinary stability this requires that small perturbations need only lead to a nearby equilibrium.

Definitions: We consider a dynamical system $\dot{x} = f(x)$ defined on an open subset of $\mathbb{R}^{m+n}$. The vector field f is assumed to be C^1 and by the existence and uniqueness theorem there is a unique maximal flow $\phi_t(x)$ satisfying $\partial\phi_t/\partial t = f(\phi_t)$ and $\phi_0(x) = x$. If x^* is an equilibrium so $f(x^*) = 0$ then the stability matrix $A \equiv Df_{x^*}$. An equilibrium x^* is strongly quasi-stable if for every $\epsilon > 0$ there is an open neighborhood $U_\epsilon \ni x^*$ such that $x \in U$ implies $\lim_{t\to\infty} \phi_t(x)$ exists and lies within ϵ of x^*. Sufficiently small perturbations from x^* lead to an equilibrium close to x^*.

The Theorem: Our objective in this section is to prove the following sufficient condition for quasi-stability:

Theorem (3-1): Suppose there is an open set $U \ni x^*$ in which the set of equilibrium $M \equiv \{x \in U \mid f(x)\} = 0$ are an m dimensional submanifold of U and that the stability matrix $A(x^*)$ has n eigenvalues with strictly negative real parts. Then there is an open set $W \ni x^*$ such that

(a) $x \in M \cap W$ implies x is strongly quasi-stable

(b) the rate of convergence in the definition of quasi-stability is exponentially fast

What this theorem says is that if the equilibria are locally a manifold of dimension equal to the number of zero eigenvalues of A and the remaining eigenvalues have strictly negative real parts then x* and nearby equilibria are all strongly quasi-stable. What we will show is

Proposition (3-2): Suppose $x = (x^1, x^2)'$; $f = (f^1, f^2)'$ where $x^1 \in \mathbb{R}^m$, $x^2 \in \mathbb{R}^n$ and that there is $U \ni 0$ with

$$f(x^1, 0) = 0 \qquad (x^1, 0) \in U$$

Df_0 has n eigenvalues with negative real parts. Then there are $g, G > 0$, $W \ni 0$ and map $p: W \to \mathbb{R}^m \times 0$ continuous at 0 such that $x \in W$ implies for $t \geq 0$

$$|\phi_t(x) - p(x)| \leq Ge^{-gt}$$

Now suppose x^* is as in theorem (3-1). Choose a local coordinate system in U such that in local coordinates $M = \mathbb{R}^m \times 0$ and $x^* = 0$. This is possible since M is by assumption an m-dimensional submanifold of U and $x^* \in M$. The coordinate change is $(y^1, y^2)' = (\psi^1(x), \psi^2(x))'$. Then

$$\dot{y} = h(y) = D\psi_x f(x) \tag{3-1}$$

and therefore

$$Dh_y = D^2\psi_x [D\psi_x]^{-1} f(x) + D\psi_x Df_x [D\psi_x]^{-1} \tag{3-2}$$

which evaluated at $y = 0 = \psi(x^*)$ yields

$$Dh_0 = D\psi_{x^*} A [D\psi_{x^*}]^{-1}. \tag{3-3}$$

This shows Dh_0 has the same eigenvalues as A and proposition (3-2) immediately implies x^* is strong quasi-stable with exponential convergence in local, and therefore canonical, coordinates.

We can now use the continuity of A(x) to show equilibria close to x^* are quasi-stable. By the eigenvalue approximation theorem in section four for x close enough to x^* A(x) has at least n eigenvalues with negative real parts. So we can find $W \ni x^*$ in which $W \cap M$ contains only equilibria satisfying the hypotheses of theorem (3-1). Since we just showed all such equilibria are strong

quasi-stable with exponential convergence theorem (3-1) follows from proposition (3-2). We now prove a series of lemmas demonstrating proposition (3-2).

Proof of the Proposition: The idea behind the proof is much like that in the usual case where all eigenvalues of A have negative real parts: we show that the linearized system is stable and that trajectories remain so close to the initial equilibrium that the linearized system is a good approximation to the actual system. We do this by examining the motion of the system in a small cylinder around the equilibrium manifold. The important extension to the ordinary stability proof is to show that the fact that the equilibrium manifold has the right dimension guarantees the system cannot drift out the end of the cylinder.

Our procedure is to define $z^1 = |x^1|$ and $z^2 = (x^2 \cdot Bx^2)^{1/2}$. The variable z^1 measures how far down the cylinder around the equilibrium manifold $x^2 = 0$ the system has drifted. The variable z^2 measures how far the system is from the equilibrium manifold in the metric induced by the positive definite matrix B. Our proof proceeds via a series of lemmas.

