İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

INVESTIGATION OF ANISOTROPIC AND THERMAL EFFECTS IN THE EARLY UNIVERSE

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FOREWORD

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LIST OF ABBREVIATIONS

UV : Ultraviolet
IR : Infrared
FRW : Friedmann Robertson Walker
SB : Symmetry Breaking
GUT : Grand Unification Theory
p. : Page
eq. : Equation

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SYMBOL LIST

ξ : Gravitational coupling constant

$\Box$: D'Alembertian

Δ : Spatial part of the D'Alembertian

β : Reciprocal of Temperature

$\Gamma^{(1)}, \Gamma_{\beta}^{(1)}$: One-loop zero temperature and thermal Effective Actions

$L^{(1)}, L_{\beta}^{(1)}$: One-loop zero temperature and thermal Lagrangians

$V^{(1)}, V_{\beta}^{(1)}$: One-loop zero temperature and thermal effective potantials

$\overline{\Gamma}^{(1)}, \overline{\Gamma}_{\beta}^{(1)}$: 1st loop quantum contribution to classical zero and thermal effective actions

$\overline{L}^{(1)}, L_{\beta}^{(1)}$: 1st loop quantum contribution to classsical zero and thermal Lagrangians

$\overline{V}^{(1)}, \overline{V}_{\beta}^{(1)}$: 1st loop quantum contribution to classical zero and thermal potantials

ϕ^{q} : 1st order quantum contribution to the classical field

A : Quantum propagator

G_F : Feynman Green function

$\overline{G}_F$: Scaled Feynman Green Function

$K(x, x')$: Heat kernel for the propagator

$a_i(x, x')$: Heat Kernel coefficients

$\overline{a}_i(x, x)$: Coincidence limit of heat coefficients

M^2 : Redefined mass square incorporating interaction

$\tilde{M}^2$: Redefined mass square incorporating interaction and curvature

$\overline{M}^2$: Redefined mass square incorporating curvature

$\mathbf{C}^2$: Square of Weyl tensor

$\mathbf{G}$: Gauss-Bonnet density

ε_i : Topological interaction couplings

$\delta\varepsilon_i$: Topological interaction counter terms

$S_{[3]}$: Spatial action

$C_i(x)$: Spatial anisotropy scale factors

d_i : Ansiotropy evolution parameter for Bianchi I spaces

Q : Effective mass anisotropy contribution

INVESTIGATION OF ANISOTROPIC AND THERMAL EFFECTS IN THE EARLY UNIVERSES

SUMMARY

Observations regarding the dynamics of the Early Universe subsequent to the time singularity called Big Bang are currently rather inadequate. Although these evidence an isotropic and an highly homogenous structure for the Universe ongoing a continuous expansion, the quantum effects dominant for the early universe era forbids any extrapolation to this period. In order to do so, one must comprise the quantum contributions, the loop approximations so to speak, to the classical action, thereby employing the effective action.

Determination of the effective action permits certain events to be investigated, such as the symmetry breaking events which're presumed to take place about 10^{-34} seconds later then the time singularity. Here, we perceive from the notion of symmetry breaking a process during which a species of interactions or particles of the same characteristics are separated to entirely different entities.

Our study considers the anisotropic and thermal effects over such symmetry breaking processes during an inflation scenario of early universe models. In this sense we'll first revise a derivation of the effective action to the first loop order for general space times. In order to discern these effects we'll be using self interacting scalar field theories. Next we'll introduce isotropic and spatially flat FRW metric, with Bianchi I spaces which constitutes an anisotropic structure for every spatial direction, and will determine their effective actions. Then the effects exerted by anisotropy over an arbitrary symmetry breaking process will be discussed. We find an interesting result where the anisotropies are seen to be favoring the symmetry restoration.

Then we'll proceed with thermal considerations and will develop a method in order to calculate the effective potential which's understood as the static part of the effective action. In this approach we arrive at an expression in terms of the modified Bessel functions and investigate the high and low temperature consequences. Using these results a discussion will be held regarding the thermal effects and validity of this method and our assumptions. We claim consistent results with the literature in the high temperature limit at least, and observe that a broken symmetry might be restored at some temperature. Also the topological effects relevant to the finite temperature study have been distincted and have been left open to discussion.

ERKEN EVRENDE ISIL VE ANİZOTROPİK ETKİLERİN İNCELENMESİ

ÖZET

Zamanın tekilliğini belirleyen Büyük Patlama sonrası Evrenin hangi süreçlerden ve nasıl geçtiği konusunda hali hazırdaki deneysel veriler oldukça yetersizdir. Eldeki bulgular, Evrenin oldukça izotropik ve homojen olduğunu gösterebilmekte, ayrıca sürekli bir genişleme içerisinde olduğu izlenimini oluşturmaktadır. Ancak bu verilerden tekillik civarı çıkarsamalarda bulunmak, bu zamanlarda hakim olan kuantum etkilerinden dolayı klasik anlayışın dışına çıkılmasını şart koşmaktadır. Bu bağlamda, klasik eyleme kuantum etkilerinin, yaygın adı ile halka yaklaşıklıklarının, dahil edilmesiyle oluşturulan etkin eylemin belirlenmesi gereklidir.

Etkin eylemin belirlenebilmesi örneğin, tekillik sonrası 10^{-34} saniye ila sonrasında gerçekleştiği düşünülen bazı simetri kırılmalarının incelenebilmesine olanak sağlar. Simetri kırılmalarından kasıt, günümüzdeki farklı etkileşmeler ve farklı parçacık gruplarının Evrenin belli bir zamanında aynı ailelere dahil olması, ancak kimi bazı koşullar neticesinde bütünselliklerini yitirerek farklı davranışlar sergilemeleridir.

Çalışmamız, henüz simetri kırılması geçirmemiş erken uzaylara anizotropik ve ısıl etkilerin tanıtılması ve bunların "hızlı genişleme" dönemindeki simetri kırılması üzerindeki etkilerini incelemeyi amaçlamaktadır. Bu amaçla ilk olarak, sıfır sıcaklık için geçerli olmak üzere birinci halka yaklaşımdaki etkin eylemin elde edilişi, genel eğri uzay-zamanlar için olmak üzere gözden geçirilmiştir. Söz konusu etkilerin ayrımsanabilmesi için kendisi ile etkileşen skalar alan teorisi kullanılmıştır. Bu hesaplara, izotropik bir uzayı temsil eden FRW metriği ile, uzay öğeleri bütünüyle anizotropik olan Bianchi I uzayları tanıtılmış ve bu uzaylar için etkin eylem hesapları belirlenmiştir. Elde edilen sonuçlar üzerinde, skaler alanın simetri kırılması üzerinde anizotropinin getirdiği etkiler tartışılmıştır. Bulguladığımız üzere, anizotropik etkiler ilginç bir şekilde simetri oluşumuna imkan tanımaktadır.

Ardından sıfır sıcaklıkta benimsenen yöntemin sonlu sıcaklıklardaki duruma kaydırılması ile, etkin eylemin statik kısmını temsil eden etkin potansiyelin hesabı için bir yöntem geliştirilmiştir. Böylece ısıl etkin potansiyel için modifiye Bessel fonkisyonları cinsinden bir ifade edilerek, yüksek ve düşük sıcaklık limitleri incelenmiş, sıcaklığın simetri kırılması üzerindeki etkileri ve kullandığımız yöntemin geçerliliği tartışılmıştır. Isıl etkileri incelemek amacı ile elde ettiğimiz sonuçlar yüksek sıcaklık limiti için literatürdeki sonuçlar ile uyumlu çıkarak simetrinin belli bir sıcaklıkta tekar oluşabileceğini göstermektedir. Ayrıca sonlu sıcaklık çalışmasında, toplojinin getirebileceği etkiler de ayrımsanmış ve tartışmaya açık tutulmuştur.

1. INTRODUCTIONEquation Chapter 1 Section 1

Our work comprises using path integral formalism in order to investigate a spontaneous symmetry breaking mechanism which's believed to occur during the inflation era [1, p.270]. We understand from this process a breakdown of certain symmetries, such as the end of Grand Unification, electroweak $SU(2)_L \otimes U(1)_\gamma \rightarrow U(1)_{EM} \oplus SU(2)_L$ symmetry breaking or baryon asymmetry [2, p.40]. Our main concern will be investigation of anisotropic and thermal effects for such events, and in order to recognize the direct consequences we'll oversimplify the matters employing self interacting scalar field theories. For these considerations we'll calculate the respective effective actions, relevance of which will be given in this chapter. However in order to render our results in a closed form, we'll be undertaking a number of approximations, short distance behavior, slow varying back-ground field and low curvature approximations being the most eminent ones.

An outline of our study might be given as follows. In the next chapter, inspired by Kirsten et.al. [3] we'll be deriving an expression for the effective action to the first order loop order at zero temperature. In the third chapter a derivation for finite temperature effective potential will be proposed. In the fourth chapter, introducing nontrivial topology by means of Bianchi I and spatially flat $k = 0$ FRW spaces, discussion of aniostropy and the implications of thermal effects will be held. We'll conclude and summarize our study in the final chapter.

As a preliminary introduction, we'll suit ourselves to discuss certain physical and mathematical aspects of quantum field theories starting from conventional zero temperature approach. It should be noted that, the following arguments apply equally well for general d-dimensional space-times and various interactions. Hence we'll refrain emphasizing definite models at this stage.

For a start, let us express the generating functional of the Green functions,

$$Z[J] = \int D\varphi \exp\left[\tfrac{i}{\hbar}\left(S[\varphi] + \int d^n x\sqrt{-g}\, J(x)\varphi(x)\right)\right] \tag{1.1}$$

which gives the Green functions,

$$G_n(x_1, x_2, ..., x_n) = (-i\hbar)^n \left.\frac{\delta^n Z[\varphi]}{\delta J(x_1)\delta J(x_2)...\delta J(x_n)}\right|_{J=0} \tag{1.2}$$

that's the expectation values of the time ordered products of fields with respect to vacuum state ($J \to 0$). It might be noted that, (1.1) could be also understood as the partition function of the system, the integrand being the probability distribution, and (1.2) will give the *n*-point correlation functions [4, p.28]. In this sense we might state the above expressions as,

$$\langle 0|A(\varphi)|0\rangle = \frac{1}{Z[0]}\int D\varphi A(\varphi)\exp\left[\tfrac{i}{\hbar}S[\varphi]\right] \tag{1.3}$$

for any operator $A(\varphi)$ and,

$$\begin{aligned} Z[J] &= \langle 0|\exp\left[\tfrac{i}{\hbar}\int d^n x\sqrt{-g}\, J\varphi\right]|0\rangle \\ G_n(x_1, x_2, ..., x_n) &= \langle 0|:\varphi(x_1)\varphi(x_2)...\varphi(x_n):|0\rangle \end{aligned}. \tag{1.4}$$

For the action we typed in the exponent of (1.1), we'll restrict ourselves to self interacting massive scalar fields,

$$S[\varphi] = \int d^d x\sqrt{-g}\left[-\tfrac{1}{2}\varphi(x)\left(\Box_x + m^2 + \xi R(x) - i\alpha\right)\varphi(x) - V(\varphi(x))\right] \tag{1.5}$$

ξ being the gravitational coupling constant and $V(\phi)$ a self interaction term[1]. Here, $i\alpha$ has been introduced solely for mathematical purposes, that's to ensure the limit of the integrand of (1.1) which will be the case for $\mathrm{Im}\alpha = 0$ and $\alpha > 0$.

[1] Further on, we'll drop the *x* dependence of field variables for convenience.

However, this procedure which we took care to outline only briefly, produces the disconnected Green functions as well. Therefore, it's more convenient to use the generating functional of the connected Green functions which's defined via,

$$Z[J] = \exp\left[\tfrac{i}{\hbar} W[J]\right] \tag{1.6}$$

and which produces the connected Green functions alone,

$$G_n^c(x_1, x_2, ..., x_n) = (-i\hbar)^n \left.\frac{\delta^n W[\varphi]}{\delta J(x_1)\delta J(x_2)...\delta J(x_n)}\right|_{J=0}. \tag{1.7}$$

Adapting statistical field theory language, we might call the generating functional of the connected Green functions as the free energy of the system, in correspondence with the above argument [4, p.93]. It might also be seen that,

$$\frac{\delta W[J]}{\delta J(x)} = \frac{-i\hbar}{Z[J]}\frac{\delta Z[J]}{\delta J(x)} = \frac{\langle 0|\varphi(x) e^{\frac{i}{\hbar}\int dv J\varphi}|0\rangle}{\langle 0|e^{\frac{i}{\hbar}\int dv J\varphi}|0\rangle} = \phi(x) \tag{1.8}$$

where ϕ is spelled as the mean field for an existing source.

Now, let us introduce another functional which's given as a Legendre trasform upon the *free energy*,

$$\Gamma[\phi] = W[J] - \int d^n x\sqrt{-g}\,\phi J \tag{1.9}$$

and using (1.8), which satisfies,

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = -J(x) \tag{1.10}$$

producing an equation of motion, and therefore playing a similar role as the classical action in case of a source. It might also be argued that the functional defined as (1.9) is the generating functional of the vertex functions [5, p.52]. In regard of the classical action (1.5) which's the generator of the classical *n*-point vertex function, it's legitimate to call the functional $\Gamma[\phi]$ as the effective action. Henceforth, we may state that the effective action is the generating functional of the vertex functions including

the classical vertex functions and all their quantum corrections. Meanwhile in the stastical physics language, from the definition held in (1.9) we may name the effective action as the *thermodynamic potential*.

From (1.10) we might extract another important property for the effective action, one we'll be using extensively while evaluating our results. In case of a vanishing source, that's for a vacuum state, r.h.s. of (1.10) becomes zero, and as a result it reads as an extremum condition for the effective action. Consequently, computing the roots of,

$$\frac{\delta\Gamma[\phi]}{\delta\phi} = 0 \tag{1.11}$$

gives us the mean fields, and if they're found to be different from zero, a broken symmetry of $\phi \to -\phi$ will be revealed. Therefore evaluation of effective action permits us to ascertain the fate of symmetry.

Next, we shall attempt to discuss some aspects of the finite temperature field theory. The relation with the above arguments might be seen considering the partition function of a thermal quantum system,

$$Z_\beta = \int dq \left\langle q \left| e^{-\beta H} \right| q \right\rangle \tag{1.12}$$

which's understood to have only one degree of freedom for convenience. If we had formerly defined the probability amplitute [6, p.82],

$$F\left(q_f, t_f; q, 0\right) = \left\langle q_f, t_f \left| q, 0 \right\rangle = \left\langle q_f \left| \exp\left[\tfrac{i}{\hbar} t_f H\right] \right| q \right\rangle$$

which'd be computed,

$$F\left(q_f, t_f; q, 0\right) = \int Dq \exp\left[\tfrac{i}{\hbar} S\left(t_f\right)\right], \quad S\left(t_f\right) = \int_0^{t_f} L\left(q, \partial_t q\right) dt$$

this would allow us to cast (1.12) as,

$$Z_\beta = \int F\left(q, i\hbar\beta; q, 0\right) dq \tag{1.13}$$

subject to the periodic boundary conditions,

$$q(0) = q(i\hbar\beta)\,. \tag{1.14}$$

On the other hand we could have directly reached (1.13) by means of the transformations,

$$dt \to i\hbar d\tau : \quad t_f \to i\hbar\beta\,. \tag{1.15}$$

Therefore we can deduce that, if the reciprocal of temperature is connected with the imaginary time according to (1.15) and if periodic configurations are proposed such as (1.14), a correspondence between thermal and zero temperature quantum systems might be constructed [7, p.21].

Generalization of this argument to field theories of *d*-dimensional space-times is similarly realized, giving the thermal partition function as [7, p.37],

$$Z_\beta[J] = \int D\varphi \exp\left[-S_\beta[\varphi] + \int_0^{\beta\hbar} d\tau \int d^{d-1}x \sqrt{-g_{[d-1]}}\, J\varphi \right] \tag{1.16}$$

where with respect to (1.15),

$$S_\beta[\varphi] = \int_0^{\hbar\beta} d\tau \int d^{d-1}x \sqrt{-g_{[d-1]}} \left[-\tfrac{1}{2}\varphi\left(\mathrm{W}_\beta + m^2 + \xi R_{[d-1]} \right)\varphi - V(\varphi) \right]. \tag{1.17}$$

Regarding the imaginary time transformation (1.15), the thermal D'Alembertian is conceived as,

$$\mathrm{W}_\beta = -\left(\frac{\partial^2}{\partial\tau^2} + \Delta_{[d-1]} \right), \quad \Delta_{[d-1]} = \nabla^a \nabla_a \quad a = 0,1,\ldots,n-1\,. \tag{1.18}$$

During these expressions we understand from the lower indices in square brackets respective values of certain objects with their time components omitted. A reason for this specification follows from the flatness of temperature (time) coordinate. Besides these maneuvers, former arguments we have supplied for zero temperature field theories apply for thermal field theories as well. Therefore, we find it appropriate to proceed with the calculation of the effective action.

