ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

PSEUDO CONHARMONICALLY SYMMETRIC MANIFOLDS

M.Sc. THESIS

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Department of Mathematical Engineering

Mathematical Engineering Programme

DECEMBER 2013

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To my family,

FOREWORD

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ABBREVIATIONS

M	**:** Manifold
g	**:** Metric tensor
$T_p(M)$	**:** The vector space of all tangent vectors at p
TM	**:** The tangent bundle of M
$\Gamma(TM)$	**:** The vector space of all vector fields on M
$C^\infty(M)$	**:** The set of all smooth functions on M
$C^\infty(M,\mathbb{R})$	**:** The set of real-valued smooth functions on M
$[,]$	**:** Lie Bracket
Γ^k_{ij}	**:** The Christoffel symbols
δ^j_i	**:** The Kronecker symbols
R^h_{kji}	**:** The Riemannian curvature tensor
S_{ij}	**:** The Ricci curvature tensor
r	**:** The scalar curvature
H_{hijk}	**:** Conharmonic curvature tensor
$(PS)_n$	**:** An n-dimensional pseudo symmetric manifold
$(PRS)_n$	**:** An n-dimensional pseudo Ricci symmetric manifold
$(PCHS)_n$	**:** Pseudo conharmonically symmetric manifold
k	**:** The gravitational constant
T	**:** The energy momentum tensor
σ	**:** The energy density
p	**:** The isotropic pressure of the fluid

PSEUDO CONHARMONICALLY SYMMETRIC MANIFOLDS

SUMMARY

The main concern of this thesis is to investigate an n-dimensional pseudo conharmonically symmetric Riemannian manifold (M,g) (non-conharmonically flat) whose the conharmonic curvature tensor H satisfies the condition

$$\begin{aligned}(\nabla_X H)(Y,Z,U,V) =& 2A(X)H(Y,Z,U,V)+A(Y)H(X,Z,U,V)+A(Z)H(Y,X,U,V)\\ &+A(U)H(Y,Z,X,V)+A(V)H(Y,Z,U,X)\end{aligned}$$

where A is a non-zero 1-form. Such a manifold is called a pseudo conharmonically symmetric manifold and denoted by $(PCHS)_n$.

In the first chapter, we give some definitions and theorems describing the definition of Riemannian manifolds. Some notions and definitions which will be used in the next chapters are expressed.

In the second chapter, some familiar definitions and theorems about conformal transformation of a Riemannian structure as an important object of study in differential geometry are stated. Let M be an n-dimensional Riemannian manifold with the metric tensor g and ρ a positive function on M. Then $(\bar{M},\bar{g})$ where

$$\bar{g} = e^{2\rho} g$$

is conformally diffeomorphic to (M,g) and the conformal diffeomorphism

$$\phi : (M,g) \longrightarrow (\bar{M},\bar{g})$$

is called a conformal transformation. By using definiton of conformal transformation, we give the Weyl (curvature) tensor or the Weyl conformal tensor, C is defined as the trace-free part of the Riemann curvature tensor,

$$\begin{aligned}C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}&[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]\\ &+\frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]\end{aligned}$$

for $X,Y,Z \in T(M)$, where R and r denote the Riemannian curvature tensor and the scalar curvature of M, respectively. The Weyl tensor has all the algebraic properties of the Riemann curvature tensor. Later, we define conharmonic transformation from the definition of harmonic function. A harmonic function f is a function whose Laplacian

$$\Delta f = g^{ji}\nabla_j \nabla_i f$$

vanishes. It is easily seen that

$$\bar{\Delta} f = \bar{g}^{ji}\bar{\nabla}_j\bar{\nabla}_i f = e^{-2\rho} g^{ji}\nabla_j\nabla_i f + (n-2)e^{-2\rho}\rho^h(\partial_h f).$$