Lemma (3-1) will give a sufficient condition on $\dot{z}^1$ and $\dot{z}^2$ such that z^1 and z^2 remain close to the initial conditions.

Lemma (3-2) recomputes the condition in lemma (3-1) in terms of x^1 and x^2.

Lemma (3-3) shows that

$$Df_0 = A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$ has A_{22} containing all n non-zero eigenvalues of A.

Lemma (3-4) shows that the negative real parts of eigenvalues of A_{22} dominate quadratic forms based on A_{22}.

Lemma (3-5) uses lemma (3-4) to show how to choose B so that z^2, the distance to the equilibrium manifold, strictly decreases in the linearized system.

Lemma (3-6) extends lemma (3-5) locally to the non-linear system.

Lemma (3-7) verifies that the conditions of lemma (3-2) hold.

Lemma (3-8) shows that the motion of the system is bounded.

Lemma (3-9) uses the Bellman-Gronwall growth lemma and lemmas (3-2), (3-6), and (3-8) to verify the existence of a function p(x) continuous at 0 to which $\phi_t(x)$ converges exponentially fast.

Lemma (3-1): Suppose $z^1(t), z^2(t) \geq 0$ and for some $A, \gamma > 0$ and $A \geq z^1, z^2 \geq 0$ $\dot{z}^2/|\dot{z}^1| < -\gamma$ then $z^2(0) \leq \min(1,\gamma)[A - z^1(0)]$ implies $0 \leq z^1(t),\ z^2(t) \leq A$ for $t \geq 0$.

Proof: See figure (3-1). The line segment connecting $(0,\ A \min(1,\gamma))$ and $(A,0)$ has slope $\geq -\gamma$. Since $\dot{z}_2/|\dot{z}_1| < -\gamma$ $z(t)$ cannot cross this line. QED

Lemma (3-2): For $z^1 = |x^1|$ $z^2 = (x^{2'}Bx^2)^{1/2}$ and B positive definite

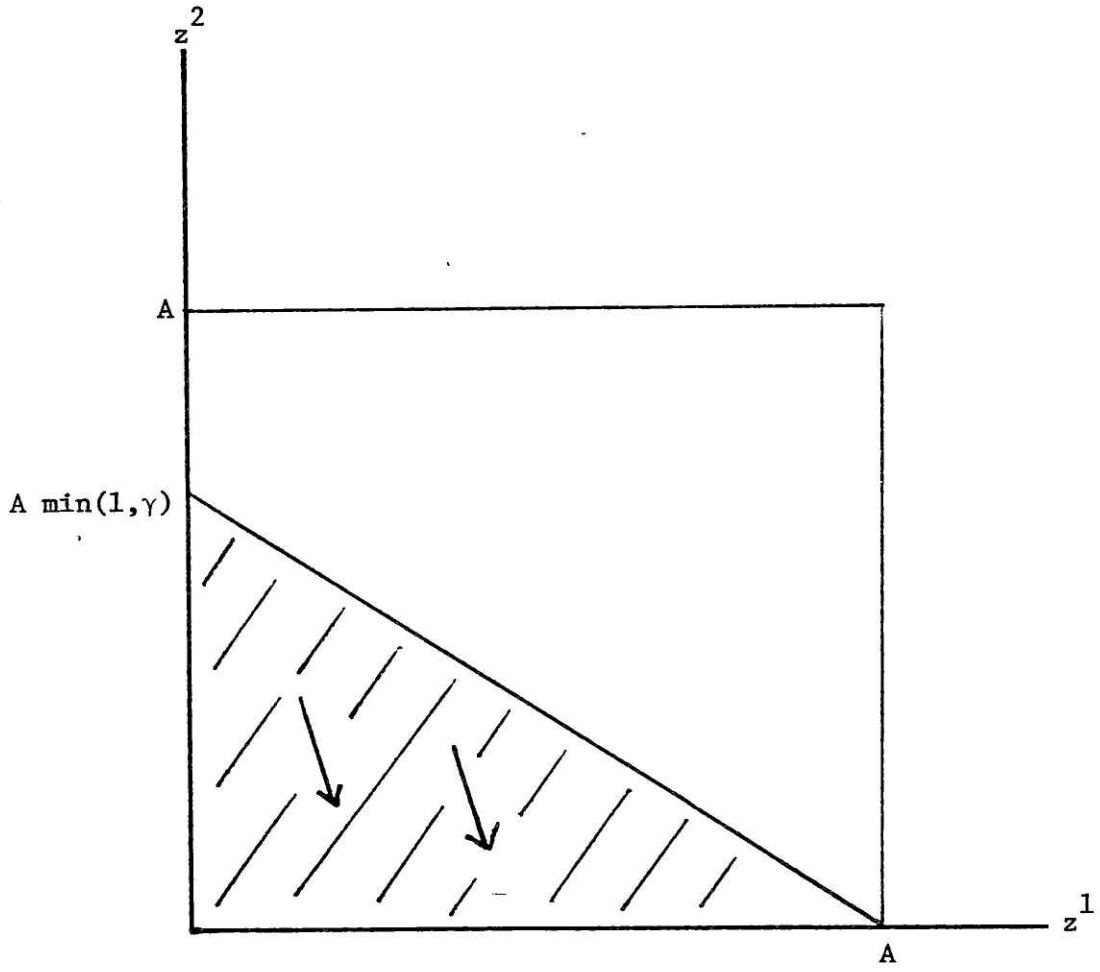
$$\dot{z}^2/|\dot{z}^1| \leq \delta \left[\frac{x^{2'}B\dot{x}^2}{|x^2|} \frac{|x^1|}{|x^{1'}\dot{x}^1|} \right]$$

where $\delta > 0$ depends only on B.

Proof:

$$\dot{z}^1 = x^{1'}\dot{x}^1/|x^1| \quad .$$

$$\dot{z}^2 = (x^{2'}B\dot{x}^2)/(x^{2'}Bx^2)^{1/2}$$



Figure(3-1): Bounding the Trajectory

$$\dot{z}^2/|\dot{z}^1| = \frac{x^{2'}B\dot{x}^2}{|x^2|}\,\frac{|x^1|}{|x^{1'}\dot{x}^1|}\cdot\frac{|x^2|}{(x^{2'}Bx^2)^{1/2}} \qquad (3\text{-}4)$$

and the lemma follows since $|x^2|/(x^{2'}Bx^2)^{1/2} < \delta$ follows from the eigenvalence of norms. QED

Lemma (3-3): Set $A = Df_0 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$;

then the eigenvalues of A_{22} are the non-zero eigenvalues of A.

Proof: Since $f(x^1,0) = 0$ $A_{11} = A_{21} = 0$ and $\det(A - \lambda I) = (-\lambda)^m \det(A_{22} - \lambda I)$. Since A has at most m zero eigenvalues the lemma follows directly. QED

Lemma (3-4): Suppose A_{22} is a real matrix and that the largest real part of any eigenvalue of A_{22} is dominated by $-\beta'$. Then there is a non-singular matrix C such that for any x^2

$$x^{2'}C'CA_{22}x^2 \leq -\beta' x^{2'}C'Cx^2$$

Proof: According to ch. 7 §1 lemma of Hirsch and Smale (4) there is a real basis such that in the corresponding inner product, $\langle\ ,\ \rangle$ $\langle A_{22}x^2, x^2\rangle \leq -\beta\langle x^2, x^2\rangle$. Letting C

transform the basis in which the coordinates of A_{22} are given to this new basis shows that this inner product can be written in the desired form. QED

Lemma (3-5): There is $\alpha', \beta' > 0$ and a positive definite matrix B such that

$$x^{2'}BA_{22}x^2 \leq -\alpha'|x^2|^2, -\beta' x^{2'}Bx^2$$

Proof: By lemmas (3-3) and (3-4) there is non-singular matrix C and $\beta' > 0$ such that

$$x^{2'}C'CA_{22}x^2 \leq -\beta' X^{2'}C'Cx^2 \qquad (3\text{-}5)$$

Taking $B = C'C$ and observing by equivalence of norms that $x^{2'}Bx^2 \geq \gamma|x^2|^2$ shows that $\alpha' = \beta'\gamma$. QED

Lemma (3-6): There is a neighborhood W of 0 with its closure $\bar{W} \subset U$ compact and a $\alpha, \beta > 0$ such that for $x \in W$

$$x^{2'}B\dot{x}^2 \leq -\alpha|x^2|^2, \beta x^{2'}Bx^2$$

Proof: Take $\bar{W}$ a closed ball of radius r at 0 with $\bar{W} \subset U$. By Taylor's theorem

$$\dot{x}^2 = f^2(x^1,x^2) = f^2(x^1,0) + A_{22}(x^1,\bar{x}^2)x^2$$

$$= A_{22}(X^1,\bar{x}^2)x^2 \qquad (3\text{-}6)$$

where $(x^1,\bar{x}^2) \in \bar{W}$. Hence, $|(x^1,\bar{x}^2)| \leq r$. Since by lemma (3-5)

$$x^{2'}BA_{22}(0)x^2 \leq -\alpha'|x^2|^2, \; -\beta' x^{2'}Bx^2 \qquad (3\text{-}7)$$

it follows that since A_{22} is continuous for any $0 < \alpha < \alpha'$ $0 < \beta < \beta'$ r can be chosen sufficiently small that the lemma holds. QED

Lemma (3-7): There is $\mu > 0$ such that for $x \in W$

$$|\dot{x}^1| \leq \mu|x^2|$$

Proof: By Taylor's theorem

$$|\dot{x}^1| = |f^1(x^1,0) + A_{12}(X^1,\bar{x}^2)x^2|$$

$$\leq \sup_{x\in\bar{W}} |A_{12}(X)||x^2| \qquad (3\text{-}8)$$

and since $\bar{W}$ compact and A_{12} continuous

$$\sup_{x\in\bar{W}} |A_{12}(x)| \leq \mu \qquad (3\text{-}9)$$

QED

Lemma (3-8): Let $V \supset 0$ be open. Then there is $V' \supset 0$ such that for $x \in V'$ $\phi_t(x) \in V$.

Proof: For $x \in W$ by lemmas (3-2), (3-6) and (3-7)

$$\dot{z}^2/|\dot{z}^1| \leq \delta \left[\frac{x^{2'} B \dot{x}^2}{|x^2||x^1|}\right]$$

$$\leq \frac{\delta}{\mu}\left[\frac{x^{2'} B \dot{x}^2}{|x^2|^2}\right]$$

$$\leq -\frac{\delta\alpha}{\mu} = -\gamma \qquad (3\text{-}10)$$

Now choose A so that the ball around zero of radius A lies in $W \cap V$ and choose V' to have radius A min $(1,\gamma)$. An application of lemma (3-1) yields the desired conclusion. QED

Lemma (3-9): There is $p: W' \to \mathbb{R}^m \times 0$ continuous at zero and $g,G > 0$ such that for $x \in W'$ and $t \geq 0$ $|\phi_t(x) - p(x)| \leq Ge^{-gt}$.

Proof: Lemmas (3-2) and (3-6) imply for some $g > 0$ and $x \in W$

$$\dot{z}^2 \leq -2gz^2 \qquad (3\text{-}11)$$

By lemma (3-8) if $x \in W'$ initially (3-11) holds at later times and by the Bellman-Gronwall growth lemma

$$|z^2| \leq \Delta e^{-gt} \quad \text{for some} \quad \Delta > 0 \tag{3-12}$$

which is to say

$$|x^2| \leq \Delta e^{-gt} \tag{3-13}$$

By lemma (3-7)

$$|\dot{x}^1| \leq \mu\Delta e^{-gt} \tag{3-14}$$

implying the existence of $p^1(x)$ such that

$$|x^1 - p^1(x)| \leq (\mu\Delta/g)e^{-gt} \tag{3-15}$$

Setting $p^2(x) = 0$ and $G = \Delta(\mu/g + 1)$ we see

$$\begin{aligned} |x - p(x)| &\leq |x^1 - p^1(x)| + |x^2| \\ &\leq Ge^{-gt} \end{aligned} \tag{3-16}$$

The continuity of p at 0 follows directly from lemma (3-8).

QED

4. AN EIGENVALUE APPROXIMATION THEOREM

It is often important to know that perturbing a matrix slightly only perturbs its eigenvalues slightly. Let A and B be matrices. Lemma VII.6.3. of Dunford and Schwartz (1) on p. 585 shows

Lemma (4-1): Let A be fixed and $\epsilon > 0$ given. Then there is a $\delta > 0$ such that if λ_B is an eigenvalue of a matrix B with $|A - B| < \delta$ there is an eigenvalue λ_A of A with $|\lambda_A - \lambda_B| < \epsilon$.

REFERENCES

(1) Dunford and Schwartz, Linear Operators, volume I, Interscience, 1958.

(2) Guilleman and Pollack, Differential Topology, Prentice Hall, 1974.

(3) Morris Hirsch, Differential Topology, Springer-Verlag, 1976.

(4) Hirsch and Smale, Differential Equations, Dynamical Systems and Linear Algebra, Academic Press, 1974.

(5) John Milnor, Topology from the Differential Viewpoint, University Press of Virginia, 1965.

BIOGRAPHICAL NOTES

Name: David Knudsen Levine

Date and Place of Birth: September 11, 1955
New Haven, Connecticut

Education: UCLA, B.A. in Mathematics, 1977
UCLA, M.A. in Economics, 1977
MIT, 1977-1981

Appointed assistant professor of economics at UCLA, beginning in academic year 1981.