2. ZERO TEMPERATURE STUDY

2.1. Formal DeEquation Chapter (Next) Section 1rivation of the Effective Action

During this section we'll derive an expression for the zero temperature effective action to the first order loop approximation, mainly following the arguments of Buchbinder, Sphapiro and Odintsov [5, p. 53-57]. From (1.9) we have**,**

$$\Gamma[\phi] + \int d^d x \sqrt{-g}\,\phi J = W[J] \tag{2.1.1}$$

and inserting this to the definition of the generating functional of connected Green functions we get,

$$\exp\left[\tfrac{i}{\hbar}\left(\Gamma[\phi] + \int d^d x \sqrt{-g}\, x\phi J\right)\right] = \int D\varphi \exp\left[\tfrac{i}{\hbar}\left(S[\varphi] + \int d^d x \sqrt{-g}\,\varphi J\right)\right]. \tag{2.1.2}$$

If we effect a translation, $\varphi \to \varphi + \phi$, ϕ being the mean field with respect to the source,

$$\exp\left[\tfrac{i}{\hbar}\left(\Gamma[\phi] + \int d^d x \sqrt{-g}\,\phi J\right)\right] = \int D\varphi \exp\left[\tfrac{i}{\hbar}\left(S[\varphi+\phi] + \int d^d x \sqrt{-g}\,(\varphi+\phi) J\right)\right]$$

and expand the classical action with respect to the mean field, we reach,

$$\exp\left[\tfrac{i}{\hbar}\left(\Gamma[\phi] - S[\phi]\right)\right] = \int D\varphi \exp\left[\tfrac{i}{\hbar}\left(\tfrac{1}{2} S_2[\phi]\varphi^2 - \varphi\left(\frac{\delta\Gamma[\phi]}{\delta\phi} - S_1[\phi]\right) + \sum_{n=3} \tfrac{1}{n!} S_n[\phi]\varphi^n\right)\right] \tag{2.1.3}$$

where we have substituted (1.10) instead of the source, and simplified our notation using,

$$\int d^n x A(\varphi) B(\varphi) = A(\varphi) B(\varphi), \qquad A_n[\phi] \equiv \left. \frac{\delta^n A[\varphi]}{\delta \varphi^n} \right|_{\phi}$$

for any field functions $A(\varphi)$ and $B(\varphi)$. Now let us make another transformation upon the field variables as,

$$\varphi \to \hbar^{\frac{1}{2}} \varphi$$

which enables us to rewrite (2.1.3) as,

$$\begin{aligned} &\exp\left[\tfrac{i}{\hbar}\bar{\Gamma}[\phi]\right] \\ &= \int D\varphi \exp\left[i\left(\tfrac{1}{2} S_2[\phi]\varphi^2 - \hbar^{-\frac{1}{2}} \varphi \left(\Gamma_1[\phi] - S_1[\phi]\right) + \sum_{n=3} \tfrac{\hbar^{\frac{n}{2}-1}}{n!} S_n[\phi]\varphi^n \right)\right]. \end{aligned} \quad (2.1.4)$$

Here, we have defined,

$$\bar{\Gamma}[\phi] = \Gamma[\phi] - S[\phi] \quad (2.1.5)$$

which might be understood as the quantum contributions to the effective action. At this point let us propose the quantum contributions as a power series with respect to $\hbar$,

$$\bar{\Gamma}[\phi] = \sum_{n=1} \hbar^n \bar{\Gamma}^{(n)}[\phi] \quad (2.1.6)$$

crowning the method with the name loop expansion. As we have been determined to find the first order loop contribution, that's $\bar{\Gamma}^{(1)}[\phi]$, if we expand both sides of (2.1.4) with respect to $\hbar$ and compare the first order terms, we find,

$$\begin{aligned} \exp\left[i\bar{\Gamma}^{(1)}[\phi]\right] &= \int D\varphi \exp\left(\tfrac{i}{2}\varphi S_2[\phi]\varphi\right) \\ &= \det{}^{-\frac{1}{2}} S_2[\phi] \end{aligned} \quad (2.1.7)$$

where in the last step we have formally effected a Gaussian integration and therefore conclude,

$$\bar{\Gamma}^{(1)}[\phi] = \frac{i}{2}\ln\left(\det\frac{S_2[\phi]}{\mu^2}\right) = \frac{i}{2}\mathrm{Tr}\left(\ln\frac{S_2[\phi]}{\mu^2}\right) \tag{2.1.8}$$

whereas for the effective action,

$$\Gamma^{(1)}[\phi] = S[\phi] + \hbar\bar{\Gamma}^{(1)}[\phi]. \tag{2.1.9}$$

As can been seen from (2.1.8), an arbitrary constant μ^2 has been introduced to render the argument of logarithm dimensionless. We'll return to this arbitrariness while we're renormalizing the effective action. Instead, now let us contemplate over (2.1.7) a while. From (1.5) we find,

$$S_2[\phi] = \left.\frac{\delta^2 S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\right|_{\phi} = -\left(\Box_x + m^2 + \xi R_x + V''(\phi) - i\alpha\right)\delta(x_1 - x_2) \tag{2.1.10}$$

which's the propagator of the quantum fields. This fact might be discerned applying the minimum action principle to the exponent of (1.1) which will yield the equation of motion for the scalar field,

$$\left(\Box_x + m^2 + \xi R(x) - i\alpha\right)\varphi(x) + V'(\varphi) = -J(x) \tag{2.1.11}$$

and which might be investigated further if the field variable is decomposed as,

$$\varphi = \phi + \hbar\varphi^q \tag{2.1.12}$$

that's, the sum of classical (mean) field and the (first order) quantum contribution. Inserting (2.1.12) into (2.1.11), in case of a vanishing source we find,

$$\left(\Box_x + m^2 + \xi R(x) - i\alpha\right)\left(\phi + \varphi^q\right) + V'\left(\phi + \varphi^q\right) = 0$$
$$\rightarrow \left(\Box_x + m^2 + \xi R(x) - i\alpha\right)\phi + V'(\phi) + \hbar\left[\left(\Box_x + m^2 + \xi R(x) - i\alpha\right)\varphi^q + V''(\phi)\varphi^q\right] = 0$$

where we can directly read the quantum equation of motion,

$$\left(\Box_x + m^2 + \xi R(x) + V''(\phi) - i\alpha\right)\varphi^q = 0 \tag{2.1.13}$$

the propagator as in (2.1.10) manifest.

(2.1.13) might also be expressed in terms of the Feynman Green function [9, p.48],

$$\left(\Box_x + m^2 + \xi R_x + V''(\phi) - i\alpha\right) G_F(x,x') = -\delta^d(x,x')\sqrt{-g_x} \qquad (2.1.14)$$

or,

$$S_2[\phi] G_F(x,x') = \delta^d(x,x')\sqrt{-g_x} \qquad (2.1.15)$$

which will be useful for the following section.

2.2. Calculation of tEquation Section (Next)he One-Loop Effective Action

There're various methods to calculate the one-loop effective potential. One might start from (2.1.8) and calculate it perturbatively with respect to curvature, enduring somewhat strenuous labor [10]. It'd also be possible to start directly from renormalization group theory approach, which produces rather simple results, if the β-functions of the system are known [11]. We'll prefer *zeta*-function requlrization and later utilize renormalization. Our choice arises from the applicability of this technique to finite temperature study in a straightforward manner, however it'll be seemingly tedious at the first look. Regarding the well established literature about this technique, we will not give specific references during our derivation if the need does not arise. Also from now on we will be working in mass dimensions, taking $\hbar = 1$, $c = 1$, $k = 1$.

For a start, we might formally represent (2.1.8) as [5, p.60],

$$\overline{\Gamma}^{(1)} = \frac{i}{2}\sum_n \ln a_n \qquad (2.2.1)$$

where it's understood,

$$\frac{1}{\mu^2} S_2[\phi] f_n = A f_n = a_n f_n \qquad (2.2.2)$$

satisfied as an eigenvalue problem. An explicit formulation for (2.2.1) might be performed via Riemann-Zeta function for a complex argument z which's defined to be,

$$\zeta_A(z) = \sum_n \frac{1}{a_n^z}. \tag{2.2.3}$$

The connection between these might be once seen taking the derivative of (2.2.3) with respect to *z*,

$$\zeta_A'(z) = -\sum_n \frac{\ln a_n}{a_n^z}$$

whence,

$$\bar{\Gamma}^{(1)}[\phi] = -\frac{i}{2}\zeta_A'(0). \tag{2.2.4}$$

We remark that, the zeta function of (2.2.3) is defined for Re $z > 1$ and clearly divergent otherwise [12, eq.9.522]. However by means of analytic continuation it might be rendered convergent for $z = 0$ [9, p.167].

Next taking in account the integral representation of the Gamma function,

$$\Gamma(z) = \int_0^\infty ids\,(is)^{z-1} e^{-is}$$

(2.2.3) might be reexpressed as,

$$\zeta_A(z) = \frac{1}{\Gamma(z)} \sum_n \int_0^\infty ids\,(is)^{z-1} e^{-ia_n s} = \frac{1}{\Gamma(z)} \int_0^\infty ids\,(is)^{z-1} \mathrm{Tr}\left(e^{-isA}\right). \tag{2.2.5}$$

In order to arrange (2.2.5) in a calculable form, we might liberate our notation from the unknown a_n's, rewriting the last equation as,

$$\zeta_A(z) = -\frac{1}{\Gamma(z)} \int_0^\infty ids\,(is)^{z-1} \int d^d x \sqrt{-g}\, K(x, x; s) \tag{2.2.6}$$

where a new function has been introduced which satisfies,

$$i\frac{\partial K_A(x, x'; s)}{\partial s} = -A_x K_A(x, x'; s) \tag{2.2.7}$$

subject to the boundary conditions,

$$\lim_{s\to o} K_A(x,x';s) = \sqrt{-g}\,\delta^d(x,x') \tag{2.2.8}$$

and hence might formally be expanded as,

$$K_A(x,x';s) = -\sum_n f_n(x)\, f_n(x') e^{-isa_n(x)}. \tag{2.2.9}$$

with respect to the eigenfunctions of (2.2.2). As to see the validity of these arguments it might be checked that,

$$-\int d^d x\sqrt{-g}\,K(x,x;s) = \int d^n x\sqrt{-g_x}\sum_n f_n(x)\, f_n(x) e^{-isa_n}$$
$$= \sum_n e^{-isa_n}$$

where orthonormality of the eigenfunctions has been effected.

Equation (2.2.6) will be seen to be convenient for computation if the kernel of the integral is somehow succesfully calculated, which we'll see to be the case in the rest of this section. We note that, by analogy of (2.2.7) with a heat equation, this kernel is sometimes referred as the heat kernel in literature [13,14].

A straightforward calculation of the kernel is possible if Riemann normal coordinates are admitted. The discussion rests on the idea of the correspondence between the Green fuction and the kernel, dictating that an expression for the kernel might be found if the Green function for the system is calculated first [13-15]. However, as we'll be remarking now and then, this technique is valid only for small distances, hence it should not be taken as a derivation that will determine the global behavior of the system. On the other hand, as we'll be interested in small curvature approximations, this will suit our expectations. Also, using this method we shall only concern ourselves with the ultraviolet divergences as we'll be probing the short wave-length behavior, freed from the infrared divergences[2].

While determining an expression for the kernel (2.2.9), we'll be presenting a derivation which might be conceived as a hybrid of [9] and [16]. However in our

results, the field interaction dependence will not appear in the heat coefficients, instead we will confine it as an exponent only. Apart from this difference, the derivation presented below produce results that agree with [13] which has been supplied for non-interacting field theories.

Returning to (2.1.15) and employing the definitions [16],

$$\bar{G}_F(x,x') = (-g_x)^{\frac{1}{4}} G_F(x,x')(-g_{x'})^{\frac{1}{4}} \tag{2.2.10}$$

and,

$$H_x = (-g_x)^{\frac{1}{4}} \left(\nabla^2 + M^2 + \xi R - i\alpha\right)(-g_x)^{-\frac{1}{4}}, \qquad M^2 \equiv m^2 + V''(\phi) \tag{2.2.11}$$

(2.1.15) might be rewritten as,

$$H\bar{G}_F(x,x') = -\delta^d(x,x') \tag{2.2.12}$$

for a close enough *x, x'* space-time seperation. Therefore, introducing the Riemann normal coordinates such as,

$$y^\mu = (x'-x)^\mu \rightarrow x'^\mu = x^\mu + y^\mu \tag{2.2.13}$$

and assuming that the components of the metric tensor are analytic functions in the neighborhood of any point *x*, we might well expand the metric with respect to *y*. According to the results which we'll borrow from the literature [17, p.36], an expansion around x up to the fourth derivative of the metric produces,

$$\begin{aligned} g_{\mu\nu}(x') = \eta_{\mu\nu}(x) + \tfrac{1}{3}R_{\mu\alpha\nu\beta}y^\alpha y^\beta + \tfrac{1}{6}R_{\mu\alpha\nu\beta;\gamma}y^\alpha y^\beta y^\gamma \\ + \left(\tfrac{1}{20}R_{\mu\alpha\nu\beta;\gamma\delta} + \tfrac{2}{45}R_{\alpha\mu\beta\lambda}R^\lambda{}_{\gamma\nu\delta}\right)y^\alpha y^\beta y^\gamma y^\delta \end{aligned} \tag{2.2.14}$$

where the curvature tensors are all calculated at x. Using this expansion, one also finds,

[2] This argument is analogous to the introduction of low limit *cut-off* for certain momentum integrals.

$$
\begin{aligned}
g^{\mu\nu} &\cong \eta^{\mu\nu} - \tfrac{1}{3}R^{\mu\ \ \nu}_{\ \alpha\ \ \beta}y^{\alpha}y^{\beta} - \tfrac{1}{6}R^{\mu\ \ \nu}_{\ \alpha\ \ \beta;\gamma}y^{\alpha}y^{\beta}y^{\gamma} \\
&\quad - \left(\tfrac{1}{20}R^{\mu\ \ \nu}_{\ \alpha\ \ \beta;\gamma\delta} + \tfrac{1}{15}R^{\mu}_{\ \alpha\beta\kappa}R^{\kappa\ \ \nu}_{\ \gamma\ \ \delta}\right)y^{\alpha}y^{\beta}y^{\gamma}y^{\delta} \\
(-g) &= 1 + \tfrac{1}{3}R_{\alpha\beta}y^{\alpha}y^{\beta} + \tfrac{1}{6}R_{\alpha\beta;\gamma}y^{\alpha}y^{\beta}y^{\gamma} \\
&\quad + \left(\tfrac{1}{18}R_{\alpha\beta}R_{\gamma\delta} + \tfrac{1}{90}R_{\lambda\alpha\beta}{}^{\kappa}R^{\lambda}_{\ \gamma\delta\kappa} + \tfrac{1}{20}R_{\alpha\beta;\gamma\delta}\right)y^{\alpha}y^{\beta}y^{\gamma}y^{\delta}
\end{aligned}
\tag{2.2.15}
$$

and,

$$
V''\left(\phi\left(x'\right)\right) = V''\left(\phi\right) + V''\left(\phi\right)_{;\mu}y^{\mu} + \tfrac{1}{2}V''\left(\phi\right)_{;\mu;\nu}y^{\mu}y^{\nu}. \tag{2.2.16}
$$

Substitution of these into (2.2.12) gives,

$$
\begin{Bmatrix}
\eta^{\mu n}\partial_{\mu}\partial_{\nu} + \left[M^{2} - i\alpha + \xi\left(1 - \tfrac{1}{6}\right)R\right] + \left[V''\left(\phi\right) + \left(\xi - \tfrac{1}{6}\right)R\right]_{;\alpha}y^{\alpha} \\
+ \left[V''\left(\phi\right) + \tfrac{1}{2}\left(\xi - \tfrac{1}{6}\right)R\right]_{;\alpha\beta}y^{\alpha}y^{\beta} \\
+ \left(\tfrac{1}{30}R_{\alpha}{}^{\lambda}R_{\lambda\beta} - \tfrac{1}{60}R^{\kappa\ \ \lambda}_{\ \alpha\ \ \beta}R_{\kappa\lambda} - \tfrac{1}{60}R^{\lambda\mu\kappa}{}_{\alpha}R_{\lambda\mu\kappa\beta} + \tfrac{1}{120}R_{;\alpha\beta} - \tfrac{1}{40}\text{WR}_{\alpha\beta}\right)y^{\alpha}y^{\beta}
\end{Bmatrix}
\times \bar{G}_{F}\left(y\right) = -\delta^{d}\left(y\right)
\tag{2.2.17}
$$

where we have converted $\partial_{\mu}g_{\mu\nu}\partial_{\nu}$ to its flat space-time counterpart regarding Lorentz invariance. It might also be stated that a Fourier transform of the Green function is applicable,

$$
\bar{G}_{F}\left(x, x'\right) = \int \frac{d^{d}k}{\left(2\pi\right)^{d}}e^{iky}\bar{G}\left(k\right), \quad ky = \eta^{\mu\nu}k_{\mu}y_{\nu} \tag{2.2.18}
$$

however which is only defined locally [18].