A harmonic function is not transformed into a harmonic function by the conformal transformation in general. However, the condition has been found upon ρ in order that the function defined by

$$\bar{f} = e^{2\alpha\rho} f$$

may become a harmonic function to the Riemannian metric $\bar{g}$ for a suitable constant α. Consequently, let f be a harmonic function on M with respect to g and $\bar{g}$ and $n > 2$. Then $\bar{f}$ is a harmonic function for $\bar{g}$ if and only if

$$\Delta\rho + \frac{n-2}{2}||\rho_k||^2 = 0.$$

In this point of view, Y.Ishi called a conharmonic transformation which is the conformal transformation $\phi : (M,g) \longrightarrow (\bar{M},\bar{g})$ satisfying

$$\Delta\rho + \frac{n-2}{2}||\rho_k||^2 = 0.$$

If we put

$$H(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{n-2}[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + g(X,U)S(Y,Z) - g(Y,U)S(X,Z)]$$

where R and S denote the Riemannian curvature tensor of type (0,4) defined by $R(X,Y,Z,U) = g(R(X,Y)Z,U)$ and the Ricci tensor of type (0,2), respectively. The curvature tensor defined by the above equation is known as the conharmonic curvature tensor.

In the third chapter, firstly a historical overview of the development of pseudo symmetric manifolds is given. In 1967, R. N. Sen and M. C. Chaki studied certain curvature restrictions on a certain kind of conformally flat space of class one and they obtained the following expression of the covariant derivative of the curvature tensor :

$$R^h_{ijk,l} = 2\lambda_l R^h_{ijk} + \lambda_i R^h_{ljk} + \lambda_j R^h_{ilk} + \lambda_k R^h_{ijl} + \lambda^h R_{lijk}$$

where R^h_{ijk} are the components of the curvature tensor R, $R_{lijk} = g_{hl}R_{ijk}$, λ_i is a non-zero covariant vector and $'$,$'$ denotes the covariant differentiation with respect to the metric tensor g_{ij}. Later in 1987, M. C. Chaki called a manifold as a pseudo symmetric manifold. A non-flat semi-Riemannian manifold (M,g), $n \geq 2$ is said to be a pseudo symmetric manifold if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = 2A(X)R(Y,Z)W + A(Y)R(X,Z)W + A(Z)R(Y,X)W + A(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho$$

where A is a non-zero 1-form, ρ is a vector field defined by

$$g(X,\rho) = A(X)$$

for all vector fields X and ∇ denotes the operator of the covariant differentiation with respect to the metric tensor g. In 1988, M. C. Chaki introduced the notion of a pseudo Ricci symmetric manifold. A non-flat Riemannian manifold is called pseudo Ricci

symmetric and denoted by $(PRS)_n$ if the Ricci tensor S of type $(0,2)$ of the manifold is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y)$$

where ∇ denotes the Levi-Civita connection and A is a non-zero 1-form. In the second part of the third chapter, we give definition of pseudo conharmonically symmetric manifolds. An n-dimensional pseudo conharmonically symmetric Riemannian manifold (M,g) (non-conharmonically flat) whose the conharmonic curvature tensor H satisfies the condition

$$\begin{aligned}(\nabla_X H)(Y,Z,U,V) =& 2A(X)H(Y,Z,U,V) + A(Y)H(X,Z,U,V) + A(Z)H(Y,X,U,V) \\ &+ A(U)H(Y,Z,X,V) + A(V)H(Y,Z,U,X)\end{aligned}$$

where A is a non-zero 1-form. Such a manifold is called a pseudo conharmonically symmetric manifold and denoted by $(PCHS)_n$. In the following parts of this chapter, we give some theorems and their proofs which are related to $(PCHS)_n$.

In the fourth chapter, certain investigations in general relativity by the coordinate free method of differential geometry are given. In this method of study, the spacetime of general relativity is a connected four-dimensional semi-Riemannian manifold (M^4,g) with Lorentz metric g with signature $(-,+,+,+)$. Here, a special type of spacetime which is called pseudo conharmonically symmetric spacetime is considered. A semi-Riemannian four-dimensional pseudo conharmonically symmetric manifold may similarly be defined by taking a Lorentz metric g with signature $(-,+,+,+)$. In this chapter, a perfect fluid pseudo conharmonically symmetric spacetime having the basic vector field as the timelike velocity vector field U of the fluid, that is

$$g(U,U) = -1$$

is considered. For a perfect fluid spacetime with the generator U as the flow vector field of the flow and k as the gravitational constant, the Einstein's equation without cosmological constant is

$$S_{ij} - \frac{r}{2}g_{ij} = kT_{ij}$$

where k is the gravitational constant, T is the energy momentum tensor of type (0,2) given by

$$T_{ij} = (\sigma + p)A_iA_j + pg_{ij}$$

with σ and p as the energy density and isotropic pressure of the fluid, respectively, and A as the associated 1-form. In the following parts of this chapter, some theorems and their proofs which are related to $(PCHS)_4$ are given.

In the last chapter, two examples for $(PCHS)_n$ satisfying the conditions

$$\begin{aligned}H(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{n-2}[&g(Y,Z)S(X,U) - g(X,Z)S(Y,U) \\ &+ g(X,U)S(Y,Z) - g(Y,U)S(X,Z)]\end{aligned}$$

and

$$\begin{aligned}(\nabla_X H)(Y,Z,U,V) =& 2A(X)H(Y,Z,U,V) + A(Y)H(X,Z,U,V) + A(Z)H(Y,X,U,V) \\ &+ A(U)H(Y,Z,X,V) + A(V)H(Y,Z,U,X)\end{aligned}$$

are given.

PSEUDO KONHARMONİK SİMETRİK MANİFOLDLAR

ÖZET

Bu tez çalışmasında, düz olmayan Riemann manifoldunun bir çeşidi olan pseudo konharmonik simetrik manifoldlar araştırılmıştır. Bu tez çalışmasının temel amacı,

$$\begin{aligned}(\nabla_X H)(Y,Z,U,V) =& 2A(X)H(Y,Z,U,V)+A(Y)H(X,Z,U,V)+A(Z)H(Y,X,U,V)\\ &+A(U)H(Y,Z,X,V)+A(V)H(Y,Z,U,X)\end{aligned}$$

bağıntısını sağlayan H konharmonik eğrilik tensörüne sahip n-boyutlu Pseudo Konharmonik Simetrik Riemann Manifoldunu incelemektir.

Bu tez beş ayrı bölümden oluşmakta olup, bu tezin birinci bölümünde, Riemann manifoldlarının genel bir tanımı yapılarak ilgili tanım ve teoremler verilmiştir. Bir n-boyutlu düzgün M manifoldu, simetrik kovariant tensör alanı G ile verilmiş ise genelleştirilmiş Riemann manifoldu adını alır ve G, M manifoldunun metrik tensörüdür. G pozitif tanımlı ise M manifoldu Riemann manifoldu olarak adlandırılır. Genelleştirilmiş bir Riemann manifoldu için

$$G(X,Y) = g_{ij}X^iY^j$$

ifadesi, $T_p(M)$ tanjant uzayı üzerinde bir iç çarpımı belirtir. Herhangi $X,Y \in T_p(M)$ için iç çarpım,

$$X \cdot Y = G(X,Y) = g_{ij}(p)X^iY^j$$

şeklinde tanımlanmıştır. Riemann eğrilik tensörünün bazı özellikleri verilmiş ve Riemann manifoldu üzerinde bazı vektör alanlarının tanımı yapılmıştır. Ayrıca tezin ilerleyen bölümlerinde kullanılacak olan temel tanım ve kavramlar ifade edilmiştir.

İkinci bölümde, Riemann uzayının önemli bir konusu olan konform ve konharmonik dönüşümlerle ilgili bazı bilinen tanım ve teoremler verilmiştir. M, g metriğiyle verilmiş bir n-boyutlu Riemann manifoldu olsun. Bu durumda $(\bar{M},\bar{g})$, $\bar{g} = e^{2\rho}g$, (M,g) manifolduna konformal olarak diffeomorfiktir ve konformal diffeomorfizm

$$\phi : (M,g) \longrightarrow (\bar{M},\bar{g})$$

konformal dönüşüm adını alır.

Konformal dönüşümün tanımı kullanılarak;

$$\begin{aligned}C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}&[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]\\ &+\frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]\end{aligned}$$

eşitliğiyle temsil edilen C konformal eğrilik tensörü verilmiştir. Bu tensöre, Weyl tensörü de denilmektedir. Weyl tensörü, Riemann eğrilik tensörünün tüm

cebirsel özelliklerine sahiptir. Bu özelliklere ek olarak, Weyl tensörü $C^a_{abc} = 0$ bağıntısını sağlar. Bir uzayın Riemann eğrilik tensörü $R(X,Y,Z,W) = 0$ ise, manifold düzdür. Bir uzayın konformal eğrilik tensörü $C(X,Y,Z,W) = 0$ ise, konformal olarak düzdür. Konformal dönüşümlerin sahip olduğu önemli özelliklerle ilgili farklı bilim dallarındaki uygulama alanlarından bahsedilmiş ve birkaç ilgi çekici örnek sunulmuştur.

Bu bölümün devamında, harmonik fonksiyonun tanımından yola çıkılarak konharmonik dönüşümün tanımı yapılmıştır. Bir harmonik fonksiyon, laplasyeni

$$\Delta f = g^{ji}\nabla_j\nabla_i f$$

sıfır olan bir fonksiyondur. Buradan kolayca aşağıdaki eşitliğe ulaşılır:

$$\bar{\Delta} f = \bar{g}^{ji}\bar{\nabla}_j\bar{\nabla}_i f = e^{-2\rho}g^{ji}\nabla_j\nabla_i f + (n-2)e^{-2\rho}\rho^h(\partial_h f).$$

Genelde bir harmonik fonksiyon konformal dönüşümle başka bir harmonik fonksiyona dönüşemez. Ancak bazı koşullar altında, bu dönüşüm gerçekleşebilir. Sonuç olarak, f (M,g) üzerinde bir harmonik fonksiyon olsun. Bu durumda $\bar{f} = e^{2\alpha\rho}f$ ile tanımlanan $\bar{f}$ fonksiyonu,

$$\Delta\rho + \frac{n-2}{2}||\rho_k||^2 = 0$$

eşitliği sağlanıyorsa, bu fonksiyon bir harmonik fonksiyondur.

Bu bölümde, konharmonik eğrilik tensörü aşağıdaki gibi tanımlanmıştır:

$$\begin{aligned} H(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{n-2}[&g(Y,Z)S(X,U) - g(X,Z)S(Y,U) \\ &+ g(X,U)S(Y,Z) - g(Y,U)S(X,Z)] \end{aligned}$$

Bu şekilde tanımlanan H eğrilik tensörü, konharmonik eğrilik tensörü olarak adlandırılır. Burada R ve S, sırasıyla Riemann eğrilik tensörü ve Ricci tensörünü temsil etmektedir. Bir manifoldun konharmonik eğrilik tensörü, manifoldun her noktasında sıfırlanıyorsa bu manifolda konharmonik olarak düzdür denir. Bu nedenle konharmonik eğrilik tensörü, manifoldun konharmonik düz olması durumundan sapmasını temsil eder. Son olarak bu bölümde, konharmonik eğrilik tensörüyle ilgili önermeler gösterilmiş ve bu alanda kimlerin çalıştığından bahsedilmiştir.

Üçüncü bölümde, ilk olarak tezin temelini oluşturan Riemann simetrik uzayları ve Pseudo simetrik manifoldlardan kısaca bahsedilmiş ve simetrik manifoldlar üzerinde çalışan yazarlardan bahsedilmiştir. Bu bölümde ayrıca, pseudo simetrik manifoldun tarihsel gelişimi hakkında bilgilere yer verilmiştir. 1967 yılında, R.N.Sen ve M.C.Chaki eğrilik tensörünün kovariant türevinin ifadesini

$$R^h_{ijk,l} = 2\lambda_l R^h_{ijk} + \lambda_i R^h_{ljk} + \lambda_j R^h_{ilk} + \lambda_k R^h_{ijl} + \lambda^h R_{lijk}$$

eşitliğiyle elde etmişlerdir. Burada R^h_{ijk} lar R eğrilik tensörünün bileşenlerini, $R_{lijk} = g_{hl}R_{ijk}$, λ_i sıfır olmayan kovariant vektörü ve $'$,$'$ ise kovariant türevi ifade etmektedir. 1987 yılında, M.C. Chaki pseudo simetrik manifoldu şu şekilde tanımlamıştır: Düz olmayan bir yarı-Riemann (M,g) $n \geq 2$ manifoldun R eğrilik tensörü

$$\begin{aligned} (\nabla_X R)(Y,Z)W =& 2A(X)R(Y,Z)W + A(Y)R(X,Z)W + A(Z)R(Y,X)W \\ &+ A(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho \end{aligned}$$

eşitliğini sağlarsa, bu manifold pseudo simetrik manifold adını alır ve $(PS)_n$ ile gösterilir. Burada A ifadeleri 1-formlar, ρ $g(X,\rho) = A(X)$ ile tanımlanan vektör alanı olup, ∇ da kovariant türevi ifade etmektedir. 1988 yılında, M.C. Chaki pseudo Ricci simetrik manifoldu şu şekilde tanımlamıştır: Düz olmayan Riemann manifoldu, eğer (0,2) tipindeki Ricci tensörü sıfırdan farklı ve

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y)$$

koşulunu sağlar ise, pseudo Ricci simetrik manifold olarak adlandırılır ve $(PRS)_n$ ifadesiyle gösterilir. Burada A ifadeleri 1-formları ve ∇ da Levi-Civita konneksiyonunu belirtir.

Üçüncü bölümün ikinci kısmında, konharmonik eğrilik tensörüne bağlı olarak pseudo konharmonik simetrik manifold tanımlanmış ve bu manifoldun sahip olduğu özellikler verilmiştir. Bir n-boyutlu (M,g) manifoldunun konharmonik eğrilik tensörü H,

$$\begin{aligned}(\nabla_X H)(Y,Z,U,V) =& 2A(X)H(Y,Z,U,V) + A(Y)H(X,Z,U,V) + A(Z)H(Y,X,U,V) \\ &+ A(U)H(Y,Z,X,V) + A(V)H(Y,Z,U,X)\end{aligned}$$

koşulunu sağlıyorsa, bu manifolda pseudo konharmonik simetrik manifold denir ve $(PCHS)_n$ ifadesiyle gösterilir.

Bu bölümün ikinci kısmında yer alan ilk teoremde, pseudo konharmonik simetrik manifoldun konharmonik eğrilik tensörünün, ikinci Bianchi özdeşliğini sağladığı ispatlanmıştır.

Bu kısımdaki üçüncü teoremde, konharmonik eğrilik tensörünün diverjansının sıfır olduğunu kabul eden bir Riemann manifoldunun sabit skaler eğriliğe sahip olduğu gösterilmiş ve devamındaki teoremde konharmonik eğrilik tensörünün diverjansının sıfır olduğunu kabul eden pseudo konharmonik simetrik manifoldun skaler eğriliğinin sıfır olduğu ispatlanmıştır.

Aynı bölümde, Pseudo konharmonik simetrik manifold rekürant kabul edildiğinde, bu manifoldun ya skaler eğriliğinin sıfıra eşit olduğu ya da rekürans vektör alanı ve 1-form A arasında

$$\lambda_l = \frac{2(n+2)}{n} A_l$$

bağıntısının bulunduğu ispatlanmıştır.

Üçüncü bölümün son kısmında, pseudo konharmonik simetrik manifoldlarla ilgili farklı teoremlerde devam verilmiştir. Bu kısımda yer alan ilk teoremde, genelleştirilmiş rekürant eğrilik tensörüne sahip pseudo konharmonik simetrik manifoldun skaler eğriliğinin

$$r = \frac{n^2(n-1)\beta_i}{(2n+4)A_i - n\alpha_i} \quad (A_i \neq \frac{n}{2n+4}\alpha_i)$$

formunda olduğu gösterilmiştