ISTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

LOCALLY CONFORMAL KAEHLER MANIFOLDS

M.Sc. Thesis by
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Program : **Mathematics Engineering**

June 2010

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Date of submission : **07 May 2010**
Date of defence examination : **07 June 2010**

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June 2010

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

LOKAL OLARAK KONFORMAL KAEHLER MANİFOLDLARI

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Tezin Enstitüye Verildiği Tarih : **07 Mayıs 2010**
Tezin Savunulduğu Tarih : **07 Haziran 2010**

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Haziran 2010

ACKNOWLEDGEMENTS

I would like to express my deep appreciation and thanks my supervisor Prof. Dr. Zerrin ŞENTÜRK for her offering invaluable guidance, continuous support and encouragement during this research and in the process of writing up this thesis.
I also would like to thank my husband for his infinite support and patience during all stages of this work.

May 2010 Pegah SARIASLANI

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LOCALLY CONFORMAL KAEHLER MANIFOLDS

SUMMARY

In this thesis, some properties of locally conformal Kaehler manifold are presented. The Lee form on l.c.K-manifolds is defined. It is proved that a Hermitian manifold is an l.c.K-manifold if and only if there exist on M a global closed 1-form α such that

$$d\Omega = 2\alpha \wedge \Omega \ .$$

It is also proved that an 2n-dimensional l.c.K-manifold characterization as a Hermitian manifold admiting a global closed 1-form α_λ whose structure $(F^\lambda_\mu, g_{\mu\lambda})$ satisfies the equation

$$\nabla_\nu F_{\mu\lambda} = -\beta_\mu g_{\nu\lambda} + \beta_\lambda g_{\nu\mu} - \alpha_\mu F_{\nu\lambda} + \alpha_\lambda F_{\nu\mu} \ .$$

Furthermore, certain properties of l.c.K-space forms and the Riemannian curvature tensor with respect to $g_{\mu\lambda}$ are presented. We get the necessary and sufficient condition for the length of the Lee form to be constant.
Moreover, we state some properties of contravariant and covariant almost analytic vector fields in l.c.K-manifolds.
Finally, we present submanifolds of l.c.K-manifolds and l.c.K-space forms. The invariant submanifold is defined on l.c.K-manifolds and space forms.

LOKAL OLARAK KONFORMAL KAEHLER MANİFOLDLARI

ÖZET

Bu tez çalışmasında, lokal olarak konformal Kaehler manifoldlarının bazı özellikleri sunulmuştur. L.c.K-manifoldlarda Lee form tanımlanmıştır. Bir Hermitian manifoldun bir l.c.K-manifold olması için gerek ve yeter koşulun

$$d\Omega = 2\alpha \wedge \Omega \ ,$$

olacak şekilde bir global kapalı 1-form α nın mevcut olması ispatlanmıştır.
Bir global kapalı 1-form α_λ yı kabul eden bir Hermitian manifold olarak, bir 2n-boyutlu l.c.K-manifoldunun karakterizasyonunun

$$\nabla_\nu F_{\mu\lambda} = -\beta_\mu g_{\nu\lambda} + \beta_\lambda g_{\nu\mu} - \alpha_\mu F_{\nu\lambda} + \alpha_\lambda F_{\nu\mu} \ ,$$

denklemini sağladığı ispatlanmıştır.
Ayrıca, l.c.K-uzay formlarının ve Riemann eğrilik tensörünün belirli özellikleri sunulmuştur. Lee formun uzunluğunun sabit olması için gerek ve yeter koşul elde edilmiştir.
Buna ilaveten, l.c.K-manifoldlarda kontravaryant ve kovaryant hemen hemen analitik vektör alanlarının bazı özellikleri ifade edilmiştir.
Son olarak, l.c.K-manifoldlar ve l.c.K-uzay formlarının altmanifoldları sunulmuştur.
Böylece, l.c.K-manifoldlar ve uzay formlarda invaryant altmanifoldlar tanımlanmıştır.

1. INTRODUCTION

The notion of a locally conformal Kaehler manifold (an l.c.K-manifold), in a Hermitian manifold has introduced by I. Vaisman [11] on 1976. And then he wrote a series of such manifolds [12], [3], etc. T. Kashiwada has determined a necessary and sufficient condition that a Hermitian manifold is an l.c.K-one by using tensor equation [4].

Moreover T. Kashiwada has determined the curvature tensor of an l.c.K-manifold with a constant holomorphic sectional curvature (an l.c.K-space form).

Furthermore, T. Kashiwada [3], [2] and K. Matsumoto [6] gave some proporties about such a manifold. Then we can see a lot of paper about these manifolds [8] and its submanifolds [5].

Let M be a real 2n-dimensional Hermitian manifold with structure (F,g) , whereby F is the almost complex structure and g is the Hermitian metric. *The fundamental 2-form* Ω is defined by $\Omega(X, Y) = g(FX, Y)$ for any vector fields X and Y tangent to M. The manifold M is called *a locally conformal Kaehler manifold* if each point x in M has an open neighborhood U with a positive differentiable function $\rho : U \to R$ such that

$$g^* = e^{-2\rho} g \mid_{\mathrm{U}}$$

is a Kaehlerian metric on U. Especially, if we can take $U = M$, then the manifold M is said to be globally conformal Kaehler [11].

A Hermitian manifold whose metric is locally conformal to a Kaehler metric is called a locally conformal Kaehler manifold. I. Vaisman [11] gives its characterization as follows:
A Hermitian manifold M is an l.c.K-manifold if and only if there exists on M a global

closed 1-form α such that

$$d\Omega = 2\alpha \wedge \Omega \ ,$$

whereby the fundamental 2-form Ω is defined as $\Omega(X, Y) = g(FX, Y)$ for any vector fields X and Y tangent to M and α is called the Lee form and introduced actually as follows: Consider at first a 1-form $\alpha = d\rho$ in a neighborhood U at any point, whereby $e^{-2\rho}g$ is a Kaehler metric in U.

The following is essential in l.c.K-manifolds [4].

An 2n-dimensional locally conformal Kaehler manifold characterization as a Hermitian manifold admiting a global closed 1-form α_λ (Lee form) whose structure $(F_\mu^\lambda, g_{\mu\lambda})$ satisfies

$$\nabla_\nu F_{\mu\lambda} = -\beta_\mu g_{\nu\lambda} + \beta_\lambda g_{\nu\mu} - \alpha_\mu F_{\nu\lambda} + \alpha_\lambda F_{\nu\mu} \ ,$$

whereby ∇_λ denotes the covariant differentiation with respect to the Hermitian metric $g_{\mu\lambda}$.

The relations between these spaces may be seen in the diagram:

Complex manifold $(F_\mu^\varepsilon F_\varepsilon^\lambda = -\delta_\mu^\lambda)$

$\downarrow$

Hermitian manifold $(F_\mu^\varepsilon F_\lambda^\gamma g_{\varepsilon\gamma} = g_{\mu\lambda})$

$\downarrow$

l.c.K-manifold $(\nabla_\nu F_{\mu\lambda} = -\beta_\mu g_{\nu\lambda} + \beta_\lambda g_{\nu\mu} - \alpha_\mu F_{\nu\lambda} + \alpha_\lambda F_{\nu\mu})$

$\downarrow$

Kaehler manifold $(\nabla_\nu F_\mu^\lambda = 0)$

In this thesis, we study l.c.K-manifolds and its submanifolds.

This thesis is divided into 6 Chapters:

Chapter 2 gives the fundamental concepts which we will use the next chapters.

In Chapter 3, we state some properties of l.c.K-manifolds. The Lee form on this

manifold is defined. Especially the main purpose of this chapter is to get a necessary and sufficient condition for a Hermitian manifold to be an l.c.K-manifold. And also we study some condition to be a Kaehler manifold and the Riemannian curvature tensor in an l.c.K-manifold [4]. Further, we present certain properties of the l.c.K-space forms and mainly get the necessary and sufficient condition for an l.c.K-space form to be an Einstein one. The Riemannian curvature tensor with respect to $g_{\mu\lambda}$ is expressed without the tensor field $P_{\mu\lambda}$. Moreover, in this chapter we get the necessary and sufficient condition for the length of the Lee form to be constant.

In Chapter 4, we study contravariant and covariant almost analytic vector fields in l.c.K-manifolds [9].

Chapter 5 is concerned with the submanifolds of l.c.K-manifolds [5]. The invariant submanifold is defined on an l.c.K-submanifold [7]. After that we present submanifold of an l.c.K-space form and invariant submanifold of this space.

Finally, in Chapter 6, we give conclusion and recommendations.

2. FUNDAMENTAL CONCEPTS IN RIEMANNIAN GEOMETRY AND COMPLEX MANIFOLDS

2.1 Riemannian Manifolds

A Riemannian metric on a smooth manifold M is a 2-tensor field g ,that is, g is a (0,2) tensor field on M. Since g is a symmetric and positive definite, Riemannian metric then determines an inner product on each tangent space T_pM , which is written by

$$\langle X, Y\rangle = g(X, Y), \qquad \forall X, Y \in T_pM.$$

Therefore if $X, Y, Z \in T_pM$, then $\langle X, Y\rangle : T_pM \times T_pM \to$ R satisfies the following properties:

(i) Symmetric provided $\langle X, Y\rangle = \langle Y, X\rangle$,

(ii) bilinear provided $\langle X + Y, Z\rangle = \langle X, Z\rangle + \langle Y, Z\rangle$

$$\langle aX, Y\rangle = a\langle Y, X\rangle \qquad , \qquad a \in \mathbb{R} \ ,$$

(iii) positive definite provided $\langle X, X\rangle = 0$ if X≠0 ,

(iv) C^∞ provided if X and Y are C^∞ fields with domain A, then

$\langle X, Y\rangle_p = \langle X_p, Y_p\rangle$ is a C^∞ function on A.

A Riemannian manifold is a C^∞ manifold M furnished with a Riemannian metric g and write (M, g) .

When (iii) replaced by

(v) Nondegenerate provided $\langle X, Y\rangle = 0 \ \forall$ Y implies X = 0 , g is called *a semi-Riemannian metric*. A C^∞ manifold with a semi-Riemannian metric is called *a semi-Riemannian manifold*.

The local components of g on an open set U ⊂ M are given by

$$g_{\mu\lambda} = g(\partial_\mu, \partial_\lambda) = \langle \partial_\mu, \partial_\lambda\rangle \ ,$$

whereby $\partial_\mu = \frac{\partial}{\partial x^\mu}$ are basis vectors on U.

A connection ∇ on a smooth Riemannian manifold M is a function ,

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

such that

(i) $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$,

(ii) $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$,

(iii) $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$,

whereby f,g $\in C^\infty(M)$, a,b $\in \mathbb{R}$, X,Y $\in T_pM$. $\nabla_X Y$ is called the covariant derivative of Y in the direction of X.

A Riemannian connection (Levi-Civita connection) ∇ on a Riemannian manifold M is a connection such that

(v) $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

(vi) $[X, Y] = \nabla_X Y - \nabla_Y X$

and it is characterized by *the Koszul formula*

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle \\ &+ \langle Y, [Z, X]\rangle - \langle Z, [X, Y]\rangle \ . \end{aligned}$$

Let T be an r-tensor. The covariant derivative ∇T of T is a tensor of order (r+1) given by

$$\begin{aligned} (\nabla T)(X_1, X_2, ..., X_r; X) &= (\nabla T)(X_1, ..., X_r) \\ &= \nabla_X(T(X_1, ..., X_r)) - \sum_{i=1}^{r} T(X_1, ..., \nabla_X X_i, ..., X_r) \ . \end{aligned}$$

Let $\{E_\mu\}$ be a local frame for TM on an open subset U⊂M. For any choices of the indices μ and λ , we can expand $\nabla_{E_\mu} E_\lambda$ interms of this frame

$$\nabla_{E_\mu} E_\lambda = \Gamma^{\varepsilon}_{\mu\lambda} E_\varepsilon. \qquad (\mathbf{2.1})$$

The Christoffel symbols $\Gamma^{\varepsilon}_{\mu\lambda}$ of ∇ with respect to this frame are given by

$$\Gamma^{\nu}_{\mu\lambda} = \frac{1}{2} g^{\nu\varepsilon}(\partial_\mu g_{\lambda\varepsilon} + \partial_\lambda g_{\mu\varepsilon} - \partial_\varepsilon g_{\mu\lambda}) \ , \qquad (\mathbf{2.2})$$

and the covariant derivative of a tensor $T_{\mu\lambda}$ is given by

$$\nabla_\nu T_{\mu\lambda} = \partial_\nu T_{\mu\lambda} - \Gamma^{\varepsilon}_{\nu\mu} T_{\varepsilon\lambda} - \Gamma^{\varepsilon}_{\nu\lambda} T_{\mu\varepsilon} \ . \qquad (\mathbf{2.3})$$

2.2 Riemannian Curvature Tensor

If X,Y $\in T_pM$, then linear operator

$$R(X,Y) : T_pM \to T_pM$$

is called *the curvature operator*.

We define *the Riemannian curvature tensor* as the covariant 4-tensor field obtained from (1,3)-tensor field R(X,Y)Z by lowering the last index. Its action on vector fields is given by

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle = g(R(X,Y)Z,W) \ ,$$

whereby

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \ . \tag{2.4}$$

In terms of local coordinates

$$R_{\omega\mu\lambda\nu} = R^{\varepsilon}_{\omega\mu\lambda}g_{\varepsilon\nu} \ ,$$

whereby

$$R^{\nu}_{\omega\mu\lambda} = \partial_\omega\Gamma^{\nu}_{\mu\lambda} - \partial_\mu\Gamma^{\nu}_{\omega\lambda} + \Gamma^{\nu}_{\omega\varepsilon}\Gamma^{\varepsilon}_{\mu\lambda} - \Gamma^{\nu}_{\mu\varepsilon}\Gamma^{\varepsilon}_{\omega\lambda} \ .$$

The curvature tensor satisfies the following symmetries :

(i) $R(X,Y)Z + R(Y,X)Z = 0$

(ii) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

which is called first Bianchi identity.

(iii) $R(X,Y,Z,W) = -R(Y,X,Z,W)$

(iv) $R(X,Y,Z,W) = -R(X,Y,W,Z)$

(v) $R(X,Y,Z,W) = R(Z,W,X,Y)$.

The total covariant derivative of the curvature tensor satisfies the following identity:

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0 \tag{2.5}$$

which is called *the second Bianchi identity*.

If $e_1, e_2, ..., e_n$ are local orthonormal vector field, then

$$R(Y,Z) = \sum_{\lambda=1}^{n} g(R(e_\lambda, Y)Z, e_\lambda)$$

defines a (0,2) tensor field with local components

$$R_{\mu\lambda} = R^{\varepsilon}_{\varepsilon\mu\lambda} = g^{\varepsilon\gamma} R_{\varepsilon\mu\lambda\gamma} \ .$$

The tensor field R(Y, Z) is called a Ricci tensor.

The scalar curvature is the function S defined as the trace of the Ricci tensor:

$$S = \sum_{\lambda=1}^{n} R(e_\lambda, e_\lambda) = traceR = R^{\lambda}_{\lambda} = g^{\mu\lambda} R_{\mu\lambda} \ . \tag{2.6}$$

The curvature tensor appears also in the Ricci identities:

$$\nabla_\omega \nabla_\mu v^\kappa - \nabla_\mu \nabla_\omega v^\kappa = R^{\kappa}_{\omega\mu\lambda} v^\lambda \ , \tag{2.7}$$

$$\nabla_\omega \nabla_\mu w_\lambda - \nabla_\mu \nabla_\omega w_\lambda = -R^{\varepsilon}_{\omega\mu\lambda} w_\varepsilon \ , \tag{2.8}$$

$$\nabla_\omega \nabla_\nu T^{\kappa}_{\mu\lambda} - \nabla_\nu \nabla_\omega T^{\kappa}_{\mu\lambda} = R^{\kappa}_{\omega\nu\varepsilon} T^{\varepsilon}_{\mu\lambda} - R^{\varepsilon}_{\omega\nu\mu} T^{\kappa}_{\varepsilon\lambda} - R^{\varepsilon}_{\omega\nu\lambda} T^{\kappa}_{\mu\varepsilon} \ . \tag{2.9}$$

A Riemannian manifold is called *an Einstein manifold* if

$$R_{\mu\lambda} = C g_{\mu\lambda} \ ,$$

whereby C is constant.

Let M be a semi-Riemannian manifold and p∈M. A two dimensional subspace π of the tangent space T_pM is called *a tangent plane* to M at p. π is determined by linearly independent vectors X and Y at p. We define *the sectional curvature* $K(\pi)$ of π spanned by X and Y at p is given by

$$K(X, Y) = K(\pi) = \frac{R(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2} \ . \tag{2.10}$$

In particular , if π is spanned by an orthonormal basis u and v at p , the sectional curvature is given by

$$K(\pi) = R(u, v, v, u).$$

In local components,

$$K(\pi) = \frac{R_{\omega\mu\lambda\nu} X^\omega Y^\mu Y^\lambda X^\nu}{(g_{\omega\nu} g_{\mu\lambda} - g_{\omega\lambda} g_{\mu\nu}) X^\omega Y^\mu Y^\lambda X^\nu} \ . \tag{2.11}$$

The sectional curvature K of M is a real-valued function on the set of all nondegenerate

tangent planes to M.

The famous theorem of Green can now be stated as follows:

GREEN'S THEOREM. In a compact orientable Riemannian manifold M, we have

$$\int_M (\nabla v) d\sigma = 0 \tag{2.12}$$

for any arbitrary vector field v, whereby $d\sigma$ is the volume element

$$d\sigma = \sqrt{g} dv^1 \wedge dv^2 \wedge ... \wedge dv^{2n}.$$

2.3 Lie Derivatives

Let M be an n-dimensional manifold and w is an r-form and $X_1, X_2, ..., X_r \in T_M$

$$\begin{aligned} (\pounds_X w)(X_1, X_2, ..., X_r) &= X(w(X_1, X_2, ..., X_r)) \\ &- \sum_{i=1}^{r} w(X_1, ..., [X, X_i], ..., X_r) \ , \end{aligned}$$

whereby $X(w(X_1, X_2, ..., X_r))$ is the derivative of the function $w(X_1, X_2, ..., X_r)$ by X_r is called *the Lie Derivative* of an r-form.

If r=1, then

$$\pounds_X w = X(w(Y)) - w([X, Y]) \ .$$

If r=0 , i.e. w is a scalar field , $\pounds_X w = Xw$.

The Lie derivatives of a contravariant vector v^λ , a covariant vector w_λ and a tensor field T^λ_μ are given by respectively

$$\pounds_X v^\lambda = X^\varepsilon \nabla_\varepsilon v^\lambda - v^\varepsilon \nabla_\varepsilon X^\lambda \ , \tag{2.13}$$

$$\pounds_X w_\lambda = X^\varepsilon \nabla_\varepsilon w_\lambda - w_\varepsilon \nabla_\lambda X^\varepsilon \ , \tag{2.14}$$

$$\pounds_X T^\lambda_\mu = X^\varepsilon \nabla_\varepsilon T^\lambda_\mu - T^\varepsilon_\mu \nabla_\varepsilon X^\lambda + T^\lambda_\varepsilon \nabla_\mu X^\varepsilon \ . \tag{2.15}$$

The equation

$$\pounds_X g_{\mu\lambda} = X^\varepsilon \nabla_\varepsilon g_{\mu\lambda} + g_{\varepsilon\lambda} \nabla_\mu X^\varepsilon + g_{\mu\varepsilon} \nabla_\lambda x^\varepsilon = \nabla_\mu X_\lambda + \nabla_\lambda X_\mu = 0 \tag{2.16}$$

is called *a Killing equation* and a vector satisfying (**2.16**) is called *a Killing vector*. Moreover for a closed vector field X , we have

$$\nabla_\mu X_\lambda - \nabla_\lambda X_\mu = 0 \ . \tag{2.17}$$

2.4 Conformal Changes of a Riemannian Metric

Let M be an 2n-dimensional Riemannian manifold with metric tensor g and ρ a positive function on M. Then

$$g^* = e^{-2\rho} g \tag{2.18}$$

defines a metric tensor on M which does not change the angle between two vectors at a point. Hence it is a conformal change of the metric.
Let ∇^* denote the operator of covariant differentiation with respect to the Christoffel symbols $\Gamma^{*\nu}_{\mu\lambda}$ formed with g^*. And g^* and g are conformally related , then we have

$$\nabla^*_X Y = \nabla_X Y + \omega(X)Y + \omega(Y)X - g(X,Y)A \tag{2.19}$$

for any vector field X and Y, whereby ω is a 1-form given by

$$\omega = d(lne^{-\rho})$$

and A is a vector field given by $g(A, X) = \omega(X)$ [1].
In local coordinates, if put $X = \partial_\mu$ and $Y = \partial_\lambda$, we have

$$\nabla^*_{\partial_\mu}\partial_\lambda = \nabla_{\partial_\mu}\partial_\lambda + \omega(\partial_\mu)\partial_\lambda + \omega(\partial_\lambda)\partial_\mu - g(\partial_\mu, \partial_\lambda)A$$

and also

$$\omega(\partial_\lambda) = d(lne^{-\rho})(\partial_\lambda) = \partial_\lambda(-\rho) = -\nabla_\lambda \rho = -\alpha_\lambda \ ,$$

whereby $\alpha = d\rho$, and from the above equation we get

$$\nabla^*_{\partial_\mu}\partial_\lambda = \nabla_{\partial_\mu}\partial_\lambda - \alpha_\mu\partial_\lambda - \alpha_\lambda\partial_\mu + g(\partial_\mu, \partial_\lambda)A \ . \tag{2.20}$$

Then we write

$$g(A, \partial_\lambda) = \omega(\partial_\lambda) = -\alpha_\lambda \ ,$$

$$A = grad(lne^{-\rho}) = grad(-\rho) = -\Sigma g^{\varepsilon\gamma}\alpha_\varepsilon \partial_\gamma = -\Sigma\alpha^\varepsilon \partial_\varepsilon \ .$$

Using (**2.20**) , we obtain

$$\Sigma\Gamma^{*\nu}_{\mu\lambda}\partial_\nu = \Sigma\Gamma^{\nu}_{\mu\lambda}\partial_\nu - \alpha_\mu\partial_\lambda - \alpha_\lambda\partial_\mu + g_{\mu\lambda}\Sigma\alpha^\varepsilon\partial_\varepsilon$$

and so

$$\Sigma\Gamma^{*\nu}_{\mu\lambda}\partial_\nu = \Sigma\Gamma^{\nu}_{\mu\lambda}\partial_\nu + \Sigma\{-\alpha_\lambda\delta^\nu_\mu - \alpha_\mu\delta^\nu_\lambda + g_{\mu\lambda}\alpha^\nu\}\partial_\nu \ .$$

Hence, the Christoffel symbols of g^* is given by

$$\Gamma^{*\nu}_{\mu\lambda} = \Gamma^{\nu}_{\mu\lambda} - \alpha_\lambda\delta^\nu_\mu - \alpha_\mu\delta^\nu_\lambda + g_{\mu\lambda}\alpha^\nu \ . \qquad (\mathbf{2.21})$$

Then we have

$$e^{-2\rho}R^*_{\mu\lambda\omega\nu} - R_{\mu\lambda\omega\nu} = P_{\mu\omega}g_{\lambda\nu} - P_{\lambda\omega}g_{\mu\nu} + g_{\mu\omega}P_{\lambda\nu} - g_{\lambda\omega}P_{\mu\nu} \ , \qquad (\mathbf{2.22})$$

whereby R^* is the Riemannian curvature tensor of g^*.

2.5 Submanifolds

Let N be a real m-dimensional manifold immersed in a real 2n-dimensional manifold M with $2n\rangle m$ and N is imbedded in M. If the manifold M is covered by a system of coordinate neighborhood $\{V, v^\lambda\}$ and N is covered by a system of coordinate neighborhood $\{U, u^i\}$, here and in the sequel the indices k, j, ..., i run over the range { 1, 2, ..., m} , then the submanifold N can be locally represented by

$$v^\lambda = v^\lambda(u^i) \ . \qquad (\mathbf{2.23})$$

In the following, we shall identify vector fields in N and their image under the differential mapping, that is , if i denotes the immersion of N in M and X is a vector field in N, we identify X and $i_*(X)$ where $i_* : T_pN \to T_pM$.

If X is a vector field in N and has the local expression $X = u^i\partial_i$ then X also has the local expression

$$X = B^\lambda_i u^i\partial_\lambda$$

in M, whereby $\partial_\lambda = \frac{\partial}{\partial v^\lambda}$ and

$$B^\lambda_i = \partial_i v^\lambda = \frac{\partial v^\lambda}{\partial u^i} \ .$$

Let X be a vector field on N. Then a vector field $\widetilde{X}$ defined on M is called *an extension* of its restriction to the submanifold N is X.

Suppose that the manifold M is a Riemannian manifold with Riemannian metric $\widetilde{g}$. Then the submanifold N is also a Riemannian manifold with Riemannian metric g given by

$$g(X, Y) = \widetilde{g}(X, Y) \tag{2.24}$$

for any vector field X and Y in N. The Riemannian metric g on N is called the induced metric on M.

In local components, it is written by

$$g_{ji} = g_{\mu\lambda} B^{\mu}_{j} B^{\lambda}_{i}$$

with $g = g_{ji} du^j du^i$ and $\widetilde{g} = g_{\mu\lambda} dv^{\mu} dv^{\lambda}$.

If a vector ξ_p of M at a point $p \in$ N satisfies

$$\widetilde{g}(u_p, \xi_p) = 0 \tag{2.25}$$

for any vector u_p of N at p, then ξ_p is called *a normal vector* of N in M at p.

Let $T^{\perp}N$ denote the vector bundle of all normal vectors of N in M. The tangent bundle of M , restricted to N, is the direct sum of the tangent bundle TN of M and the normal bundle $T^{\perp}N$ of N in M, that is

$$TM\,|_{\mathrm{N}} = TN + T^{\perp}N\ . \tag{2.26}$$

The formula

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.27}$$

is called *the Gauss formula*, whereby $\nabla_X Y$ is a vector field tangent to N and $h(X, Y)$ is a vector field normal to N.

Let ξ be a normal vector field on N and X be a vector field on N. We can decompose $\widetilde{\nabla}_X \xi$ as

$$\widetilde{\nabla}_X \xi = -A_{\xi}(X) + \nabla^{\perp}{}_X \xi\ , \tag{2.28}$$

whereby $A_{\xi}(X)$ and $\widetilde{\nabla}_X \xi$ are , respectively , the tangential component and the normal component of $\nabla_X \xi$. The equation (**2.27**) is called *the Weingarten's formula*.

The equation of Gauss and Codazzi are respectively given by

$$\begin{aligned} \langle R(X, Y)Z, W\rangle &= \langle \widetilde{R}(X, Y)Z, W\rangle + \langle h(X, W), h(Y, Z)\rangle \\ &- \langle h(Y, W), h(X, Z)\rangle \end{aligned} \tag{2.29}$$

and

$$(\widetilde{R}(X,Y)Z)^{\perp} = (\nabla^{\perp}_X h)(Y,Z) - (\nabla^{\perp}_Y h)(X,Z)\ , \tag{2.30}$$

whereby $\widetilde{R}$ denotes the curvature tensor of the submanifold N and

$$(\nabla^{\perp}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.6 Complex Manifolds

We consider a 2n-dimensional real manifold M of class C^{∞} covered by a system of coordinate neighbourhoods , v^{λ} , where $\nu, \mu, ..., \lambda$ run over the range $1, 2, ..., n, \bar{1}, \bar{2}, ..., \bar{n}$ and indices a, b, ... run over the range $1, 2, ..., n$. We can introduce in each coordinate neighbourhood (v^{λ}) complex coordinates (z^a) defined by

$$z^a = v^a + \sqrt{-1}\ v^{\bar{a}}\ . \tag{2.31}$$

We call (v^{λ}) real coordinates and z^a complex coordinates of a point with respect to these system of coordinates respectively.

If we can cover the whole manifold M by a system of coordinate neighbourhoods (z^a) in such away that, in the intersection of two complex coordinate neighbourhoods (z^a) and $(z^{\acute{a}})$, we have

$$z^{\acute{a}} = f^{\acute{a}}(z^a) \qquad , \qquad |\ A^{\acute{a}}_a\ | \neq 0\ , \tag{2.32}$$

whereby $f^{\acute{a}}$ are regular complex functions of complex variables $z^1, z^2, ..., z^n$ and $A^{\acute{a}}_a = \partial_a z^{\acute{a}}$, we say that the manifold M admits a complex structure defined by the existence of such a system of complex coordinate neighbourhoods and we call *M a complex manifold*.

Now take a general tensor in complex manifold M such as

$$T^{\lambda}_{\mu} = \begin{pmatrix} T^b_a & T^{\bar{b}}_a \\ T^b_{\bar{a}} & T^{\bar{b}}_{\bar{a}} \end{pmatrix}.$$

If a tensor T^{λ}_{μ} has components of the type

$$T^{\lambda}_{\mu} = \begin{pmatrix} T^b_a & 0 \\ 0 & T^{\bar{b}}_{\bar{a}} \end{pmatrix},$$

we say that T^{λ}_{μ} is pure and if it has components of the type

$$T^{\lambda}_{\mu} = \begin{pmatrix} 0 & T^{\bar{b}}_{a} \\ T^{b}_{\bar{a}} & 0 \end{pmatrix},$$

we say that T^{λ}_{μ} is hybrid.

We now assume that there is a given self-conjugate positive definite Riemannian metric

$$ds^2 = g_{\mu\lambda} dv^{\mu} dv^{\lambda} \tag{2.33}$$

in the complex manifold.

If the fundamental tensor $g_{\mu\lambda}$ is hybrid, that is, it has the form

$$g_{\mu\lambda} = \begin{pmatrix} 0 & g_{a\bar{b}} \\ g_{\bar{a}b} & 0 \end{pmatrix},$$

then we call such a metric *a Hermitian metric* and the complex space with a Hermitian metric *a Hermitian manifold*. Since the fundamental tensor $g_{\mu\lambda}$ is hybrid we write

$$F^{c}_{\mu} F^{b}_{\lambda} g_{cb} = g_{\mu\lambda} \ , \tag{2.34}$$

whereby the complex structure F^{λ}_{μ} has the compponents

$$F^{\lambda}_{\mu} = \begin{pmatrix} 0 & \delta^{b}_{a} \\ -\delta^{b}_{a} & 0 \end{pmatrix}$$

in the real coordinate system (v^{λ}). It satisfies

$$F^{\varepsilon}_{\mu} F^{\lambda}_{\varepsilon} = -\delta^{\lambda}_{\mu} \ .$$

A Hermitian manifold is Keahlerian if and only if

$$\nabla_{\nu} F^{\lambda}_{\mu} = 0 \qquad or \qquad \nabla_{\nu} F_{\mu\lambda} = 0 \ . \tag{2.35}$$

A vector u^{λ} at a point in a Kaehler space is always orthogonal to $F^{\lambda}_{\mu} u^{\mu}$, so they are also linearly independent. The plane element determined by u^{λ} and $F^{\lambda}_{\mu} u^{\mu}$ is called *a holomorphic section*.

The sectional curvature K of the space at this point wih respect to this holomorphic

section is given by

$$c = -\frac{R_{\omega\mu\lambda\nu}\ F_{\varepsilon}^{\omega}u^{\varepsilon}\ u^{\mu}\ F_{\gamma}^{\lambda}u^{\gamma}\ u^{\nu}}{g_{\varepsilon\mu}u^{\varepsilon}\ u^{\mu}\ g_{\gamma\nu}u^{\gamma}\ u^{\nu}}\ . \qquad (\mathbf{2.36})$$

The equation (**2.36**) is called *a holomorphic sectional curvature* with respect to the vector u^{λ}.

If (**2.36**) is always constant with respect to any vector at every point of the space, then the space is called *a space of constant holomorphic sectional curvature*.

The Riemannian curvature tensor R of an Kaehler space with the constant holomorphic sectional curvature c is given by

$$4R_{\omega\nu\mu\lambda} = c\ (g_{\omega\lambda}g_{\nu\mu} - g_{\omega\mu}g_{\nu\lambda} + F_{\omega\lambda}F_{\nu\mu} - F_{\omega\mu}F_{\nu\lambda} - 2F_{\omega\nu}F_{\mu\lambda})\ . \qquad (\mathbf{2.37})$$

If M be an even-dimensional Kaehler manifold with a constant holomorphic sectional curvature, it is called *complex space form*.

3. LOCALLY CONFORMAL KAEHLER MANIFOLDS

3.1 Locally Conformal Kaehler Manifolds

Let M be a real 2n-dimensional Hermitian manifold with structure (F,g) , whereby F is the almost complex structure and g is the Hermitian metric. *The fundamental 2-form* Ω is defined by $\Omega(X, Y) = g(FX, Y)$ for any vector fields X and Y tangent to M. The manifold M is called *a locally conformal Kaehler manifold* (*an l.c.K-manifold*) if each point x in M has an open neighborhood U with a positive differentiable function $\rho : U \to R$ such that

$$g^* = e^{-2\rho} g \mid_{\mathrm{U}} \tag{3.1}$$

is a Kaehlerian metric on U , that is, $\nabla^* F = 0$, whereby ∇^* is the covariant differentiation with respect to g^* . Especially, if we can take U = M , then the manifold M is said to be *globally conformal Kaehler*, and R is the real number space.

Theorem 3.1 [11]. *A Hermitian manifold M is an l.c.K-manifold if and only if there exists on M a global closed 1-form* α *such that*

$$d\Omega = 2\alpha \wedge \Omega \tag{3.2}$$

and α *is called the Lee form.*

Proof . If M (F,g) is an l.c.K-manifold, then on each open set U_i we have that

$$g_i = e^{-2\rho_i} g$$

is Kaehler , so that

$$\Omega_i(X, Y) = g_i(FX, Y) = e^{-2\rho_i} g(FX, Y) = e^{-2\rho_i} \Omega(X, Y) \ .$$

This gives

$$d\Omega_i = -2d\rho_i e^{-2\rho_i} \wedge \Omega + e^{-2\rho_i} \wedge d\Omega \ .$$

Since g_i is a Kaehler metric, then Ω_i is a Kaehler form and $d\Omega_i = 0$. Then we get

$$-2e^{-2\rho_i} d\rho_i \wedge \Omega + e^{-2\rho_i} \wedge d\Omega = 0$$

which implies

$$d\Omega = 2d\rho_i \wedge \Omega \ .$$

Hence, on overlaps $U_{ij} = U_i \cap U_j$, we have

$$g_i = e^{-2(\rho_i - \rho_j)} g_j \ , \qquad \Omega_i = e^{-2(\rho_i - \rho_j)} \Omega_j$$

and by exterior differential of Ω_i, we obtain

$$d\Omega_i = -2e^{-2(\rho_i - \rho_j)} d(\rho_i - \rho_j) \wedge \Omega_j + e^{-2(\rho_i - \rho_j)} d\Omega_j \ .$$

Since Ω_j is also a Kaehler form, then $d\Omega_j = 0$, we can write

$$d(\rho_i - \rho_j) \wedge \Omega_j = 0 \ .$$

By the non-degeneracy of the metric and hence the Kaehler form we must have $d\rho_i = d\rho_j$ on U_{ij}, so that the local on forms will glue up to a globally closed 1-form satisfying the definition of a Lee form.

Conversely, having a Lee form on M , there is an open cover U_i for M and family of a differentiable function ρ_i defined on U_i 's such that

$$\alpha \, |_{\mathrm{Ui}} = d\rho_i$$

for each i.

Since $d\Omega = 2\alpha \wedge \Omega$ we have by restriction that $d\Omega = 2d\rho_i \wedge \Omega$ on each U_i. This gives

$$\begin{aligned} d\Omega_i &= d(e^{-2\rho_i}\Omega) = -2e^{-2\rho_i} d\rho_i \wedge \Omega + e^{-2\rho_i} d\Omega \\ &= -e^{-2\rho_i} d\Omega + e^{-2\rho_i} d\Omega \\ &= e^{-2\rho_i}(-d\Omega + d\Omega) \\ &= 0 \ . \end{aligned}$$

Hence, $g_i = e^{-2\rho_i} g$ is Kaehler on U_i , so that $M(F, g, \alpha)$ is an l.c.K-manifold. ■

Theorem 3.2 [4]. *A Hermitian manifold M (F,g) is an l.c.K-manifold if and only if*

$$\nabla_\nu F_{\mu\lambda} = -\beta_\mu g_{\nu\lambda} + \beta_\lambda g_{\nu\mu} - \alpha_\mu F_{\nu\lambda} + \alpha_\lambda F_{\nu\mu} \ , \tag{3.3}$$

whereby α is a global closed 1-form and ∇_λ denotes the covariant differentiation with respect to the Hermitian metric $g_{\mu\lambda}$, and also

$$\beta_\lambda = \alpha^\varepsilon F_{\varepsilon\lambda} \tag{3.4}$$

and the indices $\nu, \mu, ..., \lambda$ run over the range {1,2, ... ,2n}.

Proof . Let $M(F^\lambda_\mu, g_{\mu\lambda}, \alpha_\lambda)$ be an l.c.K-manifold. By its definition at any point there exists a neighborhood in which a conformal metric $g^* = e^{-2\rho}g$ is Kaehler one. Then we write

$$\nabla^*_\nu(F^\varepsilon_\mu g^*_{\varepsilon\lambda}) = (\nabla^*_\nu F^\varepsilon_\mu) g^*_{\varepsilon\lambda} + F^\varepsilon_\mu \nabla^*_\nu g^*_{\varepsilon\lambda} = 0$$

and also

$$\nabla^*_\nu(F^\varepsilon_\mu g^*_{\varepsilon\lambda}) = \nabla^*_\nu(e^{-2\rho}F^\varepsilon_\mu g_{\varepsilon\lambda}) = \nabla^*_\nu(e^{-2\rho}F_{\mu\lambda}) = 0 \ . \tag{3.5}$$

By using (**2.3**) , we have

$$\nabla^*_\nu(e^{-2\rho}F_{\mu\lambda}) = \partial_\nu(e^{-2\rho}F_{\mu\lambda}) - \Gamma^{*\varepsilon}_{\nu\mu}(e^{-2\rho}F_{\varepsilon\lambda}) - \Gamma^{*\varepsilon}_{\nu\lambda}(e^{-2\rho}F_{\mu\varepsilon}) \ .$$

Substituting (**2.21**) into above equation, we get

$$\begin{aligned} \nabla^*_\nu(e^{-2\rho}F_{\mu\lambda}) &= \partial_\nu(e^{-2\rho}F_{\mu\lambda}) - e^{-2\rho}\Gamma^\varepsilon_{\nu\mu}F_{\varepsilon\lambda} - e^{-2\rho}\Gamma^\varepsilon_{\nu\lambda}F_{\mu\varepsilon} + e^{-2\rho}\{\alpha_\nu F_{\mu\lambda} \\ &+ \alpha_\mu F_{\nu\lambda} - g_{\nu\mu}\alpha^\varepsilon F_{\varepsilon\lambda} + \alpha_\nu F_{\mu\lambda} + \alpha_\lambda F_{\mu\nu} - g_{\nu\lambda}\alpha^\varepsilon F_{\mu\varepsilon}\} \ . \end{aligned}$$

Taking into account the following

$$\partial_\nu(e^{-2\rho}F_{\mu\lambda}) = \partial_\nu(e^{-2\rho})F_{\mu\lambda} + e^{-2\rho}\partial_\nu(F_{\mu\lambda}) = -2e^{-2\rho}\alpha_\nu F_{\mu\lambda} + e^{-2\rho}\partial_\nu F_{\mu\lambda} \ ,$$

we find

$$\nabla^*_\nu(e^{-2\rho}F_{\mu\lambda}) = e^{-2\rho}\{\nabla_\nu F_{\mu\lambda} + \alpha_\mu F_{\nu\lambda} - \alpha^\varepsilon F_{\varepsilon\lambda}g_{\nu\mu} + \alpha_\lambda F_{\mu\nu} - \alpha^\varepsilon F_{\mu\varepsilon}g_{\nu\lambda}\} \ .$$

Using (**3.4**) and (**3.5**) , we obtain the equation (**3.3**) .

Conversely, if we have (**3.3**) , then from this equation there exist a global closed 1-form α. By using previous theorem , $M(F^\lambda_\mu, g_{\mu\lambda}, \alpha_\lambda)$ is an l.c.K-manifold. This completes the proof. ■

Moreover , by using (**3.3**) and (**3.4**) in an l.c.K-manifold, we have the following formulae:

$$\nabla_\mu \beta_\lambda = -\beta_\mu \alpha_\lambda + \beta_\lambda \alpha_\mu - \|\alpha\|^2 F_{\mu\lambda} + \nabla_\mu \alpha^\varepsilon F_{\varepsilon\lambda} \ , \tag{3.6}$$

$$\nabla_\varepsilon \beta^\varepsilon = 0 \ , \tag{3.7}$$

$$\nabla_\lambda \beta^\varepsilon F_{\mu\varepsilon} = \nabla_\mu \beta^\varepsilon F_{\lambda\varepsilon} \ , \tag{3.8}$$

$$(n-2)\alpha_\lambda = F_{\varepsilon\lambda} \nabla_\gamma F^{\varepsilon\gamma} = F^{\gamma\varepsilon} \nabla_\gamma F_{\varepsilon\lambda} \ , \tag{3.9}$$

$$(n-2)\beta_\lambda = \nabla^\varepsilon F_{\varepsilon\lambda} \ , \tag{3.10}$$

$$\alpha^\varepsilon \nabla_\varepsilon F_{\mu\lambda} = \beta^\varepsilon \nabla_\varepsilon F_{\mu\lambda} = 0 \ , \tag{3.11}$$

$$\alpha^\varepsilon \nabla_\lambda F_{\varepsilon\mu} = -\beta_\lambda \alpha_\mu + \alpha_\lambda \beta_\mu - \|\alpha\|^2 F_{\mu\lambda} \ , \tag{3.12}$$

and

$$\beta^\varepsilon \nabla_\lambda F_{\varepsilon\mu} = \beta_\lambda \beta_\mu + \alpha_\lambda \alpha_\mu - \|\alpha\|^2 g_{\mu\lambda} \ , \tag{3.13}$$

whereby $\|\alpha\|$ denotes the length of the Lee form α with respect to $g_{\mu\lambda}$.
Furthermore taking into account of the Ricci's identity, we get

$$\begin{aligned} D_{\omega\nu\mu\lambda} &= \nabla_\omega \nabla_\nu F_{\mu\lambda} - \nabla_\nu \nabla_\omega F_{\mu\lambda} \\ &= -R^\varepsilon_{\omega\nu\mu} F_{\varepsilon\lambda} - R^\varepsilon_{\omega\nu\lambda} F_{\mu\varepsilon} \\ &= -R^\varepsilon_{\omega\nu\mu} F_{\varepsilon\lambda} + R^\varepsilon_{\omega\nu\lambda} F_{\varepsilon\mu} \end{aligned} \tag{3.14}$$

and by using (**3.3**) , we write

$$\begin{aligned} \nabla_\omega \nabla_\nu F_{\mu\lambda} - \nabla_\nu \nabla_\omega F_{\mu\lambda} &= \nabla_\omega \{-\beta_\mu g_{\nu\lambda} + \beta_\lambda g_{\nu\mu} - \alpha_\mu F_{\nu\lambda} + \alpha_\lambda F_{\nu\mu}\} \\ &- \nabla_\nu \{-\beta_\mu g_{\omega\lambda} + \beta_\lambda g_{\omega\mu} - \alpha_\mu F_{\omega\lambda} + \alpha_\lambda F_{\omega\mu}\} \end{aligned}$$

and from (**3.6**) , we have

$$\begin{aligned} D_{\omega\nu\mu\lambda} &= \{\beta_\omega\alpha_\mu - \beta_\mu\alpha_\omega + \|\alpha\|^2 F_{\omega\mu} - \nabla_\omega\alpha^\varepsilon F_{\varepsilon\mu}\}g_{\nu\lambda} + \{-\beta_\omega\alpha_\lambda \\ &+ \beta_\lambda\alpha_\omega - \|\alpha\|^2 F_{\omega\lambda} + \nabla_\omega\alpha^\varepsilon F_{\varepsilon\lambda}\}g_{\nu\mu} - \nabla_\omega\alpha_\mu F_{\nu\lambda} + \{\beta_\nu g_{\omega\lambda} \\ &- \beta_\lambda g_{\omega\nu} + \alpha_\nu F_{\omega\lambda} - \alpha_\lambda F_{\omega\nu}\}\alpha_\mu + \nabla_\omega\alpha_\lambda F_{\nu\mu} + \{-\beta_\nu g_{\omega\mu} \\ &+ \beta_\mu g_{\omega\nu} - \alpha_\nu F_{\omega\mu} + \alpha_\mu F_{\omega\nu}\}\alpha_\lambda + \{-\beta_\nu\alpha_\mu + \beta_\mu\alpha_\nu - \|\alpha\|^2 F_{\nu\mu} \\ &+ \nabla_\nu\alpha^\varepsilon F_{\varepsilon\mu}\}g_{\omega\lambda} - \{-\beta_\nu\alpha_\lambda + \beta_\lambda\alpha_\nu - \|\alpha\|^2 F_{\nu\lambda} + \nabla_\nu\alpha^\varepsilon F_{\varepsilon\lambda}\}g_{\omega\mu} \\ &+ \nabla_\nu\alpha_\mu F_{\omega\lambda} + \{-\beta_\omega g_{\nu\lambda} + \beta_\lambda g\nu\omega - \alpha_\omega F_{\nu\lambda} + \alpha_\lambda F_{\nu\omega}\}\alpha_\mu \\ &- \nabla_\nu\alpha_\lambda F_{\omega\mu} + \{\beta_\omega g_{\nu\mu} - \beta_\mu g\nu\omega + \alpha_\omega F_{\nu\mu} - \alpha_\mu F_{\nu\omega}\}\alpha_\lambda \end{aligned}$$

which implies that

$$\begin{aligned} D_{\omega\nu\mu\lambda} &= P_{\omega\mu}F_{\nu\lambda} - P_{\nu\mu}F_{\omega\lambda} + g_{\omega\mu}P_{\nu\varepsilon}F^\varepsilon_\lambda - g_{\nu\mu}P_{\omega\varepsilon}F^\varepsilon_\lambda \\ &- P_{\omega\lambda}F_{\nu\mu} + P_{\nu\lambda}F_{\omega\mu} - g_{\omega\lambda}P_{\nu\varepsilon}F^\varepsilon_\mu + g_{\nu\lambda}P_{\omega\varepsilon}F^\varepsilon_\mu \ , \end{aligned} \tag{3.15}$$

whereby

$$P_{\mu\lambda} = -\nabla_\mu\alpha_\lambda - \alpha_\mu\alpha_\lambda + \frac{\|\alpha\|^2}{2}g_{\mu\lambda} \tag{3.16}$$

is a symmetric (0,2)-tensor in an l.c.K-manifold.

We shall use also the tensor $G_{\omega\nu\mu\lambda}$ and we have ,

$$\begin{aligned} G_{\omega\nu\mu\lambda} &= D_{\omega\nu\mu\gamma}F^\gamma_\lambda \\ &= \{-R^\varepsilon_{\omega\nu\mu}F_{\varepsilon\gamma} + R^\varepsilon_{\omega\nu\gamma}F_{\varepsilon\mu}\}F^\gamma_\lambda \\ &= -R^\varepsilon_{\omega\nu\mu}g_{\varepsilon\lambda} - R^\varepsilon_{\omega\nu\gamma}F_{\mu\varepsilon}F^\gamma_\lambda \end{aligned} \tag{3.17}$$

and so

$$G_{\omega\nu\mu\lambda} = -R_{\omega\nu\mu\lambda} + R_{\omega\nu\varepsilon\gamma}F^\varepsilon_\mu F^\gamma_\lambda \ . \tag{3.18}$$

Making use of the tensor $H_{\mu\lambda}$ [14], defined by

$$H_{\mu\lambda} = \frac{1}{2}R_{\mu\lambda\varepsilon\gamma}F^{\varepsilon\gamma} \ , \tag{3.19}$$

by using (**3.14**) we get

$$D^\varepsilon_{\omega\varepsilon\lambda} = D_{\omega\varepsilon\lambda\nu}g^{\nu\varepsilon} = -R^\varepsilon_\omega F_{\varepsilon\lambda} - H_{\omega\lambda} \tag{3.20}$$

and also

$$\begin{aligned} D^{\varepsilon}_{\omega\varepsilon\lambda} &= D_{\omega\varepsilon\lambda\nu}g^{\nu\varepsilon} = \{\nabla_\omega\nabla_\varepsilon F_{\lambda\nu} - \nabla_\varepsilon\nabla_\omega F_{\lambda\nu}\}g^{\nu\varepsilon} \\ &= \nabla_\omega\nabla_\varepsilon F^{\varepsilon}_{\lambda} - \nabla_\varepsilon\nabla_\omega F^{\varepsilon}_{\lambda} \ . \end{aligned}$$

Consequently

$$D^{\varepsilon}_{\omega\varepsilon\lambda} = \nabla_\omega\{-\beta_\lambda\delta^{\varepsilon}_{\varepsilon} + \beta^{\varepsilon}g_{\varepsilon\lambda} - \alpha_\lambda F_{\varepsilon\gamma}g^{\varepsilon\gamma} + \alpha^{\varepsilon}F_{\varepsilon\lambda}\} - \nabla_\varepsilon\{-\beta_\lambda\delta^{\varepsilon}_{\omega} + \beta^{\varepsilon}g_{\omega\lambda} - \alpha_\lambda F^{\varepsilon}_{\omega} + \alpha^{\varepsilon}F_{\omega\lambda}\}$$

and by (**3.7**) and the fact $F_{\varepsilon\gamma}g^{\varepsilon\gamma} = 0$, we have

$$D^{\varepsilon}_{\omega\varepsilon\lambda} = (-2n+3)\nabla_\omega\beta_\lambda + \nabla_\varepsilon\alpha_\lambda F^{\varepsilon}_{\omega} - \nabla_\varepsilon\alpha^{\varepsilon}F_{\omega\lambda} + (\nabla_\varepsilon F^{\varepsilon}_{\omega})\alpha_\lambda - (\nabla_\varepsilon F_{\omega\lambda})\alpha^{\varepsilon} \ .$$

Taking into account (**3.3**) and (**3.6**) , we obtain

$$\begin{aligned} D^{\varepsilon}_{\omega\varepsilon\lambda} &= (-2n+3)\{-\beta_\omega\alpha_\lambda + \beta_\lambda\alpha_\omega - \|\alpha\|^2F_{\omega\lambda} + \nabla_\omega\alpha^{\varepsilon}F_{\omega\lambda} + \nabla_\varepsilon\alpha_\lambda F^{\varepsilon}_{\omega}\} \\ &- \nabla_\varepsilon\alpha^{\varepsilon}F_{\omega\lambda} + \{-\beta_\omega\delta^{\varepsilon}_{\varepsilon} + 2\beta_\omega\}\alpha_\lambda - \{-\beta_\omega g_{\varepsilon\lambda} + \beta_\lambda g_{\varepsilon\omega} - \alpha_\omega F_{\varepsilon\lambda} \\ &+ \alpha_\lambda F_{\varepsilon\omega}\}\alpha^{\varepsilon} \ . \end{aligned}$$

Using (**3.16**) , and the equality $P^{\varepsilon}_{\varepsilon} = -\nabla_\varepsilon\alpha^{\varepsilon} - \|\alpha\|^2 + n\|\alpha\|^2$, the above equation becomes

$$D^{\varepsilon}_{\omega\varepsilon\lambda} = -(2n-3)P_{\omega\varepsilon}F^{\varepsilon}_{\lambda} - P_{\varepsilon\lambda}F^{\varepsilon}_{\omega} + P^{\varepsilon}_{\varepsilon}F_{\omega\lambda} \ . \qquad (\mathbf{3.21})$$

In view of (**3.15**) and contracting (**3.17**) with $g^{\nu\varepsilon}$, we get

$$\begin{aligned} G^{\varepsilon}_{\omega\varepsilon\lambda} = D_{\omega\varepsilon\lambda\gamma}F^{\gamma}_{\nu}g^{\nu\varepsilon} &= \{P_{\omega\lambda}F_{\varepsilon\gamma} - P_{\varepsilon\lambda}F_{\omega\gamma} + g_{\omega\lambda}P_{\varepsilon\kappa}F^{\kappa}_{\gamma} - g_{\varepsilon\lambda}P_{\omega\kappa}F^{\kappa}_{\gamma} \\ &- P_{\omega\gamma}F_{\varepsilon\lambda} + P_{\varepsilon\gamma}F_{\omega\lambda} - g_{\omega\gamma}P_{\varepsilon\kappa}F^{\kappa}_{\lambda} + g_{\varepsilon\gamma}P_{\omega\kappa}F^{\kappa}_{\lambda}\}F^{\gamma}_{\nu}g^{\nu\varepsilon} \\ &= -(2n-3)P_{\omega\lambda} + P_{\varepsilon\kappa}F^{\varepsilon}_{\omega}F^{\kappa}_{\lambda} - P^{\varepsilon}_{\varepsilon}g_{\omega\lambda} \ . \qquad (\mathbf{3.22}) \end{aligned}$$

3.2 Condition to be a Kaehler Manifold in l.c.K-Manifolds

Theorem 3.3 [4]. *In a compact l.c.K-manifold $M(F, g, \alpha)$ $(n \neq 1)$, if*

$$\widetilde{H}^{\varepsilon}_{\varepsilon} - R^{\varepsilon}_{\varepsilon} \ (= G^{\varepsilon\gamma}_{\gamma\varepsilon}) \geqq 0$$

holds good where $\widetilde{H}_{\mu\lambda} = -H_{\mu\varepsilon}F^{\varepsilon}_{\lambda}$, then it is a Kaehler manifold. The inequality $\geqq$ in this case is naturally reduce to $=$.

Proof . From (**3.20**) , we get

$$D^{\varepsilon}_{\mu\varepsilon\omega}F^{\omega}_{\lambda} = -R^{\varepsilon}_{\mu}F_{\varepsilon\omega}F^{\omega}_{\lambda} - H_{\mu\omega}F^{\omega}_{\lambda} = \widetilde{H}_{\mu\lambda} - R_{\mu\lambda} = G^{\varepsilon}_{\mu\varepsilon\lambda}$$

and

$$G^{\varepsilon}_{\mu\varepsilon\lambda} = \widetilde{H}_{\mu\lambda} - R_{\mu\lambda} = -(2n-3)P_{\mu\lambda} + P_{\varepsilon\gamma}F^{\varepsilon}_{\mu}F^{\gamma}_{\lambda} - P^{\varepsilon}_{\varepsilon}g_{\mu\lambda} \ . \qquad (\mathbf{3.23})$$

Contracting the above equation with $g^{\lambda\mu}$, we have

$$\begin{aligned}\widetilde{H}^{\mu}_{\mu} - R^{\mu}_{\mu} &= -(2n-3)\Big\{ -\nabla_{\mu}\alpha_{\lambda} - \alpha_{\mu}\alpha_{\lambda} + \frac{\|\alpha\|^2}{2}g_{\mu\lambda}\Big\}g^{\lambda\mu} \\ &+ \Big\{ -\nabla_{\varepsilon}\alpha_{\gamma} - \alpha_{\varepsilon}\alpha_{\gamma} + \frac{\|\alpha\|^2}{2}g_{\varepsilon\gamma}\Big\}g^{\varepsilon\gamma} \\ &- \Big\{ -\nabla_{\varepsilon}\alpha^{\varepsilon} - \|\alpha\|^2 + n\|\alpha\|^2\Big\}g_{\mu\lambda}g^{\lambda\mu}\end{aligned}$$

and consequently

$$\begin{aligned}\widetilde{H}^{\mu}_{\mu} - R^{\mu}_{\mu} &= (2n-3)\nabla_{\mu}\alpha^{\mu} - \nabla_{\varepsilon}\alpha^{\varepsilon} + 2n\nabla_{\varepsilon}\alpha^{\varepsilon} + (2n-3)\alpha_{\mu}\alpha^{\mu} - 2n^2\|\alpha\|^2 \\ &+ 3n\|\alpha\|^2 - \|\alpha\|^2 + n\|\alpha\|^2 + 2n\|\alpha\|^2 - 2n^2\|\alpha\|^2 \\ &= 4n\nabla_{\mu}\alpha^{\mu} - 4\nabla_{\mu}\alpha^{\mu} + (-4n^2 + 8n - 4)\alpha_{\mu}\alpha^{\mu}\end{aligned}$$

which implies

$$\widetilde{H}^{\varepsilon}_{\varepsilon} - R^{\varepsilon}_{\varepsilon} = (2n-2)[2\nabla_{\varepsilon}\alpha^{\varepsilon} - (2n-2)\alpha_{\varepsilon}\alpha^{\varepsilon}] \ . \qquad (\mathbf{3.24})$$

By using Green's theorem, we get

$$2(2n-2)\int_M \nabla_{\varepsilon}\alpha^{\varepsilon}d\sigma - (2n-2)^2\int_M \alpha_{\varepsilon}\alpha^{\varepsilon}d\sigma = 0 \ ,$$

whereby $d\sigma$ is the volume element of the space and

$$\int_M \alpha_{\varepsilon}\alpha^{\varepsilon}d\sigma = 0 \ .$$

Hence, $\alpha = 0$, which means that the manifold is Kaehlerian. ■

Proposition 3.4 [4]. *If an 2n-dimensional l.c.K-manifold $M(F, g, \alpha)$ $(n \neq 1)$ is an Einstein space, then $P_{\mu\lambda}$ is hybrid, i.e.* ,

$$(\nabla_{\mu}\alpha_{\varepsilon} + \alpha_{\mu}\alpha_{\varepsilon})F^{\varepsilon}_{\lambda} + (\nabla_{\lambda}\alpha_{\varepsilon} + \alpha_{\lambda}\alpha_{\varepsilon})F^{\varepsilon}_{\mu} = 0 \ . \qquad (\mathbf{3.25})$$

Proof . Using the skew-symmetric property of $H_{\mu\lambda}$ in (**3.20**) , we get

$$-H_{\mu\lambda} = R^{\varepsilon}_{\mu}F_{\varepsilon\lambda} - (2n-3)P_{\mu\varepsilon}F^{\varepsilon}_{\lambda} - P_{\lambda\varepsilon}F^{\varepsilon}_{\mu} - P^{\varepsilon}_{\varepsilon}F_{\mu\lambda} = H_{\lambda\mu}$$

and then

$$R^{\varepsilon}_{\mu}F_{\varepsilon\lambda} - (2n-3)P_{\mu\varepsilon}F^{\varepsilon}_{\lambda} - P_{\lambda\varepsilon}F^{\varepsilon}_{\mu} - P^{\varepsilon}_{\varepsilon}F_{\mu\lambda} = -R^{\varepsilon}_{\lambda}F_{\varepsilon\mu} + (2n-3)P_{\lambda\varepsilon}F^{\varepsilon}_{\mu} + P_{\mu\varepsilon}F^{\varepsilon}_{\lambda} + P^{\varepsilon}_{\varepsilon}F_{\lambda\mu}$$

which gives

$$R_{\mu\varepsilon}F^{\varepsilon}_{\lambda} + R_{\lambda\varepsilon}F^{\varepsilon}_{\mu} - (2n-2)(P_{\mu\varepsilon}F^{\varepsilon}_{\lambda} + P_{\lambda\varepsilon}F^{\varepsilon}_{\mu}) = 0 \ . \qquad (\mathbf{3.26})$$

Now suppose that the l.c.K-manifold $M(F, g, \alpha)$ $(n \neq 1)$ is an Einstein space, then

$$R_{\mu\varepsilon}F^{\varepsilon}_{\lambda} + R_{\lambda\varepsilon}F^{\varepsilon}_{\mu} = 0 \ .$$

From (**3.26**) , we have

$$P_{\mu\varepsilon}F^{\varepsilon}_{\lambda} + P_{\lambda\varepsilon}F^{\varepsilon}_{\mu} = 0 \ , \tag{**3.27**}$$

which means $P_{\mu\lambda}$ is hybrid. In view of (**3.16**) , we have

$$\Big\{ -\nabla_{\mu}\alpha_{\varepsilon} - \alpha_{\mu}\alpha_{\varepsilon} + \frac{\|\alpha\|^2}{2}g_{\mu\varepsilon}\Big\}F^{\varepsilon}_{\lambda} + \Big\{ -\nabla_{\lambda}\alpha_{\varepsilon} - \alpha_{\lambda}\alpha_{\varepsilon} + \frac{\|\alpha\|^2}{2}g_{\lambda\varepsilon}\Big\}F^{\varepsilon}_{\mu} = 0$$

and therefore

$(\nabla_{\mu}\alpha_{\varepsilon} + \alpha_{\mu}\alpha_{\varepsilon})F^{\varepsilon}_{\lambda} + (\nabla_{\lambda}\alpha_{\varepsilon} + \alpha_{\lambda}\alpha_{\varepsilon})F^{\varepsilon}_{\mu} = 0$. ■

Corollary 3.5 [4]. *If an l.c.K-manifold $M(F, g, \alpha)$ $(n \neq 1)$ with $\nabla\alpha = 0$ is an Einstein space, it must be a Kaehler manifold.*

Proof . In an Einstein space, $P_{\mu\lambda}$ is hybrid, then by (**3.25**) we get $\alpha = 0$, which means that the manifold is a Kaehler manifold. ■

Proposition 3.6 [4]. *In an l.c.K-manifold $M(F, g, \alpha)$ $(n \geq 2)$ with $\nabla\alpha = 0$, $\alpha \neq 0$, it is valid for any vector field X that*

$$\widetilde{H}_{\mu\lambda}X^{\mu}X^{\lambda} - R_{\mu\lambda} \leqq 0.$$

The equality holds only if for $X = f\alpha$, $(f \in C^{\infty}(M))$.

Proof . We consider under the condition $\nabla\alpha = 0$, $\alpha \neq 0$. Taking into account (**3.23**) and (**3.16**) , we obtain

$$\widetilde{H}_{\mu\lambda} - R_{\mu\lambda} = (2n-3)\alpha_{\mu}\alpha_{\lambda} - \beta_{\mu}\beta_{\lambda} - (2n-3)\|\alpha\|^2 g_{\mu\lambda} \ ,$$

and by transvecting with a vector X , we get

$$(\widetilde{H}_{\mu\lambda} - R_{\mu\lambda})X^{\mu}X^{\lambda} = (2n-3)\alpha_{\mu}\alpha_{\lambda}X^{\mu}X^{\lambda} - \beta_{\mu}\beta_{\lambda}X^{\mu}X^{\lambda} - (2n-3)\|\alpha\|^2 X^{\mu}X^{\lambda}g_{\mu\lambda}$$

$$= (2n-3)\{(\alpha_{\varepsilon}X^{\varepsilon})^2 - \|\alpha\|^2|X|^2\} - (\beta_{\varepsilon}X^{\varepsilon})^2$$

which shows that $\widetilde{H}_{\mu\lambda} - R_{\mu\lambda}$ is negative semidefinite.

If $X = f\alpha$, $(f \in C^{\infty}(M))$, then

$$\begin{aligned}(\widetilde{H}_{\mu\lambda} - R_{\mu\lambda})X^{\mu}X^{\lambda} &= (2n-3)\alpha_{\mu}\alpha_{\lambda}f^{\mu}\alpha^{\mu}f^{\lambda}\alpha^{\lambda} - \beta_{\mu}\beta_{\lambda}f^{\mu}\alpha^{\mu}f^{\lambda}\alpha^{\lambda} \\ &- (2n-3)\|\alpha\|^2 f^{\mu}\alpha^{\mu}f^{\lambda}\alpha^{\lambda}g_{\mu\lambda} \ .\end{aligned}$$

Thus it follows

$$(\widetilde{H}_{\mu\lambda} - R_{\mu\lambda})X^\mu X^\lambda = (2n-3)\Big\{2\|\alpha\|^2 f^\mu f^\lambda - 2\|\alpha\|^2 f^\mu f^\lambda\Big\} = 0 \ . \qquad \blacksquare$$

3.3 Riemannian Curvature Tensor in l.c.K-Manifolds

In this section we treat with the following relation [13]:

$$R_{abcd}F^a_\mu F^b_\lambda F^c_\omega F^d_\nu = R_{\mu\lambda\omega\nu} \ . \qquad (\mathbf{3.28})$$

Now let

$$g^* = e^{-2\rho}g \ , \qquad P_{\mu\lambda} = -\nabla_\mu\rho_\lambda - \rho_\mu\rho_\lambda + \frac{1}{2}\rho_\varepsilon\rho^\varepsilon g_{\mu\lambda} \ ,$$

whereby ρ_λ is the differential of ρ. Since g^* and g are conformally related, then we have [1]

$$e^{-2\rho}R^*_{\mu\lambda\omega\nu} - R_{\mu\lambda\omega\nu} = P_{\mu\omega}g_{\lambda\nu} - P_{\lambda\omega}g_{\mu\nu} + g_{\mu\omega}P_{\lambda\nu} - g_{\lambda\omega}P_{\mu\nu} \ ,$$

whereby R^* is the Riemannian curvature tensor of g^*.

Proposition 3.7 [4]. *In a real 2n-dimensional (n >1) l.c.K-manifold $M(F,g,\alpha)$, the tensor field P is hybrid , i.e.* ,

$$P_{\mu\varepsilon}F^\varepsilon_\lambda + P_{\lambda\varepsilon}F^\varepsilon_\mu = 0$$

if and only if the Ricci tensor is hybrid.

Theorem 3.8 [4]. *In an l.c.K-manifold $M(F,g,\alpha)$ $(n \neq 1)$* (**3.28**) *holds good if and only if $\nabla_\mu\alpha_\lambda - \alpha_\mu\alpha_\lambda$ (or Ricci tensor) is hybrid.*

Proof . We assume that (**3.28**) holds. Choose the F-base $\{e_1,...,e_m,e_{m+1},...,e_{2m}\}$, whereby $e_{m+a} = Fe_a$, $a = 1,2,...,m$ $(= n)$, we write

$$R_{\lambda\mu^*} = \Sigma R_{\omega\lambda\mu^*\omega} \ (= \Sigma R(e_\omega, e_\lambda, Fe_\mu, e_\omega)) \ ,$$

whereby μ^*- component means the component for Fe_μ, $\mu = 1,2,...,2m$, so that

$$R_{\lambda\mu^*} = R(e_\lambda, Fe_\mu) = R_{\lambda\varepsilon}F^\varepsilon_\mu$$

and also

$$R_{\lambda\mu^*} = \Sigma R_{\omega\lambda\mu^*\omega} = -\Sigma R_{\omega^*\mu\lambda^*\omega^*} = -R_{\lambda^*\mu} = -R_{\mu\varepsilon}F^\varepsilon_\lambda \ .$$

Then it follows

$$R_{\lambda\mu^*} + R_{\lambda^*\mu} = 0$$

and consequently

$$R_{\lambda\varepsilon}F^{\varepsilon}_{\mu} + R_{\mu\varepsilon}F^{\varepsilon}_{\lambda} = 0 \ .$$

From the above equation Ricci tensor is hybrid, and hence by Proposition 3.7 we get that P is hybrid.

Conversely, if $\nabla_\mu\alpha_\lambda - \alpha_\mu\alpha_\lambda$ is hybrid, then P is hybrid, i.e.,

$$P_{\varepsilon\omega}F^{\varepsilon}_{\mu}F^{\omega}_{\lambda} - P_{\mu\lambda} = 0$$

and

$$\{e^{-2\rho}R^*_{abcd} - R_{abcd}\}F^a_\mu F^b_\lambda F^c_\omega F^d_\nu = \{P_{ac}g_{bd} - P_{bc}g_{ad} + g_{ac}P_{bd} - g_{bc}P_{ad}\}F^a_\mu F^b_\lambda F^c_\omega F^d_\nu \ .$$

Since we have

$$g_{ab}F^a_\mu F^b_\lambda = g_{\mu\lambda} \ , \qquad P_{ab}F^a_\mu F^b_\lambda = P_{\mu\lambda}$$

and by using (**3.28**) , we write

$$\{e^{-2\rho}R^*_{\mu\lambda\omega\nu} - R_{abcd}\}F^a_\mu F^b_\lambda F^c_\omega F^d_\nu = P_{\mu\omega}g_{\lambda\nu} - P_{\lambda\omega}g_{\mu\nu} + g_{\mu\omega}P_{\lambda\nu} - g_{\lambda\omega}P_{\mu\nu} \ .$$

Hence from the above equation, we obtain

$$R_{abcd}F^a_\mu F^b_\lambda F^c_\omega F^d_\nu = R_{\mu\lambda\omega\nu} \ .$$

Thus, (**3.28**) holds in an l.c.K-manifold because it holds in a Kaehler manifold and g is conformal to a Kaehler metric g^* with $P_{\mu\lambda} = -\nabla_\mu\alpha_\lambda - \alpha_\mu\alpha_\lambda + \frac{\|\alpha\|^2}{2}g_{\mu\lambda}$. ■

Theorem 3.9 [4] . *In an l.c.K-manifold $M(F, g, \alpha)$ whose $\nabla_\mu\alpha_\lambda - \alpha_\mu\alpha_\lambda$ is hybrid, if the holomorphic sectional curvature at $p \in M$ is constant c, then*

$$\begin{aligned} 4R_{\omega\nu\mu\lambda} &= c(g_{\omega\lambda}g_{\nu\mu} - g_{\omega\mu}g_{\nu\lambda} + F_{\omega\lambda}F_{\nu\mu} - F_{\omega\mu}F_{\nu\lambda} - 2F_{\omega\nu}F_{\mu\lambda}) \\ &+ 3(P_{\omega\lambda}g_{\nu\mu} - P_{\omega\mu}g_{\nu\lambda} + g_{\omega\lambda}P_{\nu\mu} - g_{\omega\mu}P_{\nu\lambda}) - \{\widetilde{P}_{\omega\lambda}F_{\nu\mu} \\ &- \widetilde{P}_{\omega\mu}F_{\nu\lambda} + F_{\mu\lambda}\widetilde{P}_{\nu\mu} - F_{\omega\mu}\widetilde{P}_{\nu\lambda} - 2(\widetilde{P}_{\omega\nu}F_{\mu\lambda} + F_{\omega\nu}\widetilde{P}_{\mu\lambda})\} \qquad (\mathbf{3.29}) \end{aligned}$$

at $p \in M$ where $P_{\mu\lambda} = -\nabla_\mu\alpha_\lambda - \alpha_\mu\alpha_\lambda + \frac{\|\alpha\|^2}{2}g_{\mu\lambda}$ and $\widetilde{P}_{\mu\lambda} = P^{\varepsilon}_{\mu}F_{\varepsilon\lambda}$.

Proof . Let K be the holomorphic sectional curvature of the section {X, FX} at each $p \in M(c)$, and $X \in T_pM$. Then we write

$$K = -\frac{R_{\varepsilon\nu\gamma\lambda}F^{\varepsilon}_{\omega}X^{\omega}X^{\nu}F^{\gamma}_{\mu}X^{\mu}X^{\lambda}}{g_{\omega\nu}X^{\omega}X^{\nu}g_{\mu\lambda}X^{\mu}X^{\lambda}} = -\frac{R_{\varepsilon\nu\gamma\lambda}F^{\varepsilon}_{\omega}F^{\gamma}_{\mu}}{g_{\omega\nu}g_{\mu\lambda}} = -\frac{R_{\omega^*\nu\mu^*\lambda}}{g_{\omega\nu}g_{\mu\lambda}} \ . \qquad \textbf{(3.30)}$$

We assume now that the holomorphic sectional curvature at $p \in M$ has the constant value c. Then we write

$$R_{\omega^*\nu\mu^*\lambda} = -c\ (g_{\omega\nu}g_{\mu\lambda}) \ ,$$

$$R_{\nu^*\mu\omega^*\lambda} = -c\ (g_{\nu\mu}g_{\omega\lambda}) \ ,$$

$$R_{\mu^*\omega\nu^*\lambda} = -c\ (g_{\mu\omega}g_{\nu\lambda}) \ .$$

Taking the sum of both sides, we have

$$R_{\omega^*\nu\mu^*\lambda} + R_{\nu^*\mu\omega^*\lambda} + R_{\mu^*\omega\nu^*\lambda} = -c\ (g_{\omega\nu}g_{\mu\lambda} + g_{\nu\mu}g_{\omega\lambda} + g_{\mu\omega}g_{\nu\lambda}) \ . \qquad \textbf{(3.31)}$$

Interchanging $\omega \longleftrightarrow \omega^*$, $\mu \longleftrightarrow \mu^*$, we get

$$R_{\omega\nu\mu\lambda} - R_{\nu^*\mu^*\omega\lambda} - R_{\mu\omega^*\nu^*\lambda} = -c\ (F_{\omega\nu}F_{\mu\lambda} + F_{\mu\nu}F_{\omega\lambda} + g_{\mu\omega}g_{\nu\lambda}) \ . \qquad \textbf{(3.32)}$$

Interchanging $\omega \longleftrightarrow \nu$, we find

$$R_{\nu\omega\mu\lambda} - R_{\omega^*\mu^*\nu\lambda} - R_{\mu\nu^*\omega^*\lambda} = -c\ (F_{\nu\omega}F_{\mu\lambda} + F_{\mu\omega}F_{\nu\lambda} + g_{\mu\nu}g_{\omega\lambda}) \ . \qquad \textbf{(3.33)}$$

Subtracting (**3.33**) from (**3.32**) , we have

$$\begin{aligned} 2R_{\omega\nu\mu\lambda} + R_{\mu^*\nu^*\omega\lambda} + R_{\omega^*\mu^*\nu\lambda} - R_{\mu\omega^*\nu^*\lambda} - R_{\nu^*\mu\omega^*\lambda} &= -c\ (2F_{\omega\nu}F_{\mu\lambda} + F_{\mu\nu}F_{\omega\lambda} - F_{\mu\omega}F_{\nu\lambda} \\ &+ g_{\mu\omega}g_{\nu\lambda} - g_{\mu\nu}g_{\omega\lambda}) \ . \end{aligned}$$

By using the first Bianchi identity , we obtain

$$\begin{aligned} 2R_{\omega\nu\mu\lambda} + R_{\mu^*\nu^*\omega\lambda} + R_{\omega^*\mu^*\nu\lambda} + R_{\omega^*\nu^*\mu\lambda} &= c\ (g_{\mu\nu}g_{\omega\lambda} - g_{\mu\omega}g_{\nu\lambda} + F_{\nu\mu}F_{\omega\lambda} \\ &- F_{\omega\mu}F_{\nu\lambda} - 2F_{\omega\nu}F_{\mu\lambda}) \ . \qquad \textbf{(3.34)} \end{aligned}$$

Since g^* is Keahler metric then, by virtue of (**3.28**) , we have

$$R^*_{\omega^*\nu^*\mu\lambda} = R^*_{\omega\nu\mu\lambda} \qquad \textbf{(3.35)}$$

or

$$R^*_{\omega\nu\mu^*\lambda^*} = R^*_{\omega\nu\mu\lambda} \ . \qquad \textbf{(3.36)}$$

In view of (**2.22**) , (**3.35**) and (**3.36**) , we obtain

$$R^{*}_{\omega^{*}\nu^{*}\mu\lambda} = R_{\omega^{*}\nu^{*}\mu\lambda} - P_{\nu^{*}\mu}g_{\omega^{*}\lambda} + P_{\omega^{*}\mu}g_{\nu^{*}\lambda} - g_{\nu^{*}\mu}P_{\omega^{*}\lambda} + g_{\omega^{*}\mu}P_{\nu^{*}\lambda}$$

$$= R_{\omega\nu\mu\lambda} - P_{\nu\mu}g_{\omega\lambda} + P_{\omega\mu}g_{\nu\lambda} - g_{\nu\mu}P_{\omega\lambda} + g_{\omega\mu}P_{\nu\lambda} \qquad (\mathbf{3.37})$$

whereby $P_{\mu^{*}\lambda} = P_{\lambda\varepsilon}F^{\varepsilon}_{\mu} = \widetilde{P}_{\mu\lambda}$ and $g_{\mu^{*}\lambda} = F^{\varepsilon}_{\mu}g_{\varepsilon\lambda} = F_{\mu\lambda}$.

From (**3.37**) , we have

$$\begin{aligned} R_{\mu^{*}\nu^{*}\omega\lambda} &= R_{\mu\nu\omega\lambda} - P_{\nu\omega}g_{\mu\lambda} + P_{\mu\omega}g_{\nu\lambda} - g_{\nu\omega}P_{\mu\lambda} + g_{\mu\omega}P_{\nu\lambda} \\ &+ \widetilde{P}_{\nu\omega}F_{\mu\lambda} - \widetilde{P}_{\mu\omega}F_{\nu\lambda} + F_{\nu\omega}\widetilde{P}_{\mu\lambda} - F_{\mu\omega}\widetilde{P}_{\nu\lambda} \ , \end{aligned}$$

$$\begin{aligned} R_{\omega^{*}\mu^{*}\nu\lambda} &= R_{\omega\mu\nu\lambda} - P_{\mu\nu}g_{\omega\lambda} + P_{\omega\nu}g_{\mu\lambda} - g_{\mu\nu}P_{\omega\lambda} + g_{\omega\nu}P_{\mu\lambda} \\ &+ \widetilde{P}_{\mu\nu}F_{\omega\lambda} - \widetilde{P}_{\omega\nu}F_{\mu\lambda} + F_{\mu\nu}\widetilde{P}_{\omega\lambda} - F_{\omega\nu}\widetilde{P}_{\mu\lambda} \ , \end{aligned}$$

$$\begin{aligned} R_{\omega^{*}\nu^{*}\mu\lambda} &= R_{\omega\nu\mu\lambda} - P_{\nu\mu}g_{\omega\lambda} + P_{\omega\mu}g_{\nu\lambda} - g_{\nu\mu}P_{\omega\lambda} + g_{\omega\mu}P_{\nu\lambda} \\ &+ \widetilde{P}_{\nu\mu}F_{\omega\lambda} - \widetilde{P}_{\omega\mu}F_{\nu\lambda} + F_{\nu\mu}\widetilde{P}_{\omega\lambda} - F_{\omega\mu}\widetilde{P}_{\nu\lambda} \end{aligned}$$

and so

$$2R_{\omega\nu\mu\lambda} + R_{\mu\nu\omega\lambda} + R_{\omega\mu\nu\lambda} + R_{\omega\nu\mu\lambda} = 4R_{\omega\nu\mu\lambda} \ .$$

Then we get

$$\begin{aligned} 4R_{\omega\nu\mu\lambda} &= c(g_{\mu\nu}g_{\omega\lambda} - g_{\mu\omega}g_{\nu\lambda} + F_{\nu\mu}F_{\omega\lambda} - F_{\omega\mu}F_{\nu\lambda} - 2F_{\omega\nu}F_{\mu\lambda}) \\ &+ P_{\nu\omega}g_{\mu\lambda} - P_{\mu\omega}g_{\nu\lambda} + g_{\nu\omega}P_{\mu\lambda} - g_{\mu\omega}P_{\nu\lambda} + P_{\mu\nu}g_{\omega\lambda} \\ &- P_{\omega\nu}g_{\mu\lambda} + g_{\mu\nu}P_{\omega\lambda} - g_{\omega\nu}P_{\mu\lambda} + P_{\nu\mu}g_{\omega\lambda} - P_{\omega\mu}g_{\nu\lambda} \\ &+ g_{\nu\mu}P_{\omega\lambda} - g_{\omega\mu}P_{\nu\lambda} - \widetilde{P}_{\nu\omega}F_{\mu\lambda} + \widetilde{P}_{\mu\omega}F_{\nu\lambda} - F_{\nu\omega}\widetilde{P}_{\mu\lambda} \\ &+ \mu\omega\widetilde{P}_{\nu\lambda} - \widetilde{P}_{\mu\nu}F_{\omega\lambda} + \widetilde{P}_{\omega\nu}F_{\mu\lambda} - F_{\mu\nu}\widetilde{P}_{\omega\lambda} + F_{\omega\nu}\widetilde{P}_{\mu\lambda} \\ &- \widetilde{P}_{\nu\mu}F_{\omega\lambda} + \widetilde{P}_{\omega\mu}F_{\nu\lambda} - F_{\nu\mu}\widetilde{P}_{\omega\lambda} + F_{\omega\mu}\widetilde{P}_{\nu\lambda} \qquad (\mathbf{3.38}) \end{aligned}$$

and

$$\begin{aligned} 4R_{\omega\nu\mu\lambda} &= c(g_{\mu\nu}g_{\omega\lambda} - g_{\mu\omega}g_{\nu\lambda} + F_{\nu\mu}F_{\omega\lambda} - F_{\omega\mu}F_{\nu\lambda} - 2F_{\omega\nu}F_{\mu\lambda}) \\ &+ 2(P_{\omega\lambda}g_{\nu\mu} - P_{\omega\mu}g_{\nu\lambda} + g_{\omega\lambda}P_{\nu\mu} - g_{\omega\mu}P_{\nu\lambda}) \\ &+ 2(\widetilde{P}_{\omega\nu}F_{\mu\lambda} + F_{\omega\nu}\widetilde{P}_{\mu\lambda}) \ . \qquad (\mathbf{3.39}) \end{aligned}$$

Consequently, we have

$$P_{\omega\lambda}g_{\nu\mu} - P_{\omega\mu}g_{\nu\lambda} + g_{\omega\lambda}P_{\nu\mu} - g_{\omega\mu}P_{\nu\lambda} = \widetilde{P}_{\omega\lambda}F_{\nu\mu} - \widetilde{P}_{\omega\mu}F_{\nu\lambda} + F_{\omega\lambda}\widetilde{P}_{\nu\mu} - F_{\omega\mu}\widetilde{P}_{\nu\lambda}$$

which completes the proof. ■

3.4 Locally Conformal Kaehler Space Form

An l.c.K-manifold $M(F, g, \alpha)$ is called *an l.c.K-space form* if it has a constant holomorphic sectional curvature. We know that the Riemannian curvature tensor R of an l.c.K-space form with the constant holomorphic sectional curvature c is given by (**3.29**) .

Proposition 3.10 [8] . *A 4-dimensional l.c.K-space form in which the tensor field $P_{\mu\lambda}$ is hybrid is an Einstein one and the scalar field P is constant . Especially, if the manifold is compact, then the length of the Lee vector field is constant.*

Proof . Let $M(c)$ be a real 2n-dimensional l.c.K-space form with constant holomorphic sectional curvature c.

Contracting (**3.29**) with $g^{\omega\lambda}$, we have

$$4R_{\omega\nu\mu\lambda}g^{\omega\lambda} = c\{2(n+1)g_{\nu\mu}\} + 3(Pg_{\nu\mu} - 2P_{\nu\mu} + 2nP_{\nu\mu}) - 6P_{\nu\mu}$$

and

$$4R_{\nu\mu} = \{2(n+1)c + 3P\}g_{\nu\mu} + 6(n-2)P_{\nu\mu} \ , \tag{3.40}$$

whereby the scalar field P is given by $P = P_{\mu\lambda}g^{\mu\lambda}$.

Contracting (**3.16**) with $g^{\mu\lambda}$, we get

$$P = -\nabla_{\varepsilon}\alpha^{\varepsilon} + (n-1)\|\alpha\|^2 \ . \tag{3.41}$$

Since $P_{\mu\lambda}$ is hybrid , taking into account (3.40) , we have

$$R_{\nu\mu} = \frac{3}{4}\ (2c + P)g_{\mu\lambda} \ .$$

Since the holomorphic sectional curvature c and the scalar field P are constant it is clear that $(2c + P)$ is constant. This shows that the l.c.K-space form is Einstein.

Especially, if the manifold is compact , by integrating (3.41) , we get

$$\int_M P d\sigma = \int_M \{-\nabla_{\varepsilon}\alpha^{\varepsilon} + (n-1)\|\alpha\|^2\} d\sigma \ .$$

Using Green's theorem we write

$$\int_M P d\sigma = (n-1)\int_M \|\alpha\|^2 d\sigma$$

and

$$\frac{P}{n-1} = \|\alpha\|^2 \ .$$

Since the scalar field P is constant, $\|\alpha\|$ is constant. ■

Theorem 3.11 [8] . *A real 2n-dimensional* $(n \neq 2)$ *l.c.K-space form* $M(c)$ *in which the tensor field* $P_{\mu\lambda}$ *is hybrid, is an Einstein one if and only if the tensor field* $P_{\mu\lambda}$ *is proportional to* $g_{\mu\lambda}$*. And then the scalar field P is constant.*

Proof . If the tensor field $P_{\mu\lambda}$ is proportional to $g_{\mu\lambda}$ and P is constant, then $P_{\mu\lambda}$ can be written as

$$P_{\mu\lambda} = \frac{P}{2n} g_{\mu\lambda} \ . \tag{3.42}$$

Substituting the above equation into (3.40) , we obtain

$$\begin{aligned} 4R_{\nu\mu} &= \Big\{2(n+1)c + 3P + 6(n-1)\frac{P}{2n}\Big\} g_{\nu\mu} \\ &= \Big\{2(n+1)c + \frac{6(n-1)}{n} P\Big\} g_{\nu\mu} \ , \end{aligned} \tag{3.43}$$

which means that the l.c.K-space form is Einstein.

Assume that the l.c.K-space form $M(c)$ is an Einstein space , say

$$R_{\nu\mu} = C g_{\nu\mu} \qquad and \qquad 4R_{\nu\mu} = 4C g_{\nu\mu} \ .$$

Using (3.40) , we get

$$\{2(n+1)c + 3P\} g_{\nu\mu} + 6(n-2) P_{\nu\mu} = 4C g_{\nu\mu}$$

and so

$$P_{\nu\mu} = \frac{4C - 2(n+1)c - 3p}{6(n-2)} g_{\nu\mu} \ .$$

This shows that $R_{\nu\mu}$ and $g_{\nu\mu}$ are proportional. ■

Corollary 3.12 [8] . *A real 2n-dimensional* $(n \neq 2)$ *Einstein l.c.K-space form* $M(c)$ *in which the tensor field* $P_{\mu\lambda}$ *is hybrid, is a complex space form if* $P = 0$.

Proposition 3.13 [8] . *In a real 2n-dimensional* $(n \neq 2)$ *l.c.K-space form* $M(c)$ *in which the tensor field* $P_{\mu\lambda}$ *is hybrid, the Riemannian curvature tensor* $R_{\omega\nu\mu\lambda}$ *can be written as*

$$\begin{aligned} R_{\omega\nu\mu\lambda} &= \frac{1}{2(n-2)}\Big[-\frac{2(n-2)c+S}{2(n-1)}(g_{\omega\lambda}g_{\nu\mu}-g_{\omega\mu}g_{\nu\lambda}) \\ &+ \frac{2(n-2)(2n-1)c+S}{6(n-1)}(F_{\omega\lambda}F_{\nu\mu}-F_{\omega\mu}F_{\nu\lambda}-2F_{\omega\nu}F_{\mu\lambda}) \\ &+ R_{\omega\lambda}g_{\nu\mu}-R_{\omega\mu}g_{\nu\lambda}+g_{\omega\lambda}R_{\nu\mu}-g_{\omega\mu}R_{\nu\lambda}+\frac{1}{3}\{\widetilde{R}_{\omega\lambda}F_{\nu\mu} \\ &- \widetilde{R}_{\omega\mu}F_{\nu\lambda}+F_{\omega\lambda}\widetilde{R}_{\nu\mu}-F_{\omega\mu}\widetilde{R}_{\nu\lambda}-2(\widetilde{R}_{\omega\nu}F_{\mu\lambda}+_{\omega\nu}\widetilde{R}_{\mu\lambda})\}\Big] \end{aligned} \tag{3.44}$$

which does not include $P_{\mu\lambda}$ *and* $\widetilde{P}_{\mu\lambda}$*, where S denotes the scalar curvature with respect to* $g_{\mu\lambda}$ *and we put* $\widetilde{R}_{\mu\lambda}=R^{\varepsilon}_{\mu}F_{\varepsilon\lambda}$.

Proof . Contracting (3.40) with $g^{\mu\lambda}$, we get

$$S = n(n+1)c+3(n-1)P\ . \tag{3.45}$$

By virtue of (3.40) and , we obtain

$$\frac{4R_{\mu\lambda}}{2(n-2)}=\frac{2(n+1)(n-1)c+3(n-1)P}{2(n-2)(n-1)}g_{\mu\lambda}+3P_{\mu\lambda}$$

and

$$\frac{4R_{\mu\lambda}}{2(n-2)}=\frac{(n-2)(n+1)c+n(n+1)c+3(n-1)P}{2(n-2)(n-1)}g_{\mu\lambda}+3P_{\mu\lambda}\ .$$

By using (3.45) , we have

$$3P_{\mu\lambda}=\frac{1}{2(n-2)}\Big\{4R_{\mu\lambda}-\frac{(n-2)(n+1)c+S}{n-1}g_{\mu\lambda}\Big\}\ . \tag{3.46}$$

Contracting (3.46) with $g^{\lambda\varepsilon}$, we have

$$3P^{\varepsilon}_{\mu}=\frac{1}{2(n-2)}\Big\{4R^{\varepsilon}_{\mu}-\frac{(n-2)(n+1)c+S}{n-1}\delta^{\varepsilon}_{\mu}\Big\} \tag{3.47}$$

and transvecting (3.47) with $F_{\varepsilon\lambda}$, we obtain

$$3\widetilde{P}_{\mu\lambda}=\frac{1}{2(n-2)}\Big\{4\widetilde{R}_{\mu\lambda}-\frac{(n-2)(n+1)c+S}{n-1}F_{\mu\lambda}\Big\}\ . \tag{3.48}$$

Substituting (3.46) and (3.48) into (**3.29**) , we get

$$\begin{aligned} 4R_{\omega\nu\mu\lambda} &= c\ (g_{\omega\lambda}g_{\nu\mu} - g_{\omega\mu}g_{\nu\lambda} + F_{\omega\lambda}F_{\nu\mu} - F_{\omega\mu}F_{\nu\lambda} - 2F_{\omega\nu}F_{\mu\lambda}) \\ &+ \frac{1}{2(n-2)}\Big[- \frac{(n-2)(n+1)c + S}{n-1} 2(g_{\omega\lambda}g_{\nu\mu} - g_{\omega\mu}g_{\nu\lambda}) \\ &+ 4(R_{\omega\lambda}g_{\nu\mu} - R_{\omega\mu}g_{\nu\lambda} + g_{\omega\lambda}R_{\nu\mu} - g_{\omega\mu}R_{\nu\lambda}) + \frac{1}{3}\Big\{2(F_{\omega\lambda}F_{\nu\mu} \\ &- F_{\omega\mu}F_{\nu\lambda}) - 2(F_{\omega\nu}F_{\mu\lambda} + F\mu\lambda F\omega\nu)\Big\} + 4\Big\{\widetilde{R}_{\omega\lambda}F_{\nu\mu} - \widetilde{R}_{\omega\mu}F_{\nu\lambda} \\ &+ \widetilde{R}_{\nu\mu}F_{\omega\lambda} - \widetilde{R}_{\nu\lambda}F_{\omega\mu} - 2(\widetilde{R}_{\omega\nu}F_{\mu\lambda} + \widetilde{R}_{\mu\lambda}F_{\omega\nu})\Big\}\Big] . \end{aligned} \tag{3.49}$$

By straightforward calculations, we get (3.44) . ■

Theorem 3.14 [8] . *In a real 2n-dimensional* $(n \neq 1, 2)$ *l.c.K-space form* $M(c)$ *in which the tensor field* $P_{\mu\lambda}$ *is hybrid, the scalar curvature S is constant, the length* $\|\alpha\|$ *of the Lee form* α *is non-zero constant if and only if*

$$\nabla_\varepsilon \alpha^\varepsilon + 3\|\alpha\|^2 + 2(n+1)c = 0. \tag{3.50}$$

Proof . Let $M(c)$ be an l.c.K-space form $M(c)$ with the scalar curvature S and the length $\|\alpha\|$ of the Lee form α is non-zero constant.

Now , differentiating (3.40) covariantly, we get

$$4\nabla_\omega R_{\nu\mu} = [2(n+1)c + 3P]\nabla_\omega g_{\nu\mu} + 6(n-2)\nabla_\omega P_{\nu\mu} .$$

Since $\nabla_\omega g_{\nu\mu} = 0$, we have

$$2\nabla_\omega R_{\nu\mu} = 3(n-2)\nabla_\omega P_{\nu\mu} . \tag{3.51}$$

Substituting (**3.16**) into (3.51) , we have

$$2\nabla_\omega R_{\nu\mu} = 3(n-2)\Big\{ - \nabla_\omega(\nabla_\nu\alpha_\mu) - (\nabla_\omega\alpha_\mu)\alpha_\nu - (\nabla_\omega\alpha_\nu)\alpha_\mu + \frac{1}{2}(\nabla_\omega\|\alpha\|^2)g_{\nu\mu}\Big\}. \tag{3.52}$$

Using the Ricci identity and the equality $\nabla_\mu\alpha_\lambda = \nabla_\lambda\alpha_\mu$, we have

$$\begin{aligned} 2(\nabla_\omega R_{\nu\mu} - \nabla_\nu R_{\omega\mu}) &= 3(n-2)\Big[R^\varepsilon_{\omega\nu\mu}\alpha_\varepsilon + (\nabla_\nu\alpha_\mu)\alpha_\omega - (\nabla_\omega\alpha_\mu)\alpha_\nu \\ &+ \frac{1}{2}\{(\nabla_\omega\|\alpha\|^2)g_{\nu\mu} - (\nabla_\nu\|\alpha\|^2)g_{\mu\omega}\}\Big]. \end{aligned} \tag{3.53}$$

Contracting (3.53) with $g^{\nu\mu}$ and taking into account $2\nabla_\varepsilon R^\varepsilon_\lambda = \nabla_\lambda S$ [14], we obtain

$$2(\nabla_\omega S - \nabla_\nu R^\varepsilon_\omega) = 3(n-2)[R^\varepsilon_\omega\alpha_\varepsilon + (\nabla_\nu\alpha^\nu)\alpha_\omega - (\nabla_\omega\alpha_\mu)\alpha^\mu + (n - \frac{1}{2})\nabla_\omega\|\alpha\|^2]$$

and

$$\nabla_\omega S = 3(n-2)[R^\varepsilon_\omega\alpha_\varepsilon + (\nabla_\varepsilon\alpha^\varepsilon)\alpha_\omega + (n-1)\nabla_\omega\|\alpha\|^2]. \tag{3.54}$$

Since S is constant , $\nabla_\omega S = 0$ and consequently for $n \neq 2$, we get

$$R^\varepsilon_\omega \alpha_\varepsilon + (\nabla_\varepsilon \alpha^\varepsilon)\alpha_\omega + (n-1)\nabla_\omega \|\alpha\|^2 = 0 \ . \tag{3.55}$$

Now contracting (3.40) with $g^{\lambda\varepsilon}$ and transvecting with α_ε, we get

$$4R^\varepsilon_\omega \alpha_\varepsilon = \{2(n+1)c + 3P\}\alpha_\omega + 6(n-2)P_{\omega\lambda}\alpha^\lambda \ . \tag{3.56}$$

From (3.41) , we get

$$3P\alpha_\omega = -3(\nabla_\varepsilon \alpha^\varepsilon)\alpha_\omega + 3(n-1)\|\alpha\|^2 \alpha_\omega \tag{3.57}$$

and transvecting (**3.16**) , with α^λ , we obtain

$$P_{\omega\lambda}\alpha^\lambda = -\frac{1}{2}\nabla_\varepsilon \|\alpha\|^2 - \frac{1}{2}\|\alpha\|^2 \alpha_\omega \ . \tag{3.58}$$

Substituting (3.57) and (3.58) into (3.56) , we have

$$4R^\varepsilon_\omega \alpha_\varepsilon = \{2(n+1)c - 3\nabla_\varepsilon \alpha^\varepsilon + 3\|\alpha\|^2\}\alpha_\omega - (3n-6)\nabla_\omega \|\alpha\|^2 \tag{3.59}$$

and substituting (3.59) into (3.55) , we find

$$\{2(n+1)c + 3\|\alpha\|^2 + \nabla_\varepsilon \alpha^\varepsilon\}\alpha_\omega + (n+2)\nabla_\omega \|\alpha\|^2 = 0 \ . \tag{3.60}$$

Since the length $\|\alpha\|$ of the Lee form α is non-zero constant, we find

$$\nabla_\omega \|\alpha\|^2 = 0$$

and so

$$2(n+1)c + 3\|\alpha\|^2 + \nabla_\varepsilon \alpha^\varepsilon = 0 \ . \tag{3.61}$$

Conversely, suppose that (3.50) fulfills. Then we have

$$2(n+1)c + 3\|\alpha\|^2 + \nabla_\varepsilon \alpha^\varepsilon = 0 \ .$$

By virtue of (3.60) , we get

$$(n+2)\nabla_\omega \|\alpha\|^2 = 0$$

and consequently $\nabla_\omega \|\alpha\|^2 = 0$, which means that $\|\alpha\|$ is non-zero constant. ■

Corollary 3.15 [8] . *Under the same assumption with Theorem 3.14, if the manifold is compact and the length of the Lee form α is non-zero constant then there exists the following relation between c and $\|\alpha\|$;*

$$2(n+1)c + 3\|\alpha\|^2 = 0. \tag{3.62}$$

Proof . By virtue of (3.61) and Green's Theorem , we find

$$\int_M \{2(n+1)c + 3\|\alpha\|^2 + \nabla_\varepsilon \alpha^\varepsilon\} d\sigma = 0$$

and

$$\int_M \{2(n+1)c + 3\|\alpha\|^2\} d\sigma = 0 \ ,$$

which gives (3.62) . ■

4. ON CERTAIN VECTOR FIELDS IN LOCALLY CONFORMAL KAEHLER MANIFOLDS

4.1 Contravariant Almost Analytic Vector Fields in an l.c.K-Manifold

A vector field v^λ in an almost complex manifold is called contravariant almost analytic if it satisfies

$$£(v)F^\lambda_\mu = 0 \ . \tag{4.1}$$

Contracting (**3.3**) with $g^{\gamma\lambda}$, we have

$$\nabla_\varepsilon F^\lambda_\mu = -\beta_\mu \delta^\lambda_\varepsilon + \beta^\lambda g_{\varepsilon\mu} - \alpha_\mu F^\lambda_\varepsilon + \alpha^\lambda F_{\varepsilon\mu} \tag{4.2}$$

and transvecting (**4.2**) with v^ε and taking into account the equality

$$\widetilde{v^\lambda} = F^\lambda_\varepsilon v^\varepsilon \ , \tag{4.3}$$

we obtain

$$v^\varepsilon \nabla_\varepsilon F^\lambda_\mu = -\beta_\mu v^\lambda + +\beta^\lambda v_\mu - \alpha_\mu \widetilde{v^\lambda} + \alpha^\lambda \widetilde{v_\mu} \ . \tag{4.4}$$

Let v^λ be a contravariant almost analytic vector field in an l.c.K-manifold M. Then substituting (**4.4**) into (**2.15**) implies

$$F^\lambda_\varepsilon \nabla_\mu v^\varepsilon - F^\varepsilon_\mu \nabla_\varepsilon v^\lambda - \beta_\mu v^\lambda + \beta^\lambda v_\mu - \alpha_\mu \widetilde{v^\lambda} + \alpha^\lambda \widetilde{v_\mu} = 0. \tag{4.5}$$

Thus, in a locally conformal Kaehler manifold, we define a contravariant almost analytic vector field as a vector which satisfies (**4.5**) .

Proposition 4.1 [8]. *For a contravariant almost analytic vector field v^λ in an l.c.K-manifold, $£(v)g_{\mu\lambda}$ is hybrid, that is*,

$$£(v)g_{\mu\lambda} - F^\varepsilon_\mu F^\gamma_\lambda £(v)g_{\varepsilon\gamma} = 0. \tag{4.6}$$

Proof . Transvecting (**4.5**) with $F_{\gamma\lambda}$ and using (**3.4**) ,we get

$$g_{\varepsilon\gamma}\nabla_\mu v^\varepsilon - F^\varepsilon_\mu F^\nu_\gamma g_{\nu\lambda}\nabla_\varepsilon v^\lambda + \beta_\mu F_{\lambda\gamma}v^\lambda + \beta^\lambda F_{\gamma\lambda}v_\mu + \alpha_\mu F^\nu_\lambda g_{\nu\gamma}F^\lambda_\varepsilon v^\varepsilon - \beta_\gamma \widetilde{v}_\mu = 0$$

and

$$\nabla_\mu v_\lambda - F^\varepsilon_\mu F^\gamma_\lambda \nabla_\varepsilon v_\gamma - \alpha_\mu v_\lambda + \alpha_\lambda v_\mu + \beta_\mu \widetilde{v}_\lambda - \beta_\lambda \widetilde{v}_\mu = 0 \ . \tag{4.7}$$

We put $\gamma \to \mu$, $\mu \to \lambda$, $\varepsilon \to \gamma$ and $\gamma \to \varepsilon$ into (**4.7**) , then

$$\nabla_\lambda v_\mu - F^\gamma_\lambda F^\varepsilon_\mu \nabla_\gamma v_\varepsilon - \alpha_\lambda v_\mu + \alpha_\mu v_\lambda + \beta_\lambda \widetilde{v}_\mu - \beta_\mu \widetilde{v}_\lambda = 0 \ . \tag{4.8}$$

By virtue of (**4.7**) and (**4.8**) , we find

$$\nabla_\mu v_\lambda + \nabla_\lambda v_\mu - F^\varepsilon_\mu F^\gamma_\lambda (\nabla_\varepsilon v_\gamma + \nabla_\gamma v_\varepsilon) = 0 \ .$$

Therefore

$$£(v)g_{\mu\lambda} - F^\varepsilon_\mu F^\gamma_\lambda (£(v)g_{\varepsilon\gamma}) = 0 \ .$$

Hence $£(v)g_{\mu\lambda}$ is hybrid. ■

Theorem 4.2 [8] . *A necessary and sufficient condition for a contravariant almost analytic vector field v^λ in an l.c.K-manifold to be Killing is that the vector field $\widetilde{v}^\lambda$ satisfies*

$$\nabla_\mu \widetilde{v}^\lambda - \nabla_\lambda \widetilde{v}^\mu = 2(\alpha_\mu \widetilde{v}^\lambda - \alpha_\lambda \widetilde{v}^\mu - \langle \alpha, v\rangle F_{\mu\lambda}). \tag{4.9}$$

Proof : Let a contravariant almost analytic vector field v^λ in an l.c.K-manifold be a Killing vector field. Contracting (**4.1**) with $g_{\nu\lambda}$, we get

$$£(v)F_{\mu\lambda} = v^\varepsilon \nabla_\varepsilon F_{\mu\lambda} - F^\varepsilon_\mu \nabla_\varepsilon v_\lambda + F_{\varepsilon\lambda}\nabla_\mu v^\varepsilon \ . \tag{4.10}$$

Using (**4.3**) , we have

$$\nabla_\mu \widetilde{v}_\lambda = F_{\varepsilon\lambda}\nabla_\mu v^\varepsilon + v^\varepsilon(\nabla_\mu F_{\varepsilon\lambda}) \ . \tag{4.11}$$

From (**4.10**) and (**4.11**) , we see

$$v^\varepsilon \nabla_\varepsilon F_{\mu\lambda} - F^\varepsilon_\mu \nabla_\varepsilon v_\lambda + \nabla_\mu \widetilde{v}_\lambda + v^\varepsilon(\nabla_\mu F_{\lambda\varepsilon}) = 0 \ . \tag{4.12}$$

By (**4.11**) , we obtain

$$v^\varepsilon \nabla_\varepsilon F_{\mu\lambda} = v^\varepsilon(\nabla_\varepsilon F_{\mu\lambda} + \nabla_\lambda F_{\varepsilon\mu}) - \nabla_\lambda \widetilde{v}_\mu - F^\varepsilon_\mu \nabla_\lambda v_\varepsilon \ . \tag{4.13}$$

Substituting (**4.13**) into (**4.12**) , we get

$$v^\varepsilon(\nabla_\varepsilon F_{\mu\lambda} + \nabla_\lambda F_{\varepsilon\mu}) - \nabla_\lambda \widetilde{v}_\mu - F^\varepsilon_\mu \nabla_\lambda v_\varepsilon - F^\varepsilon_\mu \nabla_\varepsilon v_\lambda + \nabla_\mu \widetilde{v}_\lambda + v^\varepsilon(\nabla_\mu F_{\lambda\varepsilon}) = 0 \ .$$

Thus we have

$$\nabla_\mu \widetilde{v}_\lambda - \nabla_\lambda \widetilde{v}_\mu - F^\varepsilon_\mu(\nabla_\varepsilon v_\lambda + \nabla_\lambda v_\varepsilon) + v^\varepsilon(\nabla_\varepsilon F_{\mu\lambda} + \nabla_\lambda F_{\varepsilon\mu} + \nabla_\mu F_{\lambda\varepsilon}) = 0 \ . \qquad (\mathbf{4.14})$$

From (**3.3**) , we obtain

$$\begin{aligned} v^\varepsilon[\nabla_\varepsilon F_{\mu\lambda} + \nabla_\lambda F_{\varepsilon\mu} + \nabla_\mu F_{\lambda\varepsilon}] &= 2[-v^\varepsilon \alpha_\mu F_{\varepsilon\lambda} + v^\varepsilon \alpha_\lambda F_{\varepsilon\mu} + \alpha_\varepsilon v^\varepsilon F_{\mu\lambda}] \\ &= 2(-\alpha_\mu \widetilde{v}_\lambda + \alpha_\lambda \widetilde{v}_\mu + \langle \alpha, v\rangle F_{\mu\lambda}) \ . \qquad (\mathbf{4.15}) \end{aligned}$$

Substituting (**4.15**) into (**4.14**) , we obtain

$$\nabla_\mu \widetilde{v}_\lambda - \nabla_\lambda \widetilde{v}_\mu - F^\varepsilon_\mu(\nabla_\varepsilon v_\lambda + \nabla_\lambda v_\varepsilon) = 2(\alpha_\mu \widetilde{v}_\lambda - \alpha_\lambda \widetilde{v}_\mu - \langle \alpha, v\rangle F_{\mu\lambda}) \qquad (\mathbf{4.16})$$

and using (**2.16**) ,we get

$$\nabla_\mu \widetilde{v}_\lambda - \nabla_\lambda \widetilde{v}_\mu = 2(\alpha_\mu \widetilde{v}_\lambda - \alpha_\lambda \widetilde{v}_\mu - \langle \alpha, v\rangle F_{\mu\lambda}) \ . \qquad (\mathbf{4.17})$$

Conversely, we assume that (**4.9**) holds. By using (**4.16**) , we see that

$$F^\varepsilon_\mu(\nabla_\varepsilon v_\lambda + \nabla_\lambda v_\varepsilon) = 0$$

and so

$$\nabla_\varepsilon v_\lambda + \nabla_\lambda v_\varepsilon = 0 \ ,$$

which means the contravariant almost analytic vector field v^λ is Killing in an l.c.k-manifold. ■

Theorem 4.3 [8]. *For a contravariant almost analytic vector field v^λ in an l.c.K-manifold if the vector field $\widetilde{v}^\lambda$ is closed, then the vector fields v_λ and v_λ are expressed by the linear combination of α_λ and β_λ.*

Proof . We assume that the vector field $\widetilde{v}^\lambda$ for a contravariant almost analytic vector field v^λ in an l.c.K-manifold is closed. By using (**4.16**) and (**2.17**) , we have

$$-F^\varepsilon_\mu(\nabla_\varepsilon v_\lambda + \nabla_\lambda v_\varepsilon) = 2\ (\alpha_\mu \widetilde{v}_\lambda - \alpha_\lambda \widetilde{v}_\mu - \langle \alpha, v\rangle F_{\mu\lambda}) \ . \qquad (\mathbf{4.18})$$

Transvecting (**4.18**) with F^μ_γ, we obtain

$$(\nabla_\mu v_\lambda + \nabla_\lambda v_\mu) = -2\ (\beta_\mu \widetilde{v}_\lambda + \alpha_\lambda v_\mu - \langle \alpha, v\rangle g_{\mu\lambda}) \ . \qquad (\mathbf{4.19})$$

Since the left hand side of the above equation is symmetric, we obtain

$$\beta_\mu \widetilde{v}_\lambda - \beta_\lambda \widetilde{v}_\mu + \alpha_\lambda v_\mu - \alpha_\mu v_\lambda = 0 \ . \qquad (\mathbf{4.20})$$

Transvecting (**4.20**) with α^{μ}, we have

$$\begin{aligned} \|\alpha\|^2 v_{\lambda} &= \beta_{\varepsilon} v^{\varepsilon} \beta_{\lambda} + \alpha^{\mu} v_{\mu} \alpha_{\lambda} \\ &= \langle \beta, v \rangle \beta_{\lambda} + \langle \alpha, v \rangle \alpha_{\lambda} \ . \end{aligned} \tag{4.21}$$

Hence we get

$$v_{\lambda} = \frac{\langle \alpha, v \rangle}{\|\alpha\|^2} \alpha_{\lambda} + \frac{\langle \beta, v \rangle}{\|\alpha\|^2} \beta_{\lambda} \ . \tag{4.22}$$

■

Proposition 4.4 [8]. *In an l.c.K-manifold if the vector field $\widetilde{v}_{\lambda}$ for a contravariant almost analytic vector field v^{λ} is closed, then the vector field v^{λ} satisfies*

$$\pounds(v) g_{\mu\lambda} = -\frac{2\langle \alpha, v \rangle}{\|\alpha\|^2} (\beta_{\mu}\beta_{\lambda} + \alpha_{\mu}\alpha_{\lambda} - \|\alpha\|^2 g_{\mu\lambda}). \tag{4.23}$$

Proof . Contracting (**4.22**) with $g^{\varepsilon\lambda}$, we obtain

$$v^{\varepsilon} = \frac{\langle \alpha, v \rangle}{\|\alpha\|^2} \alpha^{\varepsilon} + \frac{\langle \beta, v \rangle}{\|\alpha\|^2} \beta^{\varepsilon} \ . \tag{4.24}$$

Transvecting (**4.24**) with $F_{\varepsilon\lambda}$, we have

$$\widetilde{v}_{\lambda} = \frac{\langle \alpha, v \rangle}{\|\alpha\|^2} \beta_{\lambda} - \frac{\langle \beta, v \rangle}{\|\alpha\|^2} \alpha_{\lambda} \ . \tag{4.25}$$

Substituting (**4.22**) and (**4.25**) into (**4.19**) , we obtain

$$\begin{aligned} \pounds(v) g_{\mu\lambda} &= -2 \Big[\beta_{\mu} \Big(\frac{\langle \alpha, v \rangle}{\|\alpha\|^2} \beta_{\lambda} - \frac{\langle \beta, v \rangle}{\|\alpha\|^2} \alpha_{\lambda} \Big) + \alpha_{\lambda} \Big(\frac{\langle \alpha, v \rangle}{\|\alpha\|^2} \alpha_{\mu} + \frac{\langle \beta, v \rangle}{\|\alpha\|^2} \beta_{\mu} \Big) \\ &- \langle \alpha, v \rangle g_{\mu\lambda} \Big] \\ &= \frac{-2\langle \alpha, v \rangle}{\|\alpha\|^2} (\beta_{\mu}\beta_{\lambda} + \alpha_{\mu}\alpha_{\lambda} - \|\alpha\|^2 g_{\mu\lambda}) \ . \end{aligned} \tag{4.26}$$

■

Proposition 4.5 [8]. *In an l.c.K-manifold if the vector field $\widetilde{v}^{\lambda}$ for a contravariant almost analytic vector field v^{λ} is Killing, then the vector field v_{λ} satisfies*

$$\nabla_{\mu}v_{\lambda} - \nabla_{\lambda}v_{\mu} = 2(\alpha_{\mu}v_{\lambda} - \alpha_{\lambda}v_{\mu} - \langle\beta, v\rangle F_{\mu\lambda}) \ . \tag{4.27}$$

Proof . Let the vector field $\widetilde{v}^{\lambda}$ for a contravariant almost analytic vector field v^{λ} is Killing. In (**4.12**) , if we replace λ by μ , we get

$$v^{\varepsilon}\nabla_{\varepsilon}F_{\lambda\mu} - F^{\varepsilon}_{\lambda}\nabla_{\varepsilon}v_{\mu} + \nabla_{\lambda}\widetilde{v}_{\mu} + v^{\varepsilon}(\nabla_{\lambda}F_{\mu\varepsilon}) = 0 \tag{4.28}$$

and using (**4.12**) and (**4.28**) , we obtain

$$\nabla_{\mu}\widetilde{v}_{\lambda} + \nabla_{\lambda}\widetilde{v}_{\mu} + v^{\varepsilon}(\nabla_{\mu}F_{\lambda\varepsilon} + \nabla_{\lambda}F_{\mu\varepsilon}) - F^{\varepsilon}_{\mu}\nabla_{\varepsilon}v_{\lambda} - F^{\varepsilon}_{\lambda}\nabla_{\varepsilon}v_{\mu} = 0 \ . \tag{4.29}$$

By the straightforward calculations, we have

$$v^{\varepsilon}(\nabla_{\mu}F_{\lambda\varepsilon} + \nabla_{\lambda}F_{\mu\varepsilon}) = \alpha_{\lambda}\widetilde{v}_{\mu} + \alpha_{\mu}\widetilde{v}_{\lambda} - (\beta_{\mu}v_{\lambda} + \beta_{\lambda}v_{\mu}) + 2\langle\alpha, \widetilde{v}\rangle g_{\mu\lambda}$$

and substituting the above equation into (**4.29**) , we find

$$\nabla_{\mu}\widetilde{v}_{\lambda} + \nabla_{\lambda}\widetilde{v}_{\mu} + \alpha_{\lambda}\widetilde{v}_{\mu} + \alpha_{\mu}\widetilde{v}_{\lambda} - (\beta_{\mu}v_{\lambda} + \beta_{\lambda}v_{\mu}) + 2\langle\alpha, \widetilde{v}\rangle g_{\mu\lambda} - F^{\varepsilon}_{\mu}\nabla_{\varepsilon}v_{\lambda} - F^{\varepsilon}_{\lambda}\nabla_{\varepsilon}v_{\mu} = 0. \tag{4.30}$$

Since the vector field $\widetilde{v}^{\lambda}$ is Killing , the equation (**4.29**) implies

$$F^{\varepsilon}_{\mu}\nabla_{\varepsilon}v_{\lambda} + F^{\varepsilon}_{\lambda}\nabla_{\varepsilon}v_{\mu} = \alpha_{\mu}\widetilde{v}_{\lambda} + \alpha_{\lambda}\widetilde{v}_{\mu} - (\beta_{\mu}v_{\lambda} + \beta_{\lambda}v_{\mu}) + 2\langle\alpha, \widetilde{v}\rangle g_{\mu\lambda} \ . \tag{4.31}$$

By using (**4.5**) and (**4.31**) , we find

$$F_{\varepsilon\lambda}\nabla_{\mu}v^{\varepsilon} + F^{\varepsilon}_{\lambda}\nabla_{\varepsilon}v_{\mu} = 2(-\beta_{\mu}v_{\lambda} + \alpha_{\mu}\widetilde{v}_{\lambda} + \langle\alpha, \widetilde{v}\rangle g_{\mu\lambda}).$$

Transvecting the above equation with F^{λ}_{γ} , we get

$$\nabla_{\mu}v_{\lambda} - \nabla_{\lambda}v_{\mu} = 2(\alpha_{\mu}v_{\lambda} - \alpha_{\lambda}v_{\mu} - \langle\beta, v\rangle F_{\mu\lambda}). \qquad \blacksquare$$

4.2 Covariant Almost Analytic Vector Fields in an l.c.K-Manifold

A vector field w_{λ} in a Hermitian manifold is called a covariant almost analytic vector field if it satisfies

$$F^{\gamma}_{\mu}\nabla_{\gamma}w_{\lambda} - F^{\gamma}_{\lambda}\nabla_{\mu}w_{\gamma} - (\nabla_{\mu}F^{\gamma}_{\lambda} - \nabla_{\lambda}F^{\gamma}_{\mu})w_{\gamma} = 0 \tag{4.32}$$

[14] . In view of (**4.2**) , we have

$$(\nabla_{\mu}F^{\gamma}_{\lambda} - \nabla_{\lambda}F^{\gamma}_{\mu}) = -\beta_{\lambda}\delta^{\gamma}_{\mu} - \alpha_{\lambda}F_{\mu\varepsilon}g^{\varepsilon\gamma} + \alpha^{\gamma}F_{\mu\lambda} + \beta_{\mu}\delta^{\gamma}_{\lambda} + \alpha_{\mu}F_{\lambda\varepsilon}g^{\varepsilon\gamma} - \alpha^{\gamma}F_{\lambda\mu}$$

and

$$-w_\gamma(\nabla_\mu F^\gamma_\lambda - \nabla_\lambda F^\gamma_\mu) = \beta_\lambda w_\mu - \beta_\mu w_\lambda + \alpha_\mu \widetilde{w}_\lambda - \alpha_\lambda \widetilde{w}_\mu - 2\langle \alpha, w\rangle F_{\mu\lambda}. \tag{4.33}$$

Let w^λ be a covariant almost analytic vector field in an l.c.K-manifold.

By using (**4.33**) , (**4.32**) implies

$$F^\gamma_\mu \nabla_\gamma w_\lambda - F^\gamma_\lambda \nabla_\mu w_\gamma + \beta_\lambda w_\mu - \beta_\mu w_\lambda + \alpha_\mu \widetilde{w}_\lambda - \alpha_\lambda \widetilde{w}_\mu - 2\langle \alpha, w\rangle F_{\mu\lambda} = 0, \tag{4.34}$$

whereby $\widetilde{w}_\lambda = -F^\varepsilon_\lambda w_\varepsilon$. Hence, in a locally conformal Kaehler manifold, we define a covariant almost analytic vector field which satisfies (**4.34**) .

Transvecting (**4.34**) with β^μ gives us

$$\alpha^\varepsilon \nabla_\varepsilon w_\lambda + \beta^\gamma F^\varepsilon_\lambda \nabla_\gamma w_\varepsilon - \langle \beta, w\rangle \beta_\lambda - \langle \alpha, w\rangle \alpha_\lambda + \|\alpha\|^2 w_\lambda = 0. \tag{4.35}$$

and also transvecting (**4.34**) with β^λ, we have

$$\alpha^\varepsilon \nabla_\lambda w_\varepsilon + \beta^\gamma F^\varepsilon_\lambda \nabla_\varepsilon w_\gamma - \langle \beta, w\rangle \beta_\lambda - \langle \alpha, w\rangle \alpha_\lambda + \|\alpha\|^2 w_\lambda = 0. \tag{4.36}$$

Proposition 4.6 [8]. *For a covariant almost analytic vector field w_λ in an l.c.K-manifold the tensor field $\nabla_\mu w_\lambda - \nabla_\lambda w_\mu$ is pure in μ and λ , that is* ,

$$\nabla_\mu w_\lambda - \nabla_\lambda w_\mu + F^\varepsilon_\mu F^\gamma_\lambda (\nabla_\varepsilon w_\gamma - \nabla_\gamma w_\varepsilon) = 0. \tag{4.37}$$

Proof . Let w_λ be a covariant almost analytic vector field. Transvecting (**4.34**) with F^μ_ν, we have

$$\nabla_\mu w_\lambda + F^\varepsilon_\mu F^\gamma_\lambda \nabla_\varepsilon w_\gamma + \beta_\mu \widetilde{w}_\lambda + \beta_\lambda \widetilde{w}_\mu + \alpha_\mu w_\lambda + \alpha_\lambda w_\mu - 2\langle \alpha, w\rangle g_{\mu\lambda} = 0. \tag{4.38}$$

If we replace $\mu \to \lambda$, $\lambda \to \mu$, $\gamma \to \varepsilon$ and $\varepsilon \to \gamma$ in (**4.38**) , we have

$$\nabla_\lambda w_\mu + F^\gamma_\lambda F^\varepsilon_\mu \nabla_\gamma w_\varepsilon + \beta_\lambda \widetilde{w}_\mu + \beta_\mu \widetilde{w}_\lambda + \alpha_\lambda w_\mu + \alpha_\mu w_\lambda - 2\langle \alpha, w\rangle g_{\mu\lambda} = 0. \tag{4.39}$$

Now subtracting (**4.39**) from (**4.38**) , we have

$$\nabla_\mu w_\lambda - \nabla_\lambda w_\mu + F^\varepsilon_\mu F^\gamma_\lambda (\nabla_\varepsilon w_\gamma - \nabla_\gamma w_\varepsilon) = 0 \ . \qquad \blacksquare$$

Lemma 4.7 [14] . *If a tensor field $S_{\mu\lambda}$ is pure in μ and λ and a tensor field $T^{\mu\lambda}$ is hybrid in μ and λ, then we have $S_{\mu\lambda}T^{\mu\lambda} = 0$, identically.*

Now, the tensor $F_{\mu\lambda}$ is hybrid in μ and λ, we have from Proposition 4.6 and Lemma (4.7) , we get

$$F^{\mu\lambda}\nabla_\mu w_\lambda = 0 \tag{4.40}$$

for any covariant almost analytic vector field w_λ in an l.c.K-manifold.

Proposition 4.8 [8]. *In a real 2n-dimensional* ($n \neq 1$) *compact l.c.K-manifold, a covariant almost analytic vector field is orthogonal to the Lee vector field.*

Proof : Contracting (**4.38**) with $g^{\mu\lambda}$, we find

$$\nabla^{\lambda}w_{\lambda} + F^{\varepsilon\lambda}F^{\gamma}_{\lambda}\nabla_{\varepsilon}w_{\gamma} + \beta^{\lambda}\widetilde{w}_{\lambda} + \beta^{\mu}\widetilde{w}_{\mu} + \alpha^{\lambda}w_{\lambda} + \alpha^{\mu}w_{\mu} - 2\langle\alpha, w\rangle g_{\mu\lambda}g^{\mu\lambda} = 0$$

and using $\alpha^{\varepsilon}w_{\varepsilon} = \langle\alpha, w\rangle$, we get

$$\nabla^{\lambda}w_{\lambda} + g^{\varepsilon\gamma}\nabla_{\varepsilon}w_{\gamma} + \alpha^{\varepsilon}w_{\varepsilon} + \alpha^{\lambda}w_{\lambda} + \alpha^{\mu}w_{\mu} - 4n\langle\alpha, w\rangle = 0$$

and

$$\nabla^{\varepsilon}w_{\varepsilon} - 2(n-1)\langle\alpha, w\rangle = 0. \tag{4.41}$$

Applying Green's theorem to (**4.41**) , we find

$$\int_{M}[\nabla^{\varepsilon}w_{\varepsilon} - 2(n-1)\langle\alpha, w\rangle]d\sigma = 0$$

and hence

$$\int_{M}(2(n-1)\langle\alpha, w\rangle)d\sigma = 0 \ .$$

Since $n \neq 1$, we get

$$\langle\alpha, w\rangle = 0.$$ ■

Proposition 4.9 [8]. *If a vector field w_{λ} is contravariant and at the same time covariant almost analytic in an l.c.K-manifold , then the length of the vector field w_{λ} is constant.*

Proof . By using (**3.3**) , we get

$$(\nabla_{\varepsilon}F_{\mu\lambda})w^{\varepsilon} = -\beta_{\mu}w_{\lambda} + \beta_{\lambda}w_{\mu} - \alpha_{\mu}\widetilde{w}_{\lambda} + \alpha_{\lambda}\widetilde{w}_{\mu} \ . \tag{4.42}$$

Substituting (**4.42**) into (**2.15**) , we have

$$\pounds(w)F_{\mu\lambda} = -\beta_{\mu}w_{\lambda} + \beta_{\lambda}w_{\mu} - \alpha_{\mu}\widetilde{w}_{\lambda} + \alpha_{\lambda}\widetilde{w}_{\mu} - F^{\varepsilon}_{\lambda}\nabla_{\mu}w_{\varepsilon} + F^{\varepsilon}_{\mu}\nabla_{\lambda}w_{\varepsilon} \tag{4.43}$$

for any covariant almost analytic vector field w_{λ} in an l.c.K-manifold. By virtue of (**4.34**) and (**4.43**) , we obtain

$$\pounds(w)F_{\mu\lambda} = 2(\widetilde{w}_{\mu}\alpha_{\lambda} - \widetilde{w}_{\lambda}\alpha_{\mu} + \langle\alpha, w\rangle F_{\mu\lambda}) + F^{\varepsilon}_{\mu}(\nabla_{\lambda}w_{\varepsilon} - \nabla_{\varepsilon}w_{\lambda}) \tag{4.44}$$

Now, let a vector field w_{λ} be contravariant and the same time covariant almost analytic in an l.c.K-manifold, we have the formula

$$\pounds(w)F^{\lambda}_{\mu} = \{\pounds(w)F_{\mu\varepsilon}\}g^{\varepsilon\lambda} - F_{\mu\varepsilon}(\nabla^{\varepsilon}w^{\lambda} + \nabla^{\lambda}w^{\varepsilon})$$

and so

$$\{\pounds(w)F_{\mu\varepsilon}\}g^{\varepsilon\lambda} = 2(\widetilde{w}_\mu\alpha^\lambda - \widetilde{w}^\lambda\alpha_\mu + \langle\alpha, w\rangle F^\lambda_\mu) + F^\varepsilon_\mu(\nabla^\lambda w_\varepsilon - \nabla_\varepsilon w^\lambda) \ .$$

The we get

$$\begin{aligned}\{\pounds(w)F_{\mu\varepsilon}\}g^{\varepsilon\lambda} - F_{\mu\varepsilon}(\nabla^\varepsilon w^\lambda + \nabla^\lambda w^\varepsilon) &= 2(\widetilde{w}_\mu\alpha^\lambda - \widetilde{w}^\lambda\alpha_\mu + \langle\alpha, w\rangle F^\lambda_\mu) \\ &+ F^\varepsilon_\mu(\nabla^\lambda w_\varepsilon - \nabla_\varepsilon w^\lambda) - F_{\mu\varepsilon}(\nabla^\varepsilon w^\lambda + \nabla^\lambda w^\varepsilon) \ .\end{aligned}$$

Since the vector field w_λ be contravariant almost analytic, $\pounds(w)F^\lambda_\mu = 0$. Hence it implies

$$2(\widetilde{w}_\mu\alpha^\lambda - \widetilde{w}^\lambda\alpha_\mu + \langle\alpha, w\rangle F^\lambda_\mu - F^\varepsilon_\mu\nabla_\varepsilon w^\lambda) = 0.$$

Therefore, we write

$$2F^\varepsilon_\mu(-w_\varepsilon\alpha^\lambda - \widetilde{w}^\lambda\beta_\varepsilon - \nabla_\varepsilon w^\lambda) + 2F^\lambda_\mu\langle\alpha, w\rangle = 0$$

and consequently

$$-w_\varepsilon\alpha^\lambda - \widetilde{w}^\lambda\beta_\varepsilon - \nabla_\varepsilon w^\lambda + \langle\alpha, w\rangle = 0.$$

Contracting the above equation with $g_{\lambda\gamma}$, we get

$$-w_\varepsilon\alpha_\gamma - \widetilde{w}_\gamma\beta_\varepsilon - \nabla_\varepsilon w_\gamma + \langle\alpha, w\rangle g_{\lambda\gamma} = 0$$

and so

$$\nabla_\mu w_\lambda = -w_\mu\alpha_\lambda - \widetilde{w}_\lambda\beta_\mu + \langle\alpha, w\rangle g_{\mu\lambda} \ . \tag{4.45}$$

From this, we obtain

$$\frac{1}{2}\nabla_\mu\|w\|^2 = (\nabla_\mu w_\varepsilon)w^\varepsilon = 0 \tag{4.46}$$

which means the length of the vector field w_λ is constant. ■

Proposition 4.10 [8]. *In a 2n-dimensional l.c.K-manifold , if a covariant almost analytic vector field w_λ is closed and orthogonal to β_λ , then the vector field $\widetilde{w}_\lambda$ is harmonic.*

Proof . For a covariant almost analytic vector field w_λ, the equation (**4.32**) can be written as

$$F^\varepsilon_\mu(\nabla_\varepsilon w_\lambda - \nabla_\lambda w_\varepsilon) + \nabla_\mu\widetilde{w}_\lambda - \nabla_\lambda\widetilde{w}_\mu = 0. \tag{4.47}$$

Moreover, transvecting (**4.41**) with $F_\varepsilon\mu$, we get

$$\nabla_\varepsilon\widetilde{w}^\varepsilon = -2(n-1)\langle\beta, w\rangle \ . \tag{4.48}$$

Since w_λ is closed, using (**4.47**) , we obtain

$$\nabla_\mu \widetilde{w}_\lambda - \nabla_\lambda \widetilde{w}_\mu = 0.$$

Since w_λ is orthogonal to β_λ , $\langle \beta, w \rangle = 0$. Hence, in view of (**4.48**) ,we get

$$\nabla_\mu \widetilde{w}_\lambda - \nabla_\lambda \widetilde{w}_\mu = 0 \qquad \textit{and} \qquad \nabla_\varepsilon \widetilde{w}^\varepsilon = 0 \ ,$$

which implies that the vector field $\widetilde{w}_\lambda$ is harmonic. ■

5. SUBMANIFOLDS OF LOCALLY CONFORMAL KAEHLER MANIFOLDS

5.1 Submanifolds of Locally Conformal Kaehler Manifolds

Let N be a real m-dimensional manifold isometrically immersed in a real 2n-dimensional l.c.K-manifold $M(F^{\lambda}_{\mu}, g_{\mu\lambda}, \alpha_{\lambda})$. If the manifold M is covered by a system of coordinate neighborhoods $\{V, v^{\lambda}\}$ and N is covered by a system of coordinate neighborhoods $\{U, u^{i}\}$, whereby here and in the sequel the indices k, j, ..., i run over the range { 1, 2, ..., m} , then the submanifold N can be locally represented by

$$v^{\lambda} = v^{\lambda}(u^{i}) \ . \tag{5.1}$$

In the following, we shall identify vector fields in N and the image under the differential mapping, that is, if i denotes the immersion of N in M and X is a vector field in N, we identify X and $i_{*}(X)$ whereby

$$i_{*} : T_{p}N \to T_{p}M \ .$$

We put

$$B^{\lambda}_{i} = \partial_{i} v^{\lambda} = \frac{\partial v^{\lambda}}{\partial u^{i}} \ , \qquad \partial_{i} = \frac{\partial}{\partial u^{i}} \tag{5.2}$$

and let g_{ji} be the induced metric on N, then we have

$$g_{ji} = g_{\mu\lambda} B^{\mu}_{j} B^{\lambda}_{i} \ . \tag{5.3}$$

Let ξ^{λ}_{x} be a system of orthogonal normal vectors, whereby the indices $z, y, ..., x$ run over the range $\{1, 2, ..., 2n - m\}$. Then we have

$$g_{\mu\lambda} B^{\mu}_{i} \xi^{\lambda}_{x} = 0 \tag{5.4}$$

and also

$$g_{\mu\lambda} \xi^{\mu}_{y} \xi^{\lambda}_{x} = \delta_{yx} \ . \tag{5.5}$$

The formulas of Gauss and Weingarten are respectively given by

$$\nabla_{j} B^{\lambda}_{i} = h^{x}_{ji} \xi^{\lambda}_{x} \ , \tag{5.6}$$

$$\nabla_j \xi_x^\lambda = -h_{jx}^i B_i^\lambda + l_{jx}^y \xi_y^\lambda \ . \tag{5.7}$$

whereby h_{ji}^x and l_{jy}^x denote the second and third fundamental tensors, respectively.

The equation of Gauss and Codazzi are respectively given by

$$R_{\omega\nu\mu\lambda} B_k^\omega B_j^\nu B_i^\mu B_h^\lambda = R_{kjih} - h_{kh}^x h_{jix} + h_{ki}^x h_{jhx} \ , \tag{5.8}$$

$$R_{\omega\nu\mu\lambda} B_k^\omega B_j^\nu B_i^\mu \xi_x^\lambda = \widetilde{\nabla}_k h_{jix} - \widetilde{\nabla}_j h_{kix} \ , \tag{5.9}$$

whereby $\widetilde{\nabla}$ denotes the covariant differentiation which is defined by the covariant differentiations with respect to the induced Riemannian connection in N and the normal connection.

Now the transformation $F_\mu^\lambda B_j^\mu$ of B_j^μ by F_μ^λ can be written as

$$F_\mu^\lambda B_j^\mu = f_j^i B_i^\lambda + f_j^x \xi_x^\lambda \ , \tag{5.10}$$

whereby f_j^i and f_j^x are respectively a tensor field of type (1,1) and a normal bundle valued 1-form in N. The transformation $F_\mu^\lambda \xi_y^\mu$ of ξ_y^μ by F_μ^λ can be written as

$$F_\mu^\lambda \xi_y^\mu = -f_y^i B_i^\lambda + f_y^x \xi_x^\lambda \ , \tag{5.11}$$

whereby f_y^i and f_y^x are respectively a tangent bundle valued 1-form and a tensor field of type (1,1) of the normal bundle in N. Then we get

$$f_y^i = f_j^x g^{ji} \delta_{yx} \ . \tag{5.12}$$

A submanifold N is called *invariant* if $FT_uN = T_uN$ for any point $u \in N$, whereby T_uN denotes the tangent vector space of N at u in N, that is, a real m-dimesional N of an l.c.K-manifold M is said to be *invariant* if the tangent space at each point of N is invariant under the action of F_μ^λ.

For an invariant submanifold N, we have

$$f_j^x = 0 \ . \tag{5.13}$$

Thus from (**5.10**) and (**5.11**) , we get

$$F_\mu^\lambda B_j^\mu = f_j^i B_i^\lambda \ , \tag{5.14}$$

$$F_\mu^\lambda \xi_y^\mu = f_y^x \xi_x^\lambda \ , \tag{5.15}$$

respectively.

Next, we decompose the Lee vector field α^λ as follows

$$\alpha^\lambda = \alpha^i B_i^\lambda + \alpha^x \xi_x^\lambda \ , \tag{5.16}$$

whereby α^i and α^x are respectively the tangential and the normal components of α^λ. In an invariant submanifold N of an l.c.K-manifold M we have $\alpha^x = 0$, identically, that is , the Lee vector field α^λ is always tangent to N, say

$$\alpha^\lambda = \alpha^i B_i^\lambda .$$

Proposition 5.1 [8]. *An invariant submanifold N of an l.c.K-manifold M in which Lee vector field α^λ is tangent to N is an l.c.K-manifold with structure $(f_j^i, g_{ji}, \alpha_i)$.*

Proof . Transvecting (**5.14**) with f_k^j , we get

$$F_\mu^\lambda F_\nu^\mu B_k^\nu = f_k^j f_j^i B_i^\lambda$$

and so

$$-B_k^\lambda = f_k^j f_j^i B_i^\lambda$$

which implies that

$$f_k^j f_j^i = -\delta_k^i \ . \tag{5.17}$$

Similarly, in virtue of (**5.15**) , we get

$$f_y^z f_z^x = -\delta_y^x \ . \tag{5.18}$$

Moreover, we have

$$g_{lk} f_j^l f_i^k = g_{ji} \ . \tag{5.19}$$

The equations (**5.17**) , (**5.18**) and (**5.19**) mean that (f_j^i, g_{ji}) is a local almost complex metric structure on N.

Transvecting (**3.3**) with $B_k^\nu B_j^\mu B_i^\lambda$, we obtain

$$\nabla_k f_{ji} = -\beta_j g_{ki} + \beta_i g_{kj} - \alpha_j f_{ki} + \alpha_i f_{kj}. \tag{5.20}$$

Since Lee form α_λ is closed, we have

$$\nabla_j \alpha_i = \nabla_i \alpha_j \ , \tag{5.21}$$

that is, the 1-form α_i is closed on N, whereby $\beta_j = -\alpha_l f_j^l$. ■

Proposition 5.2 [8]. *In an invariant submanifold N of an l.c.K-manifold M in which the Lee vector field α^λ is tangent to N, if the tensor field $P_{\mu\lambda}$ is hybrid, then the tensor field p_{ji} is hybrid, too.*

Proof. In an invariant submanifold N of an l.c.K-manifold M satisfying $\alpha^x = 0$, identically, that is , the Lee vector field α^λ is always tangent to N.

Transvecting (**3.16**) with $B^\mu_j B^\lambda_i$, we get

$$P_{\mu\lambda}B^\mu_j B^\lambda_i = -(\nabla_\mu\alpha_\lambda)B^\mu_j B^\lambda_i - \alpha_\mu\alpha_\lambda B^\mu_j B^\lambda_i + \frac{\|\alpha\|^2}{2}g_{\mu\lambda}B^\mu_j B^\lambda_i$$

and so

$$p_{ji} = -\nabla_j\alpha_i - \alpha_j\alpha_i + \frac{\|\alpha\|^2}{2}g_{ji} \ . \tag{5.22}$$

Since the tensor $P_{\mu\lambda}$ is hybrid, say $P_{\mu\varepsilon}F^\varepsilon_\lambda + P_{\lambda\varepsilon}F^\varepsilon_\mu = 0$, then by using (**5.22**) , we get

$$p_{jk}f^k_i + p_{ik}f^k_j = 0.$$

This shows that, the tensor field p_{ji} is hybrid. ■

Proposition 5.3 [8]. *An invariant submanifold N of an l.c.K-manifold M is minimal, that is, $trace(h^i_{jx}) = h_{jix}g^{ji} = 0$, if and only if the Lee vector field α^λ is tangent to N.*

5.2 Invariant Submanifolds of an l.c.K-Space Form

Let M(c) be an l.c.K-space form with constant holomorphic sectional curvature c and N be a real m-dimensional invariant submanifold of M(c).

In this section, we always assume that the Lee vector field α^λ is tangent to N.

Theorem 5.4 [8]. *Let N be a real m-dimensional minimal invariant submanifold of an l.c.K-space form M(c). Then we have*

$$4R \leqslant m(m+2)c + 6(m-2)p \ . \tag{5.23}$$

The equality holds if and only if the submanifold N is totally geodesic.

Proof . Transvecting (**3.29**) with $B^\omega_k B^\nu_j B^\mu_i B^\lambda_h$, we get

$$\begin{aligned} 4R_{\omega\nu\mu\lambda}B^\omega_k B^\nu_j B^\mu_i B^\lambda_h &= c\ (g_{kh}g_{ji} - g_{ki}g_{jh} + f_{kh}f_{ji} - f_{ki}F_{jh} - 2F_{kj}F_{ih}) \\ &+ 3(P_{kh}g_{ji} - P_{ki}g_{jh} + g_{kh}P_{ji} - g_{ki}P_{jh}) - \{\widetilde{p}_{kh}f_{ji} \\ &- \widetilde{p}_{ki}f_{jh} + f_{kh}\widetilde{p}_{ji} - f_{ki}\widetilde{p}_{jh} - 2(\widetilde{p}_{kj}f_{ih} + f_{kj}\widetilde{p}_{ih})\} \ , \end{aligned}$$

whereby $\widetilde{p}_{ji} = -p_{jl}f^l_i$.

By using (**5.8**) , we obtain

$$\begin{aligned} 4R_{kjih} &= c\ (g_{kh}g_{ji} - g_{ki}g_{jh} + f_{kh}f_{ji} - f_{ki}F_{jh} - 2F_{kj}F_{ih}) \\ &+ 3(P_{kh}g_{ji} - P_{ki}g_{jh} + g_{kh}P_{ji} - g_{ki}P_{jh}) - \widetilde{p}_{kh}f_{ji} \\ &+ \widetilde{p}_{ki}f_{jh} - f_{kh}\widetilde{p}_{ji} + f_{ki}\widetilde{p}_{jh} - 2(\widetilde{p}_{kj}f_{ih} + f_{kj}\widetilde{p}_{ih}) \\ &+ 4(h^x_{kh}h_{jix} - h^x_{ki}h_{jhx})\ . \end{aligned} \qquad (\mathbf{5.24})$$

Contracting (**5.24**) with g^{kh} , we get

$$4R_{kjih}g^{kh} = c(m+2)g_{ji} + 3(pg_{ji} + mp_{ji} - 2p_{ji}) - 6p_{ji} - 4h^x_{ki}h^k_{jx}$$

and then

$$4R_{ji} = \{(m+2)c + 3p\}g_{ji} + 3(m-4)p_{ji} - 4h^x_{ki}h^k_{jx}\ , \qquad (\mathbf{5.25})$$

whereby $p_{ji}g^{ji} = p$.

Transvecting (**5.25**) with g^{ji} , we get

$$4R = m(m+2)c + 6mp - 12p - 4h^x_{ki}h^{ki}_x\ .$$

Then we have

$$4(R + ||h||^2) = m(m+2)c + 6(m-2)p\ , \qquad (\mathbf{5.26})$$

whereby R is the scalar curvature with respect to g_{ji} and $||h||$ denotes the length of the second fundamental tensor h_{jix}. By virtue of (**5.26**) , we have (**5.23**) .

If the equality holds in (**5.23**) , the we have

$$4R = m(m+2)c + 6(m-2)p\ .$$

By using (**5.26**) , we get

$$||h|| = 0 \qquad and \qquad h_{jix} = 0\ ,$$

which means that the submanifold N is totally geodesic.

Conversely, if the submanifold N is totally geodesic, then $h_{jix} = 0$ and $||h|| = 0$. Thus the equality holds in the equation (**5.23**) . ■

Proposition 5.5 [8]. *Let N be a real m-dimensional invariant closed minimal submanifold of an l.c.K-space form M(c). Then we have*

$$4\int RdN \leqslant m(m+2)cVol.N + 3(m-2)^2\int \|\alpha\|^2 dN \ . \qquad (\mathbf{5.27})$$

The equality holds if and only if the submanifold N is totally geodesic.

Proof . By transvecting (**5.22**) with g^{ji}, we find

$$p = -\nabla_j\alpha^j + \frac{m-2}{2}\|\alpha\|^2 \qquad (\mathbf{5.28})$$

and substituting (**5.28**) into (**5.26**) , we get

$$4R = m(m+2)c - 6(m-2)\nabla_j\alpha^j + 3(m-2)^2\|\alpha\|^2 - 4\|h\|^2 \ . \qquad (\mathbf{5.29})$$

Since the submanifold is compact (i.e.closed), using Green's Theorem , we have

$$4\int RdN = m(m+2)cVol.N + 3(m-2)^2\int \|\alpha\|^2 dN - 4\int \|h\|^2 dN \ , \qquad (\mathbf{5.30})$$

whereby dN and Vol. N denote the volume element and the volume of N, respectively. Then we have

$$4\int RdN \leqslant m(m+2)cVol.N + 3(m-2)^2\int \|\alpha\|^2 dN \ .$$

If the submanifold N is totally geodesic, then $\|h\| = 0$ and by virtue of (**5.30**) , we write

$$4\int RdN = m(m+2)cVol.N + 3(m-2)^2\int \|\alpha\|^2 dN \ .$$

Hence, the equality holds in (**5.27**) .

Conversely if the equality holds in (**5.27**) , from (**5.30**) , we obtain $\|h\| = 0$, that is ,the submanifold N is totally geodesic. ■

Theorem 5.6 [8]. *In a minimal invariant submanifold N of an l.c.K- space form M(c), the vector field $\widetilde{R}(Z,Y)X$ is tangent to N or equivalenty $\widetilde{\nabla}_k h_{jix} - \widetilde{\nabla}_j h_{kix} = 0$ if and only if*

$$h^s_{ix}\alpha_s = 0 \ , \qquad (\mathbf{5.31})$$

whereby $\widetilde{R}$ denotes the Riemannian curvature tensor with respect to $g_{\mu\lambda}$ and X,Y,Z are any tangent vector fields on N.

Proof . Transvecting (**3.29**) with $B^\omega_j B^\nu_i B^\mu_k \xi^\lambda_x$, we get

$$\begin{aligned} 4R_{\omega\nu\mu\lambda}B^\omega_j B^\nu_i B^\mu_k \xi^\lambda_x &= 3\Big(g_{ik}P_{\omega\lambda}B^\omega_j\xi^\lambda_x - g_{jk}P_{\nu\lambda}B^\nu_i\xi^\lambda_x\Big) \\ &- \Big\{f_{ik}\widetilde{P}_{\omega\lambda}B^\omega_j\xi^\lambda_x - f_{jk}\widetilde{P}_{\nu\lambda}B^\nu_i\xi^\lambda_x - 2f_{ji}\widetilde{P}_{\mu\lambda}B^\mu_k\xi^\lambda_x\Big\} \ . \end{aligned}$$

Using the following equality

$$P_{\omega\lambda}B^{\omega}_{j}\xi^{\lambda}_{x} = -h^{s}_{jx}\alpha_{s} = -\delta^{t}_{j}h^{s}_{tx}\alpha_{s}\ ,$$

we have

$$R_{\omega\nu\mu\lambda}B^{\omega}_{j}B^{\nu}_{i}B^{\mu}_{k}\xi^{\lambda}_{x} = \frac{1}{4}\Big\{3(g_{jk}\delta^{t}_{i} - \delta^{t}_{j}g_{ik}) + f^{t}_{j}f_{ik} - f^{t}_{j}f_{ik} - 2f_{jh}f^{t}_{h}\Big\}h^{s}_{tx}\alpha_{s}\ . \qquad (\mathbf{5.32})$$

If $\widetilde{\nabla}_{k}h_{jix} - \widetilde{\nabla}_{j}h_{kix} = 0$, by using (**5.9**) , we get (**5.31**) .

Conversely, if (**5.31**) holds , then in view of (**5.32**) , we have

$$R_{\omega\nu\mu\lambda}B^{\omega}_{j}B^{\nu}_{i}B^{\mu}_{k}\xi^{\lambda}_{x} = 0$$

and so by using (**5.9**) we have

$$\widetilde{\nabla}_{k}h_{jix} - \widetilde{\nabla}_{j}h_{kix} = 0\ ,$$

that is, the vector field $\widetilde{R}$ is tangent to N. ■

6. CONCLUSION AND RECOMMENDATIONS

Let M be a real 2n-dimensional Hermitian manifold with a metric g, a complex structure F and the fundamental 2-form Ω is defined by $\Omega(X,Y) = g(FX,Y)$. The manifold M is called *a locally conformal Kaehler manifold* (*an l.c.K-manifold*) if each point x in *M* has an open neighborhood U with a positive differentiable function $\rho : U \to R$ such that $g^* = e^{-2\rho} g\ |_{\mathrm{U}}$ is a Kaehlerian metric on U, that is , a Hermitian manifold whose metric is locally conformal to a Kaehler metric is called an l.c.K-manifold.

A Hermitian manifold M is an l.c.K-manifold if and only if there exists on M a global closed 1-form α such that $d\Omega = 2\alpha \wedge \Omega$ and α is called the Lee form.

Moreover, a necessary and sufficient condition for a Hermitian manifold is an l.c.K-manifold to be $\nabla_\nu F_{\mu\lambda} = -\beta_\mu g_{\nu\lambda} + \beta_\lambda g_{\nu\mu} - \alpha_\mu F_{\nu\lambda} + \alpha_\lambda F_{\nu\mu}$, whereby $\beta_\lambda = \alpha^\varepsilon F_{\varepsilon\lambda}$.

It is also proved that the necessary and sufficient condition for the length of the Lee form to be constant.

In future, we aim to construct some type of submanifolds of locally conformal Kaehler manifolds.

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