In order to solve (2.2.17) it's customary to propose the Green function as a perturbation series,

$$
\bar{G}_{F}\left(y\right) = \bar{G}_{0}\left(y\right) + \bar{G}_{1}\left(y\right) + \bar{G}_{2}\left(y\right) + \ldots \tag{2.2.19}
$$

with each succeding term as a function of increasing curvature, $G_0(y)$ being the flat space-time Green function, and from (2.2.18), it might be argued that each term has its respective Fourier transform. Inserting (2.2.19) into (2.2.17) and comparing terms with equal curvature dependence we find,

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + M^2 - i\alpha\right)\bar{G}_0(y) = -\int \frac{d^d k}{(2\pi)^d} e^{iky} \rightarrow \bar{G}_0(k) = \frac{1}{k^2 - M^2 + i\alpha}$$

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + M^2 - i\alpha\right)\bar{G}_1(y) + V''_{;\lambda} y^\lambda \bar{G}_0(y) = 0$$

$$\rightarrow \bar{G}_1(k) = V''_{;\lambda} y^\lambda \left(k^2 - M^2 + i\alpha\right)^{-2}$$

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + M^2 - i\alpha\right)\bar{G}_2(y) + \left[\tfrac{1}{2}V''_{;\lambda;\kappa} y^\lambda y^\kappa \left(\xi - \tfrac{1}{6}\right)R\right]\bar{G}_0(y) = 0$$

$$\rightarrow \bar{G}_2(k) = \left[\tfrac{1}{2}V''_{;\lambda;\kappa} y^\lambda y^\kappa \left(\xi - \tfrac{1}{6}\right)R\right]\left(k^2 - M^2 + i\alpha\right)^{-2}$$

$$\left(\eta^{\mu n}\partial_\mu\partial_\nu + M^2 - i\alpha\right)\bar{G}_3(y) + \left(\xi - \tfrac{1}{6}\right)R_{;\lambda} y^\lambda \bar{G}_0(y) = 0$$

$$\rightarrow \bar{G}_3(k) = i\left(\tfrac{1}{6} - \xi\right)R_{;\lambda}\left[\partial_k^\lambda \left(k^2 - M^2 + i\alpha\right)^{-1}\right]\left(k^2 - M^2 + i\alpha\right)$$

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + M^2 - i\alpha\right)\bar{G}_4(y) + \left(\xi - \tfrac{1}{6}\right)R\bar{G}_2(y) + a_{\lambda\kappa} y^\lambda y^\kappa \bar{G}_0(y)$$

$$\rightarrow \bar{G}_4(k) = \left(\tfrac{1}{6} - \xi\right)^2 R^2 \left(k^2 - M^2 + i\alpha\right)^{-3} - a_{\lambda\kappa}\left[\partial_k^\lambda \partial_k^\kappa \left(k^2 - M^2 + i\alpha\right)^{-1}\right]$$

where we have used the shorthand notations,

$$\partial_k^\lambda \equiv \frac{\partial}{\partial k_\lambda}$$

and,

$$a_{\lambda\kappa} \equiv \tfrac{1}{2}\left(\xi - \tfrac{1}{6}\right)R_{;\lambda\kappa} + \tfrac{1}{120}R_{;\lambda\kappa} - \tfrac{1}{40}\Box R_{\lambda\kappa} - \tfrac{1}{30}R_\lambda{}^\gamma R_{\gamma\kappa} + \tfrac{1}{60}R^\gamma{}_\lambda{}^\delta{}_\kappa R_{\gamma\delta} + \tfrac{1}{60}R^{\gamma\delta\sigma}{}_\lambda R_{\gamma\delta\sigma\kappa}$$

the repeating indices are understood to be summed. Taking in notice,

$$\partial_k^\lambda \left(k^2 - M^2 + i\alpha\right)^{-1} = -\left(k^2 - M^2 + i\alpha\right)^{-2}\partial_k^\lambda k^2$$

$$\rightarrow \left[\partial_k^\lambda \left(k^2 - M^2 + i\alpha\right)^{-1}\right]\left(k^2 - M^2 + i\alpha\right)^{-1} = \frac{1}{2}\partial_k^\lambda \left(k^2 - M^2 + i\alpha\right)^{-2}$$

and,

$$\left[\partial_k^\lambda \partial_k^\kappa \left(k^2 - M^2 + i\alpha\right)^{-1}\right] = \partial_k^\lambda\left[-\frac{\partial_k^\kappa k^2}{\left(k^2 - M^2 + i\alpha\right)^2}\right]$$

$$\rightarrow \left(k^2 - M^2 + i\alpha\right)^{-1}\left[\partial_k^\lambda \partial_k^\kappa \left(k^2 - M^2 + i\alpha\right)^{-1}\right]$$

$$= \frac{1}{3}\partial_k^\lambda \partial_k^\kappa \left(k^2 - M^2 + i\alpha\right)^{-2} - \frac{2}{3}\frac{\eta^{\lambda\kappa}}{\left(k^2 - M^2 + i\alpha\right)^3}$$

we might write the momentum space Green function as,

$$\begin{aligned}\bar{G}_F(k) = &\left(k^2 - M^2 + i\alpha\right)^{-1} - \left(\tfrac{1}{6} - \xi\right) R \left(k^2 - M^2 + i\alpha\right)^{-2} \\ &+ i\left[\tfrac{1}{2}\left(\tfrac{1}{6} - \xi\right) - V''_{;\lambda}\right] R_{;\lambda} \partial_k^\lambda \left(k^2 - M^2 + i\alpha\right)^{-2} \\ &- \left[\tfrac{1}{3} a_{\lambda\kappa} + V''_{;\lambda;\kappa}\right] \partial_k^\lambda \partial_k^\kappa \left(k^2 - M^2 + i\alpha\right)^{-2} \\ &+ \left[\left(\tfrac{1}{6} - \xi\right)^2 R^2 + \tfrac{2}{3} a^\lambda{}_\lambda\right] \left(k^2 - M^2 + i\alpha\right)^{-3}\end{aligned} \tag{2.2.20}$$

whereby the inverse Fouier transform gives,

$$\begin{aligned}\bar{G}_F(x, x') = &\int \frac{d^d k}{(2\pi)^d} e^{-iky} \\ &\times \left\{\left[a_0(x, x') + a_1(x, x')\left(\frac{\partial}{\partial i\alpha}\right) + a_2(x, x')\left(\frac{\partial}{\partial i\alpha}\right)^2\right]\left(k^2 - M^2 + i\alpha\right)^{-1}\right\}\end{aligned} . \tag{2.2.21}$$

A comparision with (2.2.20) shows that,

$$a_0(x, x') = 1 \tag{2.2.22}$$

and,

$$\begin{aligned}a_1(x, x') &= \left(\tfrac{1}{6} - \xi\right) R + \left[V'' + \tfrac{1}{2}\left(\tfrac{1}{6} - \xi\right) R\right]_{;\lambda} y^\lambda + \left[V''_{;\lambda;\kappa} - \tfrac{1}{3} a_{\lambda\kappa}\right] y^\lambda y^\kappa \\ a_2(x, x') &= \tfrac{1}{2}\left(\tfrac{1}{6} - \xi\right)^2 R^2 + \tfrac{1}{3} a^\lambda{}_\lambda\end{aligned} . \tag{2.2.23}$$

As might be conferred from the right most factor of (2.2.21), we might reasonably use the Schwinger representation of the momentum propagator which is well defined for $\alpha > 0$ (in accord with our convention),

$$\left(k^2 - M^2 + i\alpha\right) = -i\int_0^\infty ds e^{is\left(k^2 - M^2 + i\alpha\right)} \tag{2.2.24}$$

and interhanging the *s* integral with *k* integral of (2.2.21), explicit performation of the *k* integral gives [9, p.75],

$$\bar{G}_F = -i(4\pi)^{-\frac{d}{2}} \int_0^\infty i^{d+1} ds (is)^{-\frac{n}{2}} \exp\left[-isM^2 + \frac{\sigma^2}{4is}\right] F(x, x'; is) e^{-\alpha s} \qquad (2.2.25)$$

where $\sigma^2 \equiv y^\mu y_\mu$ is the proper arc length from *x* to x', and the *x* dependent terms has been grouped together as,

$$F(x, x'; is) = \sum_{j=0}^{\infty} (is)^j a_j(x, x'). \qquad (2.2.26)$$

Here we'll allow for the summation to extend to infinity for formal reasons only. However in reality, our calculations will not extend further than terms including the fourth derivatives of the metric tensor, that's $a_j(x, x') = 0$ for $j > 2$.

Returning to the Feynman propagator from (2.2.10),

$$G_F(x, x') = (-g_x)^{-\frac{1}{4}} \bar{G}_F(x, x')(-g_{x'})^{-\frac{1}{4}}$$

in accord with [13] we find,

$$G_F(x, x') = -i\Delta^{\frac{1}{2}}(x, x')(4\pi)^{-\frac{d}{2}} \int_0^\infty i^{d+1} ds (is)^{-\frac{d}{2}} \exp\left[-isM^2 + \frac{\sigma^2}{4is}\right] F(x, x'; is) e^{-\alpha s} \qquad (2.2.27)$$

where,

$$\Delta(x, x') = \left[g_x g_{x'}\right]^{-\frac{1}{2}} = -\det\left[\partial_\mu \partial_\nu \sigma(x, x')\right]\left[g_x g_{x'}\right]^{-\frac{1}{2}} \qquad (2.2.28)$$

is the Van Vleck determinant, technically speaking. Before resuming the computation of the first order contributions to the effective action, we'd like to remark the validity of this derivation for any dimension *d*, which will suit our purposes for subsequent chapters, that's finite temperature calculations.

The afore mentioned correspondence between the heat kernel (2.2.9) and the Green function is given by,

$$G_F(x, x') = -i \int_0^\infty K(x, x'; s) e^{-\alpha s} ds \qquad (2.2.29)$$

as might be inferred from (2.1.14). To see this, let us insert (2.2.29) into the l.h.s. of (2.1.14),

$$-i\left(\Box_x + M^2 + \xi R_x - i\alpha\right)\int_0^\infty K\left(x,x';s\right)e^{-\alpha s}ds$$
$$= \int_0^\infty \left[-i\left(\Box + M^2 + \xi R\right)K\left(x,x';s\right) - \frac{\partial K\left(x,x';s\right)}{\partial s}\right]e^{-\alpha s} - \delta\left(x,x'\right)\sqrt{-g_x}$$

where the we have effected a partial integration and then used the imposed boundary condition (2.2.8). Simplifying the Dirac-delta functions, we see that the kernel of the Green function satisfies the same heat equation as (2.2.7), therefore proving the assertion. Comparing (2.2.29) with (2.2.27), we reach a crucial result,

$$K\left(x,x';s\right) = i^{d+1}\frac{\Delta^{\frac{1}{2}}\left(x,x'\right)}{\left(4\pi is\right)^{d/2}}\exp\left[-isM^2 + \frac{\sigma^2}{4is}\right]F\left(x,x';is\right) \tag{2.2.30}$$

which gives the heat kernel in terms of the already calculated *heat* coefficients (2.2.22) and (2.2.23).

We might establish the coincidence limit of these coefficients, that's $a_j(x,x)$'s, as,

$$a_0\left(x,x\right) = 1, \quad a_1\left(x,x\right) = \left(\tfrac{1}{6} - \xi\right)R \tag{2.2.31}$$

and,

$$a_2\left(x,x\right) = \tfrac{1}{2}\left(\xi - \tfrac{1}{6}\right)^2 R^2 + \tfrac{1}{6}\left(\xi - \tfrac{1}{6}\right)\Box R + \tfrac{1}{180}\left(R^{\lambda\mu\kappa\nu}R_{\lambda\mu\kappa\nu} - \Box R - R^{\lambda\kappa}R_{\lambda\kappa}\right). \tag{2.2.32}$$

Furthermore we might even transfer the explicit R dependence of these coefficients to the exponent, thereby writing the coincidence limit of the kernel as [13,14],

$$K\left(x,x;s\right) = \frac{i^{d+1}}{\left(4\pi is\right)^{d/2}}e^{-is\left(M^2 + \left(\xi - \frac{1}{6}\right)R\right)}\overline{F}\left(x,x;is\right) \tag{2.2.33}$$

whereby comparing with (2.2.30), we find for the leading coefficients of $\overline{F}\left(x,x;is\right)$,

$$\overline{a}_0 = 1, \quad \overline{a}_1 = 0 \tag{2.2.34}$$

and,

$$\overline{a}_2 = \tfrac{1}{6}\left(\xi - \tfrac{1}{6}\right)\text{WR} + \tfrac{1}{180}\left(R^{\lambda\mu\kappa\nu}R_{\lambda\mu\kappa\nu} - \text{WR} - R^{\lambda\kappa}R_{\lambda\kappa}\right). \tag{2.2.35}$$

Returning to (2.2.6) and armed with these results, we find for the Riemann-Zeta function,

$$\begin{aligned}\zeta_A(z) = &-\frac{i^{-d+1}}{(4\pi)^{\frac{d}{2}}\Gamma(z)} \\ &\times\sum_{j=0}\Gamma\left(z+j-\tfrac{d}{2}\right)\int d^4x\sqrt{-g}\,\overline{a}_j(x,x)\left[M^2+\left(\xi-\tfrac{1}{6}\right)\right]^{\frac{d}{2}-j-z}\end{aligned} \tag{2.2.36}$$

where we have performed the *s* integration. From now on we'll concentate on conventional $d = 4$ dimensions, and thereby will rewrite (2.2.36) as,

$$\zeta_A(z) = \frac{-i}{(4\pi)^2\Gamma(z)}\int d^4x\sqrt{-g}\left\{\begin{array}{l}\Gamma(z-2)\overline{a}_0\text{M}^{4-2z} + \Gamma(z-1)\overline{a}_1\text{M}^{2-2z} \\ +\Gamma(z)\overline{a}_2\text{M}^{-2z} + \Gamma(z+1)\overline{a}_3\text{M}^{-2-2z} + \ldots\end{array}\right\} \tag{2.2.37}$$

$\text{M}^2 \equiv \left[M^2 + \left(\xi - \tfrac{1}{6}\right)R\right]$ has been defined. Using the relation $\Gamma(z+1) = z\Gamma(z)$ we get,

$$\zeta_A(z) = \frac{-i}{4\pi^2}\int d^4x\sqrt{-g}\left\{\begin{array}{l}\dfrac{\text{M}^{4-2z}\overline{a}_0}{(z-2)(z-1)} + \dfrac{\text{M}^{2-2z}\overline{a}_1}{(z-1)} + \text{M}^{-2z}\overline{a}_2 \\ +z\text{M}^{-2-2z}\overline{a}_3 + z(z+1)\text{M}^{-4-2z}\overline{a}_4 + \text{K}\end{array}\right\}$$

so that,

$$\begin{aligned}\frac{\partial\zeta_A(z)}{\partial z} = &\frac{-i}{16\pi^2}\int d^4x\sqrt{-g} \\ &\times\left\{\begin{array}{l}-\frac{2z-3}{(z-2)^2(z-1)^2}\text{M}^{4-2z}\overline{a}_0 - \frac{1}{(z-1)^2}\text{M}^{2-2z}\overline{a}_1 - \log\frac{\text{M}^2}{\mu^2}\left[\overline{a}_0\frac{\text{M}^{4-2z}}{(z-2)(z-1)} + \overline{a}_1\frac{\text{M}^{2-2z}}{(z-1)} + \overline{a}_2\text{M}^{-2z}\right] \\ +\text{M}^{-2-2z}\overline{a}_3 + (2z+1)\text{M}^{-4-2z}\overline{a}_4 + \text{L} \;- z\log\frac{\text{M}^2}{\mu^2}\left[\overline{a}_3\text{M}^{-2-2z} + (z+1)\overline{a}_4\text{M}^{-4-2z} + \text{L}\right]\end{array}\right\}\end{aligned}$$

whence,

$$\zeta'(0)=\frac{i}{16\pi^2}\int d^4x\sqrt{-g}$$
$$\times\left\{\frac{\bar{a}_0 M^4}{2}\left(\log\frac{M^2}{\mu^2}-\frac{3}{2}\right)+\bar{a}_1 M^2\left(\log\frac{M^2}{\mu^2}-1\right)+\bar{a}_2\log\frac{M^2}{\mu^2}+\sum_{j=1}(l-1)!\frac{\bar{a}_{2+j}}{M^{2j}}\right\}. \quad (2.2.38)$$

Now using (2.2.4) the first order contribution to the effective Lagrangian becomes,

$$\bar{\Gamma}^{(1)}[\phi]=\frac{1}{32\pi^2}\int d^4x\sqrt{-g}$$
$$\times\left\{\frac{\bar{a}_0 M^4}{2}\left(\log\frac{M^2}{\mu^2}-\frac{3}{2}\right)+\bar{a}_1 M^2\left(\log\frac{M^2}{\mu^2}-1\right)+\bar{a}_2\log\frac{M^2}{\mu^2}+\sum_{j=1}(l-1)!\frac{\bar{a}_{2+j}}{M^{2j}}\right\} \quad (2.2.39)$$

and inserting the heat kernel coefficients (2.2.34) and (2.2.35) in place, we see that the first-loop effective action will yield,

$$\bar{\Gamma}^{(1)}[\phi]=\frac{1}{32\pi^2}\int d^4x\sqrt{-g}\left\{\frac{M^4}{2}\left(\log\frac{M^2}{\mu^2}-\frac{3}{2}\right)+\bar{a}_2\log\frac{M^2}{\mu^2}\right\} \quad (2.2.40)$$

which might be seen as the final expression for our zero temperature calculations. We might remark the compliance of this result with that of flat space-times' [19] if the curvature terms are taken as zero.

2.3 RenormalizaEquation Section (Next)tion of the Effective Action

We'll take in account the self interaction,

$$V(\phi)=\frac{\lambda_{(4)}\phi^4}{4!} \quad (2.3.1)$$

This quartic interaction constitutes one of the simplest models in field theory and has been extensively studied in literature, inspring our calculations in so far [3].

Apart from the interaction, we should also define the topology of the space-time. As we have stated earlier we'll be considering Bianchi I and Robertson-Walker space-times as to investigate their respective effects over a phase transition. However an explicit application with these spaces will be given in the discussion chapter, so that

we may compare them with thermal derivations. Therefore we find it sufficient to carry on with the renormalization of the tree level Lagrangian and the first loop contribution (2.2.40), as to relieve the theory from ultraviolent divergences that'll appear for the small distance study we have been carrying.

However before engaging in calculations we shall consider one final important fact. That's, we should propose the classical action in a certain way in order to render the final theory renormalizable. In another saying, the classical action should contain terms other than the already existing ones in (1.5), which may cancel with divergences resulting from the gravitational contributions in addition to the well-known quantum divergences. In order to understand this fact, it might help to cast the divergent part of the effective action, which comes solely from the classical action,

$$\Gamma_{div} = -\int d^4x\sqrt{-g}\left\{\begin{array}{l}\left(m^2-\xi R\right)Z_\Lambda+\left(m^2-\xi R\right)RZ_\kappa \\ \qquad +Z_{a_1}R^2+Z_{a_2}R_{\mu\nu}R^{\mu\nu}+Z_{a_3}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}+Z_{a_4}\Box R\end{array}\right\} \tag{2.3.2}$$

as understood from general coordinate invariance and locality considerations. Here all of the coefficients $Z_\Lambda, Z_\kappa, Z_{a_i}$ are divergent functions of the self interaction constant only [5, p.113][3] . Therefore, in order to render the theory renormalizable we shall start with a classical action which contains the same kind of terms as that of above, for us to oppose these divergences. Hence, the general form of the classical Lagrangian must be given as,

$$\begin{array}{l}L=\Lambda-\frac{1}{2}\phi\left(\Box+m^2+\xi R\right)\phi-V(\phi) \\ \qquad +\varepsilon_0R+\frac{1}{2}\varepsilon_1R^2+\varepsilon_2R_{\mu\nu}R^{\mu\nu}+\varepsilon_3R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}+\varepsilon_4\Box R\end{array} \tag{2.3.3}$$

where we're introducing the cosmological constant for computation purposes only, which we'll later take it to be zero . Instead of (2.3.3) it'll be more convenient to express this Lagrangian in terms of certain space invariants,

$$L=\Lambda-\frac{1}{2}\phi\left(\Box+m^2+\xi R\right)\phi-V(\phi)+\varepsilon_0R+\frac{1}{2}\varepsilon_1R^2+\varepsilon_2\mathbf{C}^2+\varepsilon_3\mathbf{G}+\varepsilon_4\Box R \tag{2.3.4}$$

where,

[3] It might be noted that these divergences have no counterpart in flat space-times.

$$\mathbf{C}^2 = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{3}R^2, \quad \mathbf{G} = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \qquad (2.3.5)$$

are the square of the Weyl tensor and the Gauss-Bonnet density of the space, respectively. It might be argued that, (2.3.4) and (2.3.3) accomodate terms of the same forms, with equal number of coefficients.

Effecting the renormalization conditions that will negate the divergences arising from gravitation (2.3.2) and from self interactions alike, the renormalized coupling constants will be given as [3,20],

$$\begin{aligned}
&\Lambda = \left(L + \overline{L}^{(1)}\right)\Big|_{\varphi_0, R=0} \qquad \xi = \left.\frac{\partial^3\left(L + \overline{L}^{(1)}\right)}{\partial\phi^2\partial R}\right| \qquad \varepsilon_2 = \left.\frac{\partial\left(L + \overline{L}^{(1)}\right)}{\partial \mathrm{C}^2}\right|_{\phi=0, R_6} \\
&\lambda = \left.\frac{\partial^4\left(L + \overline{L}^{(1)}\right)}{\partial\phi^4}\right|_{\varphi_1, R=0} \qquad \varepsilon_0 = \left.\frac{\partial\left(L + \overline{L}^{(1)}\right)}{\partial R}\right|_{\phi=0, R=0} \qquad \varepsilon_3 = \left.\frac{\partial\left(L + \overline{L}^{(1)}\right)}{\partial \mathrm{G}}\right|_{\phi=0, R_7} \\
&m^2 = \left.\frac{\partial^2\left(L + \overline{L}^{(1)}\right)}{\partial\phi^2}\right|_{\phi=0, R=0} \qquad \varepsilon_1 = \left.\frac{\partial^2\left(L + \overline{L}^{(1)}\right)}{\partial R^2}\right|_{\phi=0, R_5} \qquad \varepsilon_4 = \left.\frac{\partial\left(L + \overline{L}^{(1)}\right)}{\partial \mathrm{W}R}\right|_{\phi=0, R_8}
\end{aligned} \qquad (2.3.6)$$

where we understand $\overline{L}^{(1)}$ as the integrand of (2.2.40), that's the Lagrangian including the first order contributions. Here, the renormalized coupling constants have been defined at energy scales which may be in general all different. However, how these constants behave under different scales are determined with their respective renormalization group equations, whose concern is beyond our study. Therefore the corresponding counterterms will be found as:

$$\begin{aligned}
m^2 = m^2 + \delta m^2 + \frac{1}{64\pi^2}\frac{\partial^2}{\partial\phi^2}\left\{M^4\left(\log\frac{M^2}{\mu^2} - \frac{3}{2}\right)\right\}_{\phi=0, R=0} \qquad \left(\frac{\partial a_2}{\partial\phi} = 0\right) \\
\rightarrow 64\pi^2\delta m^2 = 2m^2\lambda\left(1 - \log\frac{m^2}{\mu^2}\right)
\end{aligned} \qquad (2.3.7)$$

$$\lambda = \lambda + \delta\lambda + \frac{1}{64\pi^2}\frac{\partial^4}{\partial\phi^4}\left\{M^4\left(\log\frac{M^2}{\mu^2} - \frac{3}{2}\right)\right\}_{\varphi_1, R=0}$$

$$\rightarrow 64\pi^2\delta\lambda = -6\lambda^2 \log\frac{M_1^2}{\mu^2} - \frac{12\lambda^3\phi_1^2}{M_1^2} + \frac{2\lambda^4\phi_1^4}{M_1^4} \tag{2.3.8}$$

$$\xi = \xi + \delta\xi + \frac{\partial^3}{\partial R\partial^2\phi}\left\{M^4\left(\log\frac{M^2}{\mu^2} - \frac{3}{2}\right)\right\}_{\varphi_3, R_3}$$

$$\rightarrow 64\pi^2\delta\xi = -2\left(\xi - \tfrac{1}{6}\right)\lambda \log\frac{M_3^2}{\mu^2} - \frac{2\lambda^2\hat{\xi}\phi_3^2}{M_3^2} \tag{2.3.9}$$

$$\varepsilon_0 = \varepsilon_0 + \delta\varepsilon_0 + \frac{1}{64\pi^2}\frac{\partial}{\partial R}\left\{M^4\left(\log\frac{M^2}{\mu^2} - \frac{3}{2}\right)\right\}_{\phi=0, R=0}$$

$$\rightarrow 64\pi^2\delta\varepsilon_0 = 2m^2\left(\xi - \tfrac{1}{6}\right)\left(1 - \log\frac{m^2}{\mu^2}\right) \tag{2.3.10}$$

$$\varepsilon_1 = \varepsilon_1 + \delta\varepsilon_1 + \frac{1}{64\pi^2}\frac{\partial^2}{\partial R^2}\left\{M^4\left(\log\frac{M^2}{\mu^2} - \frac{3}{2}\right)\right\}_{\phi=0, R_5}$$

$$\rightarrow 64\pi^2\delta\varepsilon_1 = -2\left(\xi - \tfrac{1}{6}\right)^2 \log\frac{\bar{M}_5^2}{\mu^2} \tag{2.3.11}$$

meanwhile, obtaining an expression for $\bar{a}_2$ in terms of $\mathbf{C}^2$ and $\mathbf{G}$ as,

$$\bar{a}_2 = \tfrac{1}{6}\left(\xi - \tfrac{1}{5}\right)\Box R + \tfrac{1}{360}\left(\mathbf{C}^2 - \mathbf{G}\right)$$

remaining renormalization conditions will read,

$$\varepsilon_2 = \varepsilon_2 + \delta\varepsilon_2 + \frac{1}{32\pi^2}\frac{\partial}{\partial \mathrm{C}^2}\left\{\bar{a}_2 \log\tfrac{M^2}{\mu^2}\right\}_{\phi=0, R_6}$$

$$\rightarrow 64\pi^2\delta\varepsilon_2 = -\tfrac{1}{60}\log\frac{\bar{M}_6^2}{\mu^2} \tag{2.3.12}$$

$$\varepsilon_3 = \varepsilon_3 + \delta\varepsilon_3 + \frac{1}{32\pi^2}\frac{\partial}{\partial \mathrm{G}}\left\{\bar{a}_2 \log\tfrac{M^2}{\mu^2}\right\}_{\phi=0, R_7}$$

$$\rightarrow 64\pi^2\delta\varepsilon_3 = \tfrac{1}{180}\log\frac{\bar{M}_7^2}{\mu^2} \tag{2.3.13}$$

$$\varepsilon_4 = \varepsilon_4 + \delta\varepsilon_4 + \frac{1}{32\pi^2}\frac{\partial}{\partial \Box R}\left\{\bar{a}_2 \log\tfrac{M^2}{\mu^2}\right\}_{\phi=0, R_8}$$

$$\rightarrow 64\pi^2\delta\varepsilon_4 = \tfrac{1}{3}\left(\xi - \tfrac{1}{6}\right)\log\frac{\bar{M}_8^2}{\mu^2} \tag{2.3.14}$$

and using (2.3.7) with (2.3.8), the final counterterm given by the renormalization condition of the cosmological constant will be (with $\Lambda = 0$),

$$0 = \delta\Lambda + \tfrac{1}{2}\left(m^2 + \delta m^2\right)\varphi_0^2 + \tfrac{1}{24}\left(\lambda + \delta\lambda\right)\varphi_0^4 + \tfrac{1}{64\pi^2}M_0^4\left(\log\frac{M_0^2}{\mu^2} - \frac{3}{2}\right)$$

$$\begin{aligned}\rightarrow 64\pi^2\delta\Lambda &= \lambda m^2\varphi_0^2\log\frac{m^2}{\mu^2} - 64\pi^2\frac{m^2\varphi_0^2}{2} - 64\pi^2\frac{\lambda\varphi_0^4}{24} \\ &- \lambda m^2\varphi_0^2 + \frac{\lambda^2\varphi_0^4}{4}\log\frac{\tilde{M}_1^2}{\mu^2} - \tilde{M}_0^4\log\frac{\tilde{M}_0^2}{\mu^2} + \frac{3\tilde{M}_0^4}{2} - \frac{\lambda^4\varphi_0^4\varphi_1^4}{12\tilde{M}_1^4} + \frac{\lambda^3\varphi_0^4\varphi_1^2}{2\tilde{M}_1^2}\end{aligned} \tag{2.3.15}$$

where throughout these calculations we've shortened our notation defining,

$$\begin{gathered}M_i^2 = m^2 + \tfrac{1}{2}\lambda\phi_i^2, \quad \bar{M}_i^2 = m^2 + \left(\xi - \tfrac{1}{6}\right)R_i, \\ \tilde{M}_i^2 = m^2 + \left(\xi - \tfrac{1}{6}\right)R_i + \tfrac{1}{2}\lambda\phi_i^2\end{gathered} \quad . \tag{2.3.16}$$

Now we can write the renormalized effective action. However, originating from a former argument, during all these calculations we've been using a yet undefinite constant μ^2. Fixing its value at an arbitrary value, say M^2 [3,20], the renormalized effective Lagrangian will be,

$$L_{eff,r}^{(1)} = L_r + \bar{L}_r^{(1)} \tag{2.3.17}$$

the first term comprising renormalized interaction terms, and the other signifying the renormalized first order contributions including the counter terms. From the renormalization results, at the energy scale M^2 we find for the second term,

$$
\begin{aligned}
64\pi^2\bar{L}_r^{(1)} = {} & \frac{\mathbf{C}^2}{60}\log\frac{M^2}{\bar{M}_6^2} - \frac{\mathbf{G}}{180}\log\frac{M^2}{\bar{M}_7^2} - \left(\xi - \tfrac{1}{5}\right)\frac{\mathbf{W}R}{3}\log\frac{M^2}{\bar{M}_8^2} - 32\pi^2 m^2\varphi_0^2 \\
& - \frac{8\pi^2}{3}\lambda\varphi_0^4 + m^4\log\frac{M^2}{M_0^2} + \lambda m^2\varphi_0^2\left(-\log\frac{M^2}{m^2} + \log\frac{M^2}{M_0^2} + \frac{3}{2}\right) \\
& + 2\hat{\xi}Rm^2\left(\log\frac{M^2}{m^2} - \frac{1}{2}\right) + \hat{\xi}^2R^2\left(\log\frac{M^2}{\bar{M}_5^2} - \frac{3}{2}\right) \\
& + \frac{\lambda^2\varphi_0^4}{4}\left[\log\frac{M_0^2}{M_1^2} - \frac{3}{2} + \frac{\lambda^2\varphi_1^4}{3M_1^4} - \frac{2\lambda\varphi_1^2}{M_1^2}\right] \\
& + \lambda\phi^2\left\{m^2\left(1 + \log\frac{M^2}{m^2} - \frac{3}{2}\right) + \hat{\xi}R\left(\log\frac{M^2}{M_3^2} - \frac{\lambda\varphi_3^2}{M_3^2} - \frac{3}{2}\right)\right\} \\
& + \frac{\lambda^2\phi^4}{4}\left[\log\frac{M^2}{M_1^2} - \frac{3}{2} - \frac{\lambda^2\varphi_1^4}{3M_1^4} - \frac{2\lambda\varphi_1^2}{M_1^2}\right]
\end{aligned}
\qquad . \qquad (2.3.18)
$$

In order to simplify this lengthy expression, we might effect some simplifications we have foretold during the introduction part. One of these is to take the mean field very small and thereby expand certain terms as,

$$
\log\frac{M^2}{m^2} = \log\frac{\left(m^2 + \frac{\lambda}{2}\phi^2 + \hat{\xi}R\right)}{m^2} \cong \log\left(1 + \frac{\hat{\xi}R}{m^2}\right) + \frac{1}{2}\frac{\lambda}{m^2 + \hat{\xi}R}\phi^2
$$

which enables us to rewrite (2.3.18) as,

$$
\begin{aligned}
64\pi^2\bar{L}_r^{(1)} \cong {} & \Lambda_{eff} + \frac{\lambda\phi^2}{m^2 + \hat{\xi}R}\left(\frac{\mathbf{C}^2}{120} - \frac{\mathbf{G}}{360} - \left(\xi - \tfrac{1}{5}\right)\frac{\mathbf{W}R}{6} + \frac{m^4}{2} + m^2\hat{\xi}R + \frac{\hat{\xi}^2R^2}{2}\right) \\
& + \lambda\phi^2\left\{m^2\left[\log\left(1 + \frac{\hat{\xi}R}{m^2}\right) - \frac{1}{2}\right] + \hat{\xi}R\left(\log\frac{m^2 + \hat{\xi}R}{M_3^2} - \frac{\lambda\varphi_3^2}{M_3^2} - \frac{3}{2}\right)\right\} + O\left(\phi^4\right)
\end{aligned}
\qquad (2.3.19)
$$

with Λ_{eff} being a complicated function of curvature terms alone. If we rearrange this expression as,

$$
\begin{aligned}
\bar{L}_r^{(1)} \cong {} & \Lambda'_{eff} \\
& + \frac{\lambda\phi^2}{64\pi^2}\left\{\begin{aligned} & m^2\log\left(1 + \frac{\hat{\xi}R}{m^2}\right) + \hat{\xi}R\left(\log\frac{m^2 + \hat{\xi}R}{M_3^2} - \frac{\lambda\varphi_3^2}{M_3^2} - 1\right) \\ & - \frac{1}{12\left(m^2 + \hat{\xi}R\right)}\left[-\frac{\mathbf{C}^2}{10} + \frac{\mathbf{G}}{30} + 2\left(\xi - \tfrac{1}{5}\right)\mathbf{W}R\right] \end{aligned}\right\} + O\left(\phi^4\right)
\end{aligned}
\qquad (2.3.20)
$$

whereas we see, a mass term appears. A further simplification is even possible, that's to take the curvature small. Expanding certain terms as ,

$$\log\left(1+\frac{\hat{\xi}R}{m^2}\right)\cong\frac{\hat{\xi}}{m^2}R-\frac{\hat{\xi}^2}{2m^4}R^2,\quad \frac{1}{\left(m^2+\hat{\xi}R\right)}\cong\frac{1}{m^2}-\frac{\hat{\xi}R}{m^4}+\frac{\hat{\xi}^2R^2}{m^6}$$

we finally get,

$$\overline{L}_r^{(1)}\cong\Lambda'_{eff}+\frac{\lambda\phi^2}{64\pi^2}\left\{\begin{array}{l}\left(\xi-\frac{1}{6}\right)R\left(\log\frac{m^2}{M_3^2}-\frac{\lambda\varphi_3^2}{M_3^2}\right)+\frac{\left(\xi-\frac{1}{6}\right)^2R^2}{2m^2}\\ +\frac{\mathbf{C}^2}{120m^2}-\frac{\mathbf{G}}{360m^2}-\frac{\left(\xi-\frac{1}{5}\right)\square R}{6m^2}\end{array}\right\}. \qquad (2.3.21)$$

Therefore the first order loop renormalized effective Lagangian will read as,

$$L_r^{(1)}(\phi)\cong\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}m_{eff}^2\phi^2-\frac{1}{24}\lambda\phi^4+\overline{L}_r^{(1)} \qquad (2.3.22)$$

where from (2.3.21) the effective mass term is found to be as,

$$m_{eff}^2=m^2+\xi R+\frac{\lambda}{32\pi^2}\left\{\begin{array}{l}\left(\xi-\frac{1}{6}\right)R\left(\log\frac{m^2}{M_3^2}-\frac{\lambda\varphi_3^2}{M_3^2}\right)+\frac{\left(\xi-\frac{1}{6}\right)^2R^2}{2m^2}\\ +\frac{\mathbf{C}^2}{120m^2}-\frac{\mathbf{G}}{360m^2}-\frac{\left(\xi-\frac{1}{5}\right)\square R}{6m^2}\end{array}\right\}. \qquad (2.3.23)$$

This expression will be the leading one while we examine the physical implications of the effective Lagrangian for a symmetry breaking.

3. FINITE TEMPERATURE STEquation Chapter (Next) Section 1UDY

Since the early universe is mostly radiation dominated [1,2], thermal effects over the symmetry breaking might be rather important [21]. Thereby in this chapter we'll try to discern this thermal effects, and in this sense we'll derive an expression for the thermal effective action valid for both high temperature and low temperature limits. Similar to the previous chapter we'll be undertaking a number of assumptions, the most relevant one being the small distance behavior. As we have discussed, this technique is in no position to determine the global behavior. On the other hand though, as it transforms an arbitrary space to a conformally invariant space time, while considering the thermal effects this manipulation is rather indispensable, since it enables one to use the periodic boundary conditions over the imaginary time which's only valid for conformally invariant space times [22].

The plan of study for this chapter is similar to the zero temperature study. First we'll try to derive the thermal effective action. Then we'll be probing certain temperature limits in order to render the expressions in a closed form and try to discern the thermal effective mass.

3.1 Calculation of the One-Loop Thermal Effective Potantial

Moving onward to four dimensional theory and from (1.17) we will write the relevant classical action as,

$$S_{\beta}\left[\phi\right]=\int_{0}^{\beta}d\tau\int\sqrt{-g_{[3]}}d^{3}x\left[\phi\,\tfrac{1}{2}\left(\tfrac{\partial^{2}}{\partial\tau^{2}}+\Delta-m^{2}-\xi R_{[3]}-V\left(\phi\right)+i\alpha\right)\phi\right] \qquad (3.1.1)$$

where, the lower indices in square brackets over certain objects represent their respective values in the three dimensional space. As we did in the previous chapter, we again introduce an $i\alpha$ parameter; however in contrast with the zero temperature

approach we will use it for computational purposes only and later on we will ignore it[4]. In retrospect of the discussion we overtook in the first chapter and regarding the trasformation (1.15), the thermal variant of (2.1.8) becomes,

$$\exp\left[-\overline{\Gamma}_{\beta}^{(1)}[\phi]\right] = \int D\varphi \exp\left[-\tfrac{1}{2}\varphi \tfrac{\delta^2 S_\beta[\phi]}{\delta\phi^2}\varphi\right] \tag{3.1.2}$$

which allows us to formally express the finite temperature first order contribution for the effective action as,

$$\overline{\Gamma}_{\beta}^{(1)} = \mathrm{Det}^{-\frac{1}{2}}\left(\frac{\delta^2 S_\beta}{\delta\phi^2}\right) \tag{3.1.3}$$

or,

$$\overline{\Gamma}_{\beta}^{(1)}[\phi] = -\frac{1}{2}\zeta_A'(0) \tag{3.1.4}$$

where we have redefined *A*,

$$A = \frac{\delta^2 S_\beta[\phi]}{\delta\phi^2} = \tfrac{\partial^2}{\partial\tau^2} + H_{[3]}, \quad H_{[3]} = \Delta - m^2 - \xi R_{[3]} - V''(\phi) \tag{3.1.5}$$

as an operator sum of temperature and three-dimensional spatial parts. Meanwhile we might express the thermal effective as,

$$\Gamma_{\beta}^{(1)} = S_\beta + \overline{\Gamma}_{\beta}^{(1)} = \int_0^\beta d\tau \int d^3x \sqrt{-g^{[3]}} L_\beta^{(1)}\left(\phi, \partial_\mu\phi\right) \tag{3.1.6}$$

where we understand,

$$L_\beta^{(1)}\left(\phi, \partial_\mu\phi\right) = \tfrac{1}{2}\partial_\mu\phi Z^{\mu\nu}\partial_\nu\phi - V_\beta^{(1)}(\phi) \tag{3.1.7}$$

a compositon of kinetic and static parts, $V_\beta^{(1)}$ being the thermal effective potential to the first order loop. If we assume that the field variables are changing very slowly with respect to the temperature we may neglect the kinetic part in (3.1.6), and effecting the τ integral over the effective potential,

[4] Since, it'll be the only imaginary term present for the whole thermal study, we find this approach fair.

$$-\int_0^{\beta} d\tau \int d^3x \sqrt{-g_{[3]}} V_{\beta}^{(1)}(\phi) = -\beta \int d^3x \sqrt{-g^{[3]}} V_{\beta}^{(1)}(\phi)$$

we find,

$$\int d^3x \sqrt{-g_{[3]}} V_{\beta}^{(1)} \cong -\frac{1}{\beta} \Gamma_{\beta}^{(1)}$$

and for the first order loop effective potential,

$$\int d^3x \sqrt{-g_{[3]}} \overline{V}_{\beta}^{(1)} = \frac{1}{2\beta} \zeta_A'(0). \tag{3.1.8}$$

In this chapter we'll be using the effective potential formalism instead of the effective action, neglecting the kinetic terms. For our consideration, we observe that computation of the effective potential instead of the effective action gives equally well the symmetry breaking condition as an extremum problem as done in (1.11).

Returning back to (3.1.5) we see that we must follow a different path for the calculation of the thermal first loop contribution, since we now have an operator consisting of two distinct parts which satisfy [23; 24, p.90],

$$\left(\frac{\partial^2}{\partial \tau^2} + H_{[3]}\right) \phi_{jn}(x,\tau) = \left[\omega_j^2(x) + \left(\frac{2\pi n}{\beta}\right)^2\right] \phi_{jn}(x,\tau) \tag{3.1.9}$$

the first part in the square brackets being the spatial eigenvalue, while the second part is understood from the periodic boundary conditions imposed over the temperature coordinate. Therefore we shall use instead of the zeta function in (2.2.3),

$$\zeta_A(z) = \sum_{n=-\infty}^{\infty} \sum_j \left[\omega_j^2 + (2\pi n/\beta)^2\right]^{-z} = \sum_{n=-\infty}^{\infty} \zeta\left(z \mid S_{[3]} + [2\pi n/\beta]^2\right). \tag{3.1.10}$$

Note that, we've now changed our notation for the zeta function in the summation in order to discriminate it as a generalized Riemann zeta function which's defined via,

$$\zeta\left(z\,|\,H_{[3]}+\left[2\pi n/\beta\right]^2\right)=\sum_j\frac{1}{\left[\omega_j^2+\left(\frac{2\pi n}{\beta}\right)^2\right]^z}. \tag{3.1.11}$$

Using the Poisson summation formula [25],

$$\sum_{n=-\infty}^{\infty}f(an)=\frac{2}{a}\sum_{n=-\infty}^{\infty}\int_0^{\infty}dyf\left(\tfrac{y}{a}\right)\cos\left(\tfrac{2\pi ny}{a}\right) \tag{3.1.12}$$

which's valid for an even function of *n*, we might refine (3.1.10) as,

$$\zeta_A(z)=\frac{\beta}{\pi}\sum_{n=-\infty}^{\infty}\int_0^{\infty}dy\zeta\left(z\,|\,H_{[3]}+y^2\right)\cos(n\beta y). \tag{3.1.13}$$

Furthermore, regarding the integral representation of the gamma function once more, the generalized zeta function in (3.1.13) might be rewritten as,

$$\zeta\left(z\,|\,H_{[3]}+y^2\right)=\frac{1}{\Gamma(z)}\int_0^{\infty}ids(is)^{z-1}e^{-isy^2}\mathrm{Tr}\left[\exp\left(-isH_{[3]}\right)\right]. \tag{3.1.14}$$

Inserting this expression into (3.1.13), and casting the cosine term as an Euler representation we'll get,

$$\begin{aligned}\zeta_A(z)&=\frac{\beta}{\pi}\frac{1}{\Gamma(z)}\sum_{n=-\infty}^{\infty}\int_0^{\infty}ids(is)^{z-1}\mathrm{Tr}\left[\exp\left(-isH_{[3]}\right)\right]\left\{\int_0^{\infty}dye^{-isy^2}\frac{e^{in\beta y}+e^{-in\beta y}}{2}\right\}\\&=\frac{\beta}{\sqrt{\pi}}\frac{1}{\Gamma(z)}\sum_{n=-\infty}^{\infty}\int_0^{\infty}ids(is)^{z-\frac{3}{2}}\exp\left[-\tfrac{\beta^2n^2}{4is}\right]\mathrm{Tr}\left[\exp\left(-isH_{[3]}\right)\right]\end{aligned} \tag{3.1.15}$$

where we have effected a pair of Gaussian integrations in the last step. To this point, we've followed closely the arguments presented in [23]. However from now on we'll recourse to the arguments we've provided in the previous chapter. In order to so, a comparision with (2.2.5), reveals that the trace part of (3.1.15) might be represented as an integration over a kernel which satisfies an equation of the form,

$$i\frac{\partial K_{[3]}}{\partial s}=H_{[3]}K_{[3]} \tag{3.1.16}$$

subject to the boundary condition,

$$\lim_{s\to o} K_{[3]}(x,x';s) = \sqrt{-g_{[3]}}\,\delta^3(x,x'). \qquad (3.1.17)$$

Therefore, the same arguments of the last section will apply here as well. That's, if we can once discern this kernel, (3.1.15) will readily be calculated as,

$$\zeta_A(z) = \frac{\beta}{\sqrt{\pi}}\frac{1}{\Gamma(z)}\sum_{n=-\infty}^{\infty}\int_0^{\infty} ids\,(is)^{z-\frac{3}{2}}\exp\left[-\tfrac{\beta^2 n^2}{4is}\right]\int d^3x\sqrt{-g_{[3]}}K_{[3]}(x,x;s). \qquad (3.1.18)$$

Comparing this result with (2.2.6), we see that the calculation of the thermal effective action might proceed in a similar manner as that of zero temperature calculations. Indeed, we can even use certain results of the previous section which have been given for general space times at an arbitrary dimension.

First allow us to cast the respective propagator equation of (2.1.14) for a three dimensional space,

$$\left(H_{[3]} - i\alpha\right)G_{[3]}(x,x') = \delta^3(x-x')\sqrt{-g_{[3]}}\,. \qquad (3.1.19)$$

Next we'll propose the solution of (3.1.19) similar to (2.2.29)[5],

$$G_{[3]} = i\int_0^{\infty} K_{[3]}e^{-\alpha s} \qquad (3.1.20)$$

and using (2.2.33) we get for the thermal kernel of (3.1.18),

$$K_{[3]}(x,x;s) = \frac{1}{(4\pi is)^{\frac{3}{2}}}\exp\left[-isM^2 - is\left(\xi-\tfrac{1}{6}\right)R_{[3]}\right]\bar{F}_{[3]}(x,x;is) \qquad (3.1.21)$$

where as usual we have defined,

$$M^2 = m^2 + V''(\phi)$$

and,

$$\bar{F}_{[3]}(x,x;is) = \sum_{j=0}(is)^j\,\bar{a}_j^{[3]}(x,x) \qquad (3.1.22)$$

[5] It might also be checked that, this porposition satisfies (3.1.19) using (3.1.16).

with the three-dimensional heat coefficients the same as that of (2.2.34) and (2.2.35), except that all of the geometric expressions are to be calculated at a three dimensional space. Also we shall note that the definition of heat coefficients of (3.1.21) are valid only for a static space, that's for a space whose metric is independent of time (temperature). We'll come back to the implications of this specifications later while discussing the symmetry breaking at the next chapter.

Before inserting the solution (3.1.21) into (3.1.18), we might distinguish the $n = 0$ term in the summation as it will give the zero temperature contibution to the effective Lagrangian, which's merely an extension of the results already calculated at the last chapter. Therefore, if we concentrate to the thermal contributions only (that's the terms except $n = 0$), from (3.1.21) we find,

$$\zeta_A^{\beta}(z) = \frac{\beta}{8\pi^2\Gamma(z)} \sum_{\substack{n=1 \\ j=0}} \int d^3x \sqrt{-g_{[3]}}\, \bar{a}_j^{[3]} \int_0^{\infty} ids\,(is)^{z+j-3} \exp\left[-\frac{\beta^2 n^2}{4is} - is\bar{M}_{[3]}^2 \right]$$

where,

$$\bar{M}_{[3]}^2 \equiv M^2 + \left(\xi - \tfrac{1}{6}\right) R^{[3]}.$$

Now we'll perform a change of variable in the is integration as $is \to is^{-1}$ which produces,

$$\zeta_A^{\beta}(z) = \frac{\beta}{8\pi^2\Gamma(z)} \sum_{\substack{n=1 \\ j=0}} \int d^3x \sqrt{-g_{[3]}}\, \bar{a}_j^{[3]} \int_0^{\infty} ids\,(is)^{1-z-j} \exp\left[-\frac{is\beta^2 n^2}{4} - \frac{\bar{M}^2}{is} \right]$$

and then expanding the component of the exponential inversely proportional to is,

$$\zeta_A^{\beta}(z) = \frac{\beta}{8\pi^2\Gamma(z)} \sum_{\substack{n=1 \\ j=0,k=0}} \int d^3x \sqrt{-g_{[3]}}\, \bar{a}_j^{[3]} \frac{(-1)^k \bar{M}^{2k}}{k!} \int_0^{\infty} ids\,(is)^{1-z-j-k} \exp\left[-\frac{is\beta^2 n^2}{4} \right]$$

where we have dropped the dimensional indicative from $\bar{M}^2$ for brevity. Next we'll manage the is integration,

$$\zeta_A^{\beta}(z) = \frac{\beta}{8\pi^2\Gamma(z)} \sum_{\substack{n=1 \\ j=0,k=0}} \int d^3x \sqrt{-g_{[3]}}\, \overline{a}_j^{[3]} \frac{(-1)^k M^{2k}}{k!} \frac{\Gamma(2-z-j-k)}{\left(\frac{\beta^2 n^2}{4}\right)^{2-z-j-k}}.$$

At this point we'll undertake a simplification, forbidding terms exceeding the second derivative of the three- space metric, that's $a_j \approx 0$ for $j > 2$. This allows us,

$$\zeta_A^{\beta}(z) \cong \frac{\beta}{8\pi^2\Gamma(z)} \sum_{\substack{n=1 \\ k=0}} \int d^3x \sqrt{-g_{[3]}} \frac{(-1)^k M^{2k}}{k!} \left\{ \begin{array}{l} \Gamma(2-z-k)\left(\frac{\beta n}{2}\right)^{2z+2k-4} \\ +\Gamma(-z-k)\,\overline{a}_2^{[3]}\left(\frac{\beta n}{2}\right)^{2z+2k} \end{array} \right\}$$

whereby, using the recurrence relations for the Gamma functions we arrive at,

$$\begin{aligned} \zeta_A^{\beta}(z) = \frac{\beta}{8\pi^2} \sum_{\substack{n=1 \\ k=0}} \int d^3x \sqrt{-g_{[3]}} \frac{(-1)^k M^{2k}}{k!} \\ \times \left\{ \frac{\left(\frac{\beta n}{2}\right)^{2z+2k-4}(-z-k)(1-z-k) + \overline{a}_2^{[3]}\left(\frac{\beta n}{2}\right)^{2z+2k}}{\prod_{j=1}^{2z+k}(z-j)} \right\} \end{aligned} \qquad (3.1.23)$$

and differentiation of this expression with respect to z will give,

$$\begin{aligned} \zeta_A'^{\beta}(z) = \frac{\beta}{8\pi^2} \sum_{n=1,k=0} \int d^3x \sqrt{-g_{[3]}} \frac{(-1)^k M^{2k}}{k!} \\ \times \left\{ \begin{array}{l} \dfrac{\left(\frac{\beta n}{2}\right)^{2z+2k-4}\left[2(z+k)(z+k-1)\log\frac{\beta n\mu}{2} + (2z+2k-1)\right] + 2\overline{a}_2^{[3]}\left(\frac{\beta n}{2}\right)^{2z+2k}\log\frac{\beta n\mu}{2}}{\prod_{j=1}^{2z+k}(z-j)} \\ \quad - \dfrac{\left[\frac{d}{dz}\prod_{j=1}^{2z+k}(z-j)\right]\left[\left(\frac{\beta n}{2}\right)^{2z+2k-4}(z+k)(z+k-1) + \overline{a}_2^{[3]}\left(\frac{\beta n}{2}\right)^{2z+2k}\right]}{\left[\prod_{j=1}^{2z+k}(z-j)\right]^2} \end{array} \right\} \end{aligned}$$

μ being an arbitrary term having mass dimension. Using the relations,

$$\lim_{z\to 0}\left[\prod_{j=1}^{2z+k}(z-j)\right] = \lim_{z\to 0}\left[(-1)^{2z+k}(2z+k)!\right] = (-1)^k\, k!$$

and from Gradstheyn&Rhyzik [12, eq.8.365],

$$\lim_{z\to 0}\left\{\frac{d}{dz}\left[\prod_{j=1}^{2z+k}(z-j)\right]\right\}=\lim_{z\to 0}\left\{\frac{d}{dz}\left[z(-1)^{2z+k-1}(2z+k)!\sum_{j=1}^{2z+k}\tfrac{1}{j}\right]\right\},\quad z'=2z+k$$

$$=\lim_{z'\to k}\left\{(-1)^{z'-1}z'!\sum_{j=1}^{z'}\tfrac{1}{j}+\left(\tfrac{z'-k}{2}\right)(-1)^{z'-1}\left[\left(2\tfrac{d}{dz'}z'!\right)\sum_{j=1}^{z'}\tfrac{1}{j}+z'!\left(2\tfrac{d}{dz'}\sum_{j=1}^{z'}\tfrac{1}{j}\right)\right]\right\}$$

$$=(-1)^{k-1}k!\sum_{j=1}^{k}\tfrac{1}{j}(-1)^{k-1}$$

we finally reach,

$$\zeta_A'^{\beta}(0)=\frac{\beta}{8\pi^2}\sum_{n=1,k=0}\int d^3x\sqrt{-g_{[3]}}\frac{(-1)^k M^{2k}}{k!}$$
$$\times\left\{\begin{array}{l}\dfrac{\left(\frac{\beta n}{2}\right)^{2k-4}\left[2k(k-1)\log\frac{\beta n\mu}{2}+2k-1\right]+2\bar{a}_2^{[3]}\left(\frac{\beta n}{2}\right)^{2k}\log\frac{\beta n\mu}{2}}{(-1)^k k!}\\ \qquad+\dfrac{(-1)^k\sum_{j=1}^{k}\frac{1}{j}\left[\left(\frac{\beta n}{2}\right)^{2k-4}k(k-1)+\bar{a}_2^{[3]}\left(\frac{\beta n}{2}\right)^{2k}\right]}{k!}\end{array}\right\}. \qquad (3.1.24)$$

We'll be taking (3.1.24) as our final expression for the derivative of the thermal zeta function in a general sense, though for further calculations we'll effect a couple of manipulations. First, we'll translate the summation over *k* by minus two,

$$\zeta_A'^{\beta}(0)=\frac{\beta}{8\pi^2}\sum_{n=1,k=-2}\int d^3x\sqrt{-g_{[3]}}\frac{M^{2k+4}}{(k+2)!}$$
$$\times\left\{\begin{array}{l}\dfrac{\left(\frac{\beta n}{2}\right)^{2k}\left[2(k+2)(k+1)\log\frac{\beta n\mu}{2}+2k+3\right]+2\bar{a}_2^{[3]}\left(\frac{\beta n}{2}\right)^{2k+4}\log\frac{\beta n\mu}{2}}{(k+2)!}\\ \qquad+\dfrac{\left[\sum_{j=1}^{k+2}\frac{1}{j}\right]\left[\left(\frac{\beta n}{2}\right)^{2k}(k+1)(k+2)+\bar{a}_2^{[3]}\left(\frac{\beta n}{2}\right)^{2k+4}\right]}{(k+2)!}\end{array}\right\}$$

or rearranging,

$$\zeta_A'^{\beta}(0)=\frac{\beta}{8\pi^2}\sum_{\substack{n=1\\k=0}}\int d^3x\sqrt{-g_{[3]}}\,\frac{M^{2k+4}}{(k+2)!}$$
$$\times\left\{\begin{array}{l}\dfrac{\left(\frac{\beta n}{2}\right)^{2k}\left[2(k+2)(k+1)\log\frac{\beta n\mu}{2}+2k+3\right]+2\bar{a}_2^{[3]}\left(\frac{\beta n}{2}\right)^{2k+4}\log\frac{\beta n\mu}{2}}{(k+2)!}\\ \qquad+\dfrac{\left[\sum_{j=1}^{k+2}\frac{1}{j}\right]\left[\left(\frac{\beta n}{2}\right)^{2k}(k+1)(k+2)+\bar{a}_2^{[3]}\left(\frac{\beta n}{2}\right)^{2k+4}\right]}{(k+2)!}\end{array}\right\}\quad .\qquad (3.1.25)$$
$$+\frac{\beta}{8\pi^2}\sum_{n=1}^{\infty}\int d^3x\sqrt{-g^{[3]}}\left\{\begin{array}{l}\left[-\left(\frac{\beta n}{2}\right)^{-4}+2\bar{a}_2^{[3]}\log\frac{\beta n\mu}{2}\right]+\\ M^2\left[\left(\frac{\beta n}{2}\right)^{-2}+2\bar{a}_2^{[3]}\left(\frac{\beta n}{2}\right)^{2}\log\frac{\beta n\mu}{2}+\bar{a}_2^{[3]}\left(\frac{\beta n}{2}\right)^{2}\right]\end{array}\right\}$$

Next we'll carry out the *k* summations, however before engaging with these computations, we find it convenient to deal with the zero-temperature contribution to the effective action for consistence.

Reading the $n = 0$ term from (3.1.18) and using (3.1.21) we'll have,

$$\zeta_A^0(z)=\frac{\beta}{8\pi^2\Gamma(z)}\sum_{j=0}\int d^3x\sqrt{-g_{[3]}}\,\bar{a}_j^{[3]}\int_0^{\infty}ids\,(is)^{z+j-3}e^{-isM^2}$$
$$=\frac{\beta}{8\pi^2\Gamma(z)}\sum_{j=0}\int d^3x\sqrt{-g_{[3]}}\,\bar{a}_j^{[3]}\,\frac{\Gamma(z+j-2)}{M^{2z+2j-4}}\quad .$$

Comparing this with (2.2.40), we see that the integrals are same except the dimensional difference. Hence we'll find,

$$\zeta_A'^{0}(0)\cong-\frac{\beta}{8\pi^2}\int d^3x\sqrt{-g_{[3]}}\left\{\frac{M^4}{2}\left(\log\frac{M^2}{\mu^2}-\frac{3}{2}\right)+\bar{a}_2^{[3]}\log\frac{M^2}{\mu^2}\right\}.\qquad (3.1.26)$$

Therefore the overall first order contribution for the effective potential will be,

$$\bar{V}_{\beta}^{(1)}=\frac{1}{2\beta}\left(\zeta_A'^{0}(0)+\zeta_A'^{\beta}(0)\right)\qquad (3.1.27)$$

where the zero temperature and thermal contributions are given by (3.1.25) and (3.1.26) respectively. However we're not complete with the thermal contributions, the *k* summations in (3.1.25) should yet to be performed. In order to so we shall

resort to the series representations of the modified Bessel functions [12, eq.8.445-446],

$$I_n(x) = \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n} \tag{3.1.28}$$

$$\begin{aligned} K_n(x) = (-1)^{n+1} I_n(x) \ln \tfrac{xe^C}{2} + \tfrac{1}{2}(-1)^n \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n} \left(\sum_{l=1}^{k} \tfrac{1}{l} + \sum_{l=1}^{k+n} \tfrac{1}{l}\right) \\ + \tfrac{1}{2}\sum_{l=0}^{n-1} \frac{(-1)^l (n-l-1)!}{l!}\left(\frac{x}{2}\right)^{2l-n} \end{aligned} \tag{3.1.29}$$

C being the Euler's constant; on the other hand we shall note the recurrence relations [12, eq.8.486],

$$\begin{gathered} I_{n-1}(x) - I_{n+1}(x) = \tfrac{2n}{x} I_n(x), \qquad I_{n-1}(x) + I_{n+1}(x) = 2I_n'(x) \\ I_n(x) = I_{-n}(x) \end{gathered} . \tag{3.1.30}$$

Denoting each distinct part of the k summation in (3.1.25) with z_i's, i = 1,2,3,4,5 and a defining a new variable $u = \beta M$, a straightforward calculation gives,

$$\begin{aligned} z_1 &= \frac{\beta}{8\pi^2} \sum_{\substack{n=1 \\ k=0}} \int d^3x \sqrt{-g_{[3]}} \frac{M^{2k+4}}{(k+2)!} \left\{ 2 \frac{\left(\frac{\beta n}{2}\right)^{2k}}{(k)!} \log \tfrac{\beta n \mu}{2} \right\} \\ &= \frac{\beta}{8\pi^2} \sum_{n=1} \int d^3x \sqrt{-g_{[3]}} \frac{8M^4}{(un)^2} \log \tfrac{un\mu}{2M} I_2(un) \end{aligned}$$

$$\begin{aligned} z_2 &= \frac{\beta}{8\pi^2} \sum_{\substack{n=1 \\ k=0}} \int d^3x \sqrt{-g_{[3]}} \frac{M^{2k+4}}{(k+2)!} \left\{ \frac{\left(\frac{\beta n}{2}\right)^{2k} (2k+3)}{(k+2)!} \right\} \\ &= \frac{\beta}{8\pi^2} \sum_{n=1} \int d^3x \sqrt{-g_{[3]}} \frac{4M^4}{(un)^2} \left\{ \sum_{k=0}^{\infty} \frac{1}{(k+2)!(k+2)!} \left[\left(\tfrac{un}{2}\right)^{-1} \tfrac{\partial}{\partial(un/2)} \left(\tfrac{un}{2}\right)^{2k+4} - \left(\tfrac{un}{2}\right)^{2k+2} \right] \right\} \\ &= \frac{\beta}{8\pi^2} \sum_{n=1} \int d^3x \sqrt{-g_{[3]}} \frac{4M^4}{(un)^2} \left\{ 2\left(\tfrac{un}{2}\right)^{-1} I_1(un) - \left(\tfrac{un}{2}\right)^{-2} I_0(un) + \left(\tfrac{un}{2}\right)^{-2} - 1 \right\} \end{aligned}$$

$$z_3 = \frac{\beta}{8\pi^2}\sum_{\substack{n=1\\k=0}}\int d^3x\sqrt{-g_{[3]}}\,\frac{M^{2k+4}}{(k+2)!}\left\{\frac{2\bar{a}_2^{[3]}\left(\frac{\beta n}{2}\right)^{2k+4}\log\frac{\beta n\mu}{2}}{(k+2)!}\right\}$$

$$= \frac{\beta}{8\pi^2}\sum_{n=1}\int d^3x\sqrt{-g_{[3]}}\,2\bar{a}_2^{[3]}\log\frac{un\mu}{M}\left\{\sum_{k=2}^{\infty}\frac{\left(\frac{un}{2}\right)^{2k}}{k!k!}\right\}$$

$$= \frac{\beta}{8\pi^2}\sum_{n=1}\int d^3x\sqrt{-g_{[3]}}\,2\bar{a}_2^{[3]}\log\frac{un\mu}{2M}\left\{\sum_{k=0}^{\infty}\frac{\left(\frac{un}{2}\right)^{2k}}{k!k!}-1-\left(\frac{un}{2}\right)^2\right\}$$

$$z_4 = \frac{\beta}{8\pi^2}\sum_{\substack{n=1,\\k=0}}\int d^3x\sqrt{-g_{[3]}}\,\frac{M^{2k+4}}{(k+2)!}\left\{\left[\sum_{j=1}^{k+2}\frac{1}{j}\right]\frac{\left[\left(\frac{\beta n}{2}\right)^{2k}(k+1)(k+2)\right]}{(k+2)!}\right\}$$

$$= \frac{\beta}{8\pi^2}\sum_{n=1}\int d^3x\sqrt{-g_{[3]}}\,\frac{4M^4}{(un)^2}\left\{\sum_{k=0}^{\infty}\left[\frac{\left(\frac{un}{2}\right)^{2k+2}}{(k)!(k+2)!}\left(\sum_{j=1}^{k+2}\frac{1}{j}\right)\right]\right\}$$

$$= \frac{\beta}{8\pi^2}\sum_{n=1}\int d^3x\sqrt{-g_{[3]}}\,\frac{4M^4}{(un)^2}\left\{K_2(un)+I_2(un)\ln\frac{une^C}{2}+\frac{2}{un}I_1(un)-\frac{2}{(un)^2}I_0(un)\right\}$$

the conversion to the Bessel functions in the last step is conceived from (3.1.29) as,

$$K_2(x) = -I_2(x)\ln\frac{xe^C}{2}+\frac{1}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2k+2}}{k!(k+2)!}\left(2\sum_{j=1}^{k+2}\frac{1}{j}-\frac{1}{k+2}-\frac{1}{k+1}\right)+\frac{1}{2}\left[\left(\frac{x}{2}\right)^{-2}-1\right]$$

$$\rightarrow \sum_{k=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2k+2}}{k!(k+2)!}\left(\sum_{j=1}^{k+2}\frac{1}{j}\right) = K_2(x)+I_2(x)\ln\frac{xe^C}{2}+\frac{1}{2}\left[1-\frac{4}{x^2}\right]+\frac{1}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2k+2}}{k!(k+2)!}\left(\frac{1}{k+2}+\frac{1}{k+1}\right)$$

$$= (\ldots)+\frac{1}{2}\sum_{k=2}^{\infty}\frac{(k-1)\left(\frac{x}{2}\right)^{2k-2}}{(k)!(k)!}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{\left(\frac{x}{2}\right)^{2k}}{(k)!(k+1)!}$$

$$= (\ldots)+\frac{1}{2}\left(\frac{x}{2}\right)\frac{1}{2}\frac{\partial}{\partial(x/2)}\sum_{k=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2k-2}}{(k)!(k)!}+\frac{1}{2}\left(\frac{x}{2}\right)^{-1}\sum_{k=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2k+1}}{(k)!(k+1)!}+\frac{1}{2}\left[\frac{4}{x^2}-1\right]$$

$$\rightarrow \sum_{k=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2k+2}}{k!(k+2)!}\left(\sum_{j=1}^{k+2}\frac{1}{j}\right) = K_2(x)+I_2(x)\ln\frac{xe^C}{2}+\frac{2}{x}I_1(x)-\frac{2}{x^2}I_0(x) \qquad (3.1.31)$$

where extensive use of (3.1.30) has been made; finally for the last term,

$$z_5 = \frac{\beta}{8\pi^2}\sum_{\substack{n=1\\k=0}}\int d^3x\sqrt{-g_{[3]}}\,\frac{\bar{M}^{2k+4}}{(k+2)!}\left\{\left[\sum_{j=1}^{k+2}\frac{1}{j}\right]\frac{\bar{a}_2^{[3]}\left(\frac{\beta n}{2}\right)^{2k+4}}{(k+2)!}\right\}$$

$$= \frac{\beta}{8\pi^2}\sum_{n=1}\int d^3x\sqrt{-g^{[3]}}\,\bar{a}_2^{[3]}\left\{\sum_{k=2}^{\infty}\left[\frac{\left(\frac{un}{2}\right)^{2k}}{k!k!}\left(\sum_{j=1}^{k}\frac{1}{j}\right)\right]\right\} \qquad .$$

$$= \frac{\beta}{8\pi^2}\sum_{n=1}\int d^3x\sqrt{-g_{[3]}}\,\bar{a}_2^{[3]}\left\{K_0(un)+I_0(un)\log\frac{une^C}{2}-\left(\frac{un}{2}\right)^2\right\}$$

Putting these pieces together and reexpressing the k independent part of (3.1.25) in terms of u, after simplifications we'll arrive at,

$$\zeta_A'^{\beta}(0) = \frac{\beta}{8\pi^2}\sum_{n=1}\int d^3x\sqrt{-g_{[3]}} \times\left\{\begin{array}{l}\frac{4\bar{M}^4}{(un)^2}\left[K_2(un)+I_2(un)\log\frac{u^3n^3\mu^2e^C}{8\bar{M}^2}+\frac{6}{un}I_1(un)-\frac{6}{(un)^2}I_0(un)\right]\\ +\bar{a}_2^{[3]}\left[K_0(un)+I_0(un)\log\frac{u^3n^3\mu^2e^C}{8\bar{M}^2}\right]\end{array}\right\} \qquad (3.1.32)$$

while for the first loop thermal effective potential,

$$\bar{V}_{\beta}^{(1)} = \frac{1}{16\pi^2}\sum_{n=1}\left\{\begin{array}{l}\frac{4\bar{M}^4}{(un)^2}\left[K_2(un)+I_2(un)\log\frac{u^3n^3\mu^2e^C}{8\bar{M}^2}+\frac{6}{un}I_1(un)-\frac{6}{(un)^2}I_0(un)\right]\\ +\bar{a}_2^{[3]}\left[K_0(un)+I_0(un)\log\frac{u^3n^3\mu^2e^C}{8\bar{M}^2}\right]\end{array}\right\} \qquad (3.1.33)$$

which is ready to be probed for certain physical circumstances such as low and high temperature situations, that's ($\beta\to\infty$) and ($\beta\to 0$) limits.

3.2. Low TempEquation Section (Next)erature Limit

For a constant mass, a quick glance at (3.1.33) reveals a divergence at the low temperature limit ($u\to\infty$) due to the modified Bessel functions $I_n(u)$s. If we scale the temperature by $T\to T\bar{M}$ and furnishing $T^{-1}=u$, we may demonstrate this divergence by means of a graph,

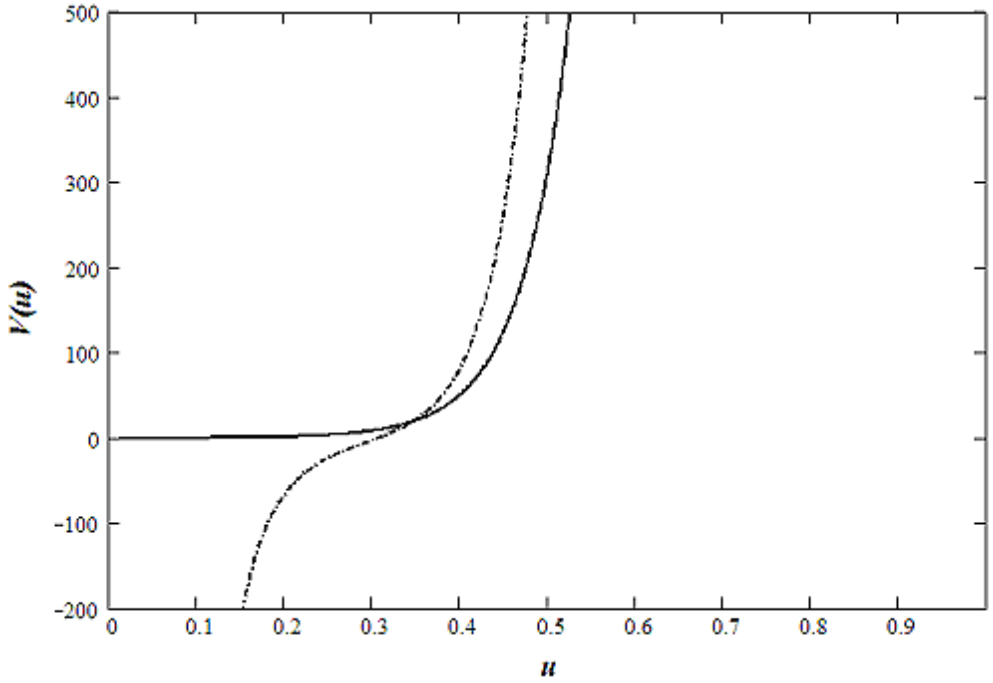


Figure 3.1.:Effective potential vs. reciprocal of scaled temperature for the low temperature limit

which has been numerically computed for n = 30. Here the solid curve represents the behavior of gravitational contributed terms associated with $\bar{a}_2$, while the dashed curve represents the first term of (3.1.33). As we increase the number of terms we observe that the positive divergences shift to left. Confronted with this behavior we'll not pursue an analytic study for this regime.

On the other hand, let us remind the reader the analogy of the effective action with the thermodynamic potential as we have argued in the introduction chapter. This will permit us to compute the entropy using,

$$S = -\left(\frac{\partial \Gamma}{\partial T}\right)_P = \frac{\partial}{\partial T}\left(\frac{\mathrm{V}_S V_\beta}{T}\right) \tag{3.2.1}$$

which's valid for constant pressure. Here we have assumed for the effective potential not to bear any coordinate dependence and using (3.1.8) we have extracted the spatial integration as V_S, that's the spatial volume at the time concerned. This assumption might be justifed if the variation of curvature and field variables are relatively small. With these in mind we may evaluate the entropy per volume,

$$s(u)=\frac{S}{\mathrm{V}_S}=\frac{\partial}{\partial T}\left(\frac{V_\beta}{T}\right)=-\frac{u^2V_\beta}{\bar{M}^2}-\frac{u^2}{\bar{M}}\frac{\partial V_\beta}{\partial u} \tag{3.2.2}$$

which we may interpret graphically,

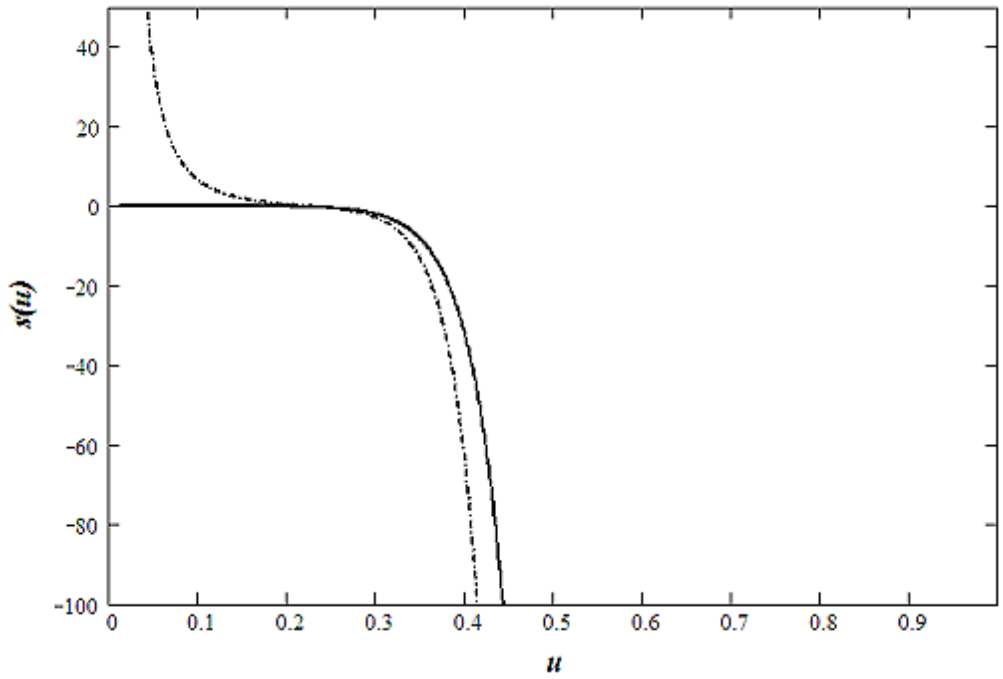


Figure 3.2.:Variation of entropy per volume vs. reciprocal of scaled temperature for the low temperature limit

whereby the definition of curves goes as in Fig. 3.1. The physical relevance of these figures will be discussed in the fourth chapter, thermal effects section.

3.3 High TeEquation Section (Next)mperature Limit

From (3.1.33)**,** the $u\to 0$ limit might easily be conceived as the behaviors of Bessel functions in this limit are well known. Using once more the series representations (3.1.28) and (3.1.29) we observe,

$$I_0(un)\;;\;1+\frac{(un)^2}{4}+\frac{(un)^4}{64}+\ldots,\quad I_1\left(n\beta\bar{M}\right)\;;\;\frac{(un)}{2}+\frac{(un)^3}{16}+\ldots,$$
$$I_2(un)\;;\;\frac{(un)^2}{8}+\frac{(un)^4}{96}+\ldots$$

and,

$$K_2(un):\ -I_2\ln\frac{une^C}{2\bar{M}}+\frac{3(un)^2}{32}+\frac{2}{(un)^2}-\frac{1}{2},\quad K_0(un):\ -I_0\log\frac{une^C}{2\bar{M}}+\frac{(un)^2}{4}.$$

Substitution of these into (3.1.33) gives,

$$\bar{V}_\beta^{(1)}(0)\cong\frac{\beta}{16\pi^2}\sum_{n=1}\left\{\bar{M}\bar{\sigma}^4\log\frac{un\mu}{2\bar{M}}+\frac{4\bar{M}\bar{\sigma}^4}{(un)^2}-\frac{16\bar{M}\bar{\sigma}^4}{(un)^4}+3\bar{M}\bar{\sigma}^4+2\bar{a}_2^{[3]}\log\frac{un\mu}{2\bar{M}}\right\}. \qquad (3.3.1)$$

The infinite summations over *n* except the logarithms might at once be performed using [12, eq.0.233],

$$\sum_{n=1}^{\infty}\frac{1}{n^{2m}}=\frac{\pi^{2m}2^{2m-1}}{(2m)!}\left|B_{2m}\right|$$

for any integer value *m* with B_{2m} being the Bernoulli numbers $B_2=\frac{1}{6},\ B_4=-\frac{1}{30}$. On the other hand the series of logarithms might be summed observing the identity,

$$\lim_{y\to 0}\left\{-\frac{d}{dy}\left(\sum_{n=1}^{\infty}(nx)^{-y}\right)\right\}=\sum_{n=1}^{\infty}\log nx$$

and which might be cast in the terms of zeta functions valid for a complex variable *y*,

$$\sum_{n=1}^{\infty}\log nx=\lim_{y\to 0}\left\{\log x\zeta(y)-\zeta'(y)\right\}$$

yielding [12, eq. 9.542],

$$\sum_{n=1}^{\infty}\log nx=-\frac{1}{2}\log\frac{x}{2\pi}+\Phi$$

where we understand an infinite however *x* independent term from Φ. Using these relations we get for (3.3.1),

$$\bar{V}_\beta^{(1)}=-\frac{\bar{M}\bar{\sigma}^4}{32\pi^2}\left(\ln\frac{\mu}{4\pi T}+3\right)+\frac{\bar{M}\bar{\sigma}^2T^2}{24}-\frac{\pi^2T^4}{90}-\frac{\bar{a}_2^{[3]}}{16\pi^2}\ln\frac{\mu}{4\pi T} \qquad (3.3.2)$$

where we have disregarded β and field independent terms like Φ appearing above. We may remark that this result agrees with flat space-time results [24, p.98], as $R_{[3]}, \bar{a}_2^{[3]} \to 0$, however we have a curvature attributed term in addition.

Combining (3.3.2) with zero temperature results (3.1.26), the thermal effective potential for the high temperature limit will be found as,

$$V_\beta^{(1)}(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\tilde{M}^2 T^2}{24} - \frac{\pi^2 T^4}{90} - \frac{\tilde{M}^4}{32\pi^2}\log\frac{\tilde{M}^2}{4\pi T\mu} - \frac{\bar{a}_2^{[3]}}{16\pi^2}\log\frac{\tilde{M}^2}{4\pi T\mu} . \tag{3.3.3}$$

Before ending this chapter, we note that the UV divergences present for the four dimensional theory developes for the three dimensional case either. In order to render the final theory finite one should transfer counter divergences by means of renormalized coupling constants as we have had done in the zero temperature study. However for the current finite temperature study we'll not be employing explicit renormalization conditions. This's because, before embarking on such a renormalization prescription we'd have to analyze the character of the expected divergences, a study which we didn't yet took. Therefore, at this stage we'll not advance further than fixing the value of μ by setting a convenient energy scale which we choose $\mu = \tilde{M}$, Then the effective potential will be computed to be,

$$V_\beta^{(1)}(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\tilde{M}^2 T^2}{24} - \frac{\pi^2 T^4}{90} - \frac{\tilde{M}^4}{32\pi^2}\log\frac{\tilde{M}}{4\pi T} - \frac{\bar{a}_2^{[3]}}{16\pi^2}\log\frac{\tilde{M}}{4\pi T} \tag{3.3.4}$$

and for the thermal effective mass term,

$$m_{\beta,eff}^2 = m^2 + \hat{\xi}R_{[3]} + \frac{\lambda}{2}\left\{\frac{T^2\left(m^2+\hat{\xi}R_{[3]}\right)}{6} + \frac{\left(m^2+\hat{\xi}R_{[3]}\right)}{8\pi^2}\log\frac{4\pi T}{m} - \frac{\hat{\xi}^2 R_{[3]}}{16m^2} - \frac{\hat{\xi}^2 R_{[3]}^2}{32\pi^2 m^2} - \frac{\bar{a}_2^{[3]}}{m^2}\right\} . \tag{3.3.5}$$

4. DISCUSSION

4.1. Notion of Symmetry BreEquation Chapter (Next) Section 1aking

So far we have been trying to find an expression for the effective action of the self interacting scalar fields for both zero and finite temperature cases. Our reasons for this procedure follow from the argument provided below (1.11), that's the solution of this equation permits one to find the values of the mean field as a function of curvature and space time coordinates. Besides, there's even a relevance between this argument and the effective masses that's been derived. This might be understood if one diferentiates (2.3.22) or (3.3.4) with respect to the field variable and then apply the extremum condition (1.11),

$$\phi^3 = -\frac{12 m_{eff}^2}{\lambda}\phi \,. \tag{4.1.1}$$

We see that for a positive mass term the only solution is,

$$m_{eff}^2 > 0: \quad \phi_1 = 0 \tag{4.1.2}$$

while for a negative mass term two other solutions appears,

$$m_{eff}^2 < 0: \quad \phi_1 = 0 \;\; \& \;\; \phi_{2,3} = \pm\sqrt{\frac{12\left|m_{eff}^2\right|}{\lambda}} \,. \tag{4.1.3}$$

Hence while the mass term changes sign a transition occurs, ϕ being the order parameter. We note that a transition of this sort is a *second order* phase transition since the symmetric ($\phi = 0$) and asymmetric ($\phi \neq 0$) cases may coexist at a certain time.

Connection of these ideas to our discussion is that, observing the respective effective masses, the curvature, the value of gravitational coupling constant ξ and temperature are seen to be in position to determine the signature of the mass term, therefore the possibility of a symmetry breaking.

Before proceeding with the effects of topology and temperature over symmetry breaking, we find it beneficial to draw a physical sketch first, extending the notions given in the introduction. As have been stated, this work considers pre-inflation and the onset of inflation eras which're believed to occur between 10^{-42} and 10^{-34} to 10^{-32} seconds following the time singularity called Big Bang [2, p.47]. During these intervals there're a number of events presumed to take place which we'll try to give an account for. Although far from precise, it's believed that the phase transiton starts just before the inflation. At this point the symmetric $\phi = 0$ state is outfavored with respect to non-symmetric $\phi \neq 0$ state. Also during this phase, the universe enters an inflation era due to the present vacuum energy, where the radius of the space is supposed to grow exponentially. Meanwhile the phase transition occurs resulting in a nonsymmetric state, and as the fluctuations around the new ground state produces lighter particles this reduces vacuum energy and thereby ending the inflation process [1, p.272].

This crude picture reveals some facts relevant to the arguments outlined before, especially the small field variable and low curvature limits we have extensively used. As is argued in the literature, the symmetry breaking is rather slow with respect to inflation time scale. This phenomenon is referred as "slow rollover", one of the key features of inflation models. As the field variable starts from zero, then it'll be reasonable to take it very small at the onset of inflation, justfying the small field expansion we have made use of. On the other hand, regarding exponential expansion of the universe, one might argue for the low curvature limit. This might be perceived from a comoving frame which will attain a somewhat flat surface just after the beginning of inflation, due to the exponential growth of its radius. In short, we see that these features of the inflation model are in accord with the assumptions used while arriving at (2.3.23) and (3.3.5). Now we'll discuss the topological and thermal effects over the symmetry breaking, starting with the zero temperature case whereby our main interest will be consideration of anisotropy.

4.2Equation Section (Next). Anisotropic Effects, Zero Temperature Discussion

To this point we have refrained ourselves employing definite space times, as the arguments we have been following are valid for a general class of space times. Now let us return back to determination of the topology of the space time at the onset of inflation. Although this problem is irrelevant for the inflation models, as the exponential growth renders the need for special initial conditions obsolete, we'll be examining the role of topology over symmetry breaking which occurs at the start of this era. To be more precise, we'll discuss the behavior of anisotropic space-times versus isotropic spaces, investigating the effect of respective curvature effects to the dynamics of the phase transition.

As is well known, observations such as cosmic background radiations reveal that the present universe is isotropic and homogenous to a large extent [25]. However Wald has shown that, certain anisotropic models might also evolve to the present universe at the stage of inflation [26]. These models are known as Bianchi Models, and this assertion is valid except for the Bianchi IX model which might be conceived as the anisotropic version of $k > 0$ FRW model [1. p,304]. As we're interested in the effect of initial anisotropy over the symmetry breaking we'll not concern ourselves with the classification of Bianchi Models, and instead study the Bianchi I model which reduces to spatially flat $k = 0$ FRW models for zero anisotropy, motivating our choice.

Therefore starting with a Bianchi I model, which constitutes an anisotropic space with different scales for each spatial direction, the line element of the four dimensional space-time might be cast as [9, p.142],

$$ds^2 = C(\eta)\left[d\eta^2 - \frac{C_i(\eta)}{C(\eta)}dx_i^2\right], \qquad C(\eta) = (C_1C_2C_3)^{\frac{1}{3}} \tag{4.2.1}$$

where η reduces to the conformal time in the isotropic limit $C_1 = C_2 = C_3 = C$, with $dt = C(\eta)d\eta$ and $-\infty < \eta < \infty$[6]. Moreover, in the isotropic limit we see that the overall scale factor reduces to the square of the radius of the universe. Hence it's evident that (4.2.1) is the anisotropic counterpart of $k = 0$ FRW line element.

On general grounds, that's without specifications for the scale factors, certain geometric invariants will be computed as,

$$\mathbf{C}^2 = 4C^{-2}\left[3U - 4\dot{S} + 12Q^2\right] \tag{4.2.2}$$

where it's been defined,

$$S = \tfrac{1}{144}\sum_{\substack{i<j \\ k\neq i,j}} \left(d_k - D\right)\left(d_i - d_j\right)^2, \qquad U = \tfrac{1}{72}\sum_{i<j}\left(\dot{d}_i - \dot{d}_j\right)^2 \tag{4.2.3}$$

where we call d_i's as the aniotropy evolution parameters. In terms of these functions we find for the Gauss-Bonnet density,

$$\mathbf{G} = 4C^{-2}\left[\begin{array}{l} 3U - 4\dot{S} + \tfrac{3}{8}\dot{D}^2 - 3\dot{D}Q + \tfrac{3}{8}D^4 + 3D^2Q + \tfrac{3}{2}\dot{D}D^2 \\ \qquad\qquad + \tfrac{1}{8}\left[\sum_i\left(\dot{d}_i + d_iD\right)\right]^2 \end{array}\right]. \tag{4.2.4}$$

Finally, the D'Alembertian over the Ricci scalar will be conceived as,

$$\begin{aligned} \Box R &= \left[\frac{D}{C}\partial_\eta + \frac{1}{C}\partial_\eta^2\right]\left[\frac{1}{C}\left(3\dot{D} + \tfrac{3}{2}D^2 + 6Q\right)\right] \\ &= -\frac{\dot{D}}{C^2}\left(3\dot{D} + \tfrac{3}{2}D^2 + 6Q\right) - \frac{2D}{C^2}\left(3\ddot{D} + 3\dot{D}D + 6\dot{Q}\right) \\ &\qquad + \frac{1}{C}\left(3\dddot{D} + 3\dot{D}^2 + 3\ddot{D}D + 6\ddot{Q}\right) \end{aligned}. \tag{4.2.5}$$

Now let us we assume for d_i's to be constant. We may argue that this's a very physical assumption, since by definition these anisotropy evolution parameters execute the same role as the Hubble constant, which's believed to be constant during the inflation era we have so far been investigating [2, p.48]. With this assumption we may simplify above expressions as,

$$\begin{aligned} &R \cong C^{-1}\left[\tfrac{3}{2}D^2 + 6Q\right], \quad \mathbf{C}^2 \cong 48C^{-2}Q^2 \\ &\mathbf{G} \cong 4C^{-2}\left[\tfrac{3}{8}D^4 + 3D^2Q + \tfrac{3}{8}D^2\right], \quad \Box R \cong 0 \end{aligned} \tag{4.2.6}$$

---

[6] The overall scale factor C is not to be confused with the square of Weyl Tensor $\mathbf{C}^2$.

and the topological dependent part of the zero temperature effective mass will read,

$$\frac{\mathbf{C}^2}{120m^2}-\frac{\mathbf{G}}{360m^2}+\frac{\nabla R}{30m^2}\cong\frac{1}{360m^2C^2}\left[144Q^2-3D^4-12D^2Q\right].$$

Substituting this into (2.3.23) the effective mass in terms of anisotropy evolution parameters will be found as,

$$m_{eff}^2=m^2+\xi R+\frac{\lambda}{32\pi^2}\left\{\begin{array}{r}\left(\xi-\frac{1}{6}\right)R\left(\log\frac{m^2}{M_3^2}-\frac{\lambda\varphi_3^2}{M_3^2}\right)+\frac{\xi R^2\left(\xi-\frac{1}{3}\right)}{2m^2}\\ +\frac{1}{360m^2C^2}\left[324Q^2+78D^2Q+\frac{33}{4}D^4\right]\end{array}\right\}. \qquad (4.2.7)$$

On the other hand, for a spatially flat isotropic FRW space the respective topological invariants are computed to be,

$$\mathbf{C}^2=0,\quad \mathbf{G}=\tfrac{1}{6}R^2 \qquad (4.2.8)$$

with R a positive constant [3]. Therefore the effective mass for the isotropic case will be found as,

$$m_{eff}^2=m^2+\xi R+\frac{\lambda}{32\pi^2}\left\{\left(\xi-\tfrac{1}{6}\right)R\left(\log\frac{m^2}{M_3^2}-\frac{\lambda\varphi_3^2}{M_3^2}\right)+\frac{\left(\xi^2-\frac{1}{3}\xi+\frac{7}{270}\right)R^2}{2m^2}\right\}. \qquad (4.2.9)$$

From these results we are finally in position for the physical implications, that's the effect of anisotropy over the symmetry breaking. We suit ourselves to carry our discussion over qualitive grounds as we hadn't outlined d_i's precisely which determines the evolution of anisotropies.

First let us discuss the situation for the presence of isotropy. From (4.2.9) and the restrictions over R for a spatially flat isotropic space, we find that the only determining factor over symmetry breaking is the gravitational coupling constant. Regarding the terms in the curly parantheses of (4.2.9), we might summarize role of ξ over symmetry breaking (SB) via a table,

Table 4.1: Effect of ξ over< the SB for k =0 FRW space

$\xi < \frac{1}{6} - \varepsilon$	$\frac{1}{6} - \varepsilon < \xi < \frac{1}{6}$	$\frac{1}{6} < \xi < \frac{1}{6} + \varepsilon$	$\frac{1}{6} + \varepsilon < \xi$
Both linear and quadratic terms help for SB.	Linear term helps for SB.	Linear term opposes SB.	Both of the terms opposes SB, no SB occurs.
	Quadratic term helps for SB.		

where we calculate $\varepsilon = \sqrt{\frac{1}{540}}$. One might note that this discussion sets an upper limit for the value of the gravitational coupling constant as the rightmost value bars a symmetry breaking which's in contadiction with GUTs.

On the anisotropy front, for constant and real valued anisotropy evolution factors d_i's we see that the situation is somewhat different than the isotropy case. If we once more recourse to a table, inspecton of (4.2.9) reveals,

Table 4.2: Effect of ξ over the SB process for anisotropic Bianchi I spaces

$\xi < 0$	$0 < \xi < \frac{1}{6}$	$\frac{1}{6} < \xi < \frac{1}{3}$	$\frac{1}{6} < \xi$
Linear term helps and the quadratic term opposes SB.	Both linear and quadratic terms help SB.	Linear term opposes and the quadratic term helps for SB.	Both of the terms opposes SB, no SB occurs.

where we once more recognize an upper limit for ξ. However for the anisotropic case the value of the coupling constant is not the only determining factor for the fate of the symmetry. As we observe from (4.2.7) there're other factors as well, especially the Q related terms which do not have any counterpart for the isotropic spaces. If we write the relevant topological contributions in terms of anisotropy evolution parameters d_i's,

$$\frac{1}{360m^2C^2}\left[324Q^2 + 78D^2Q + \tfrac{33}{4}D^4\right] = \frac{1}{360m^2C^2}\begin{bmatrix} \frac{16}{27}\left(d_1^4 + d_2^4 + d_3^4\right) + \frac{121}{108}\left(d_1^2d_2^2 + d_1^2d_3^2 + d_2^2d_3^2\right) \\ + \frac{53}{54}\left(d_1^2d_2d_3 + d_1d_2^2d_3 + d_1d_2d_3^2\right) \\ + \frac{4}{27}\left(d_1^3d_2 + d_1^3d_3 + d_2^3d_1 + d_2^3d_3 + d_3^3d_1 + d_3^3d_2\right) \end{bmatrix} \quad (4.2.10)$$

assuming d_i's are negative in value (in order to smooth the anisotropies with time) we see that the above expression is positive definite, and therefore these contributions do

always tend to preserve symmetry. Therefore we may phrase that, anisotropies favor symmetry.

4.3. Thermal Effects Equation Section (Next)

We couldn't have arrived at analytic expression for the low temperature limit. Even, we sadly shall confess that the results we have derived for this limit is seemingly in contradiction with current ideas held about early universes. First of all, as we observe from Fig.3.1, for constant field ϕ the behavior of the effective potential forbids a phase transition as it's been minimized in tranquility with respect to this configuration. Also we see from Fig. 3.2. that the entropy per volume decreases almost exponentially. Not considering the character of its divergence, we may note that this decrease contradicts present universe models, whereby the entropy is argued to increase [1, p.273]. With this negative result we may argue for the invalidity of small distances and static spaces assumptions at the low temperature limit. This may even be related to the exponential growth which signifies large distance effects, that's IR behaviors.

Meanwhile our high remperature results are in agreement with other authors [21,23]. Upon comparing the thermal effective mass (3.3.5) for the high temperature limit with that of zero temperature result (2.3.23), we may deduce that the inclusion of thermal effects always help to restore the symmetry, independent from the topology of the spatial space. Yet we observe that the topological terms do exert a non trivial independent effect over the fate of the symmetry.

However our derivation has one fundemental drawback: while deriving an expression for the effective action by means of heat coefficients we'd been using static spaces. This's an intrinsic problem met by other authors too [27]. In order to remedy this problem we may impose a condition relating the temperature of the system with its radius. We'll not do so however, since a condition of this kind deserves a through examination of the dynamics for the early universe, yet which consist of not that well established predictions based on limited observations.

We may also inspect three dimensional variants of the anisotropic and isotropic spaces considered in the previous section. That's we may propose the three dimensional variant of (4.2.1) as,

$$ds^2 = C_1(x)\,dx_1^2 + C_2(x)\,dx_2^2 + C_3(x)\,dx_3^2 \tag{4.3.1}$$

where all components are understood being functions of spatial coordinates only; whereby the respective isotropic space might be heeded as the $C_i(x) = C(x)$ limit. However in order to extract physically relevant results from these line elements, as to understand the topological contributions for the thermal case, non-trivial metrics shall be proposed. Currently we shall refrain from such a discussion either, and leave it to another study. However we observe that for constant C_i s, (4.3.1) will generate flat space results, negating all topological terms.

5. Conclusions

The study we have foretook aimed the investigation of the quantum effects during early universe. Our main concern was the anisotropic and thermal effects over a SB mechanism. As we saw, loop contributions coming from such effects do help for a symmetry restoration. In accordance with the broad literature, we may state the importance of such considerations as they provide mechanisms for GUT scenarios.

If we were to summarize our work, we should begin with emphasizing the assumptions we have so far employed. From these, the short distance inquiry might be regarded as foregoing one. We shall remark the competence of this method, since it enables one to execute many techniques inherited from flat space theories. In our study, we have enjoyed a momentum space representation which's generally not applicable to curved spaces due to IR divergences.Yet as we have stated, it fails to incorporate the general behaviors of space time dynamics. The extent of this hindrance with respect to phase transitions are open to discussion. Also as we have observed, this method fails for low temperature considerations. Perhaps a renormalization group approach would better suit for such considerations, since it permits the computation of renormalized theory constants in a much easier way [11], and even more attractive if one aims to ascertain the dynamics of symmetry breakings.

The remaining assumptions, which we remind as the small curvature and small mean field approximations are relatively physical.

While evaluating our results we have prefered qualitive grounds, since we lacked a reference point, existence of which is open to discussion either. However, we may summarize our results as follows: 1) The anisotropies introduce additional terms into the effective mass (4.2.7) which have no counterpart in isotropic spaces; 2) The high temperature limit effective mass (3.3.5) clearly reveals a topological term which may play an important role during the symmetry breaking process. These observations

may be important for the discussions held about initial conditions of the Early Universe.

Finally we may say, confronted with the demanding prerequisities the works we've mostly referenced dated about a decade in average. Though the current area of interest is very active [28], and even the heat kernel approach we have employed will be seen to be rather popular [31,32], the current effort aiming to remove the local effects.

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BIOGRAPHY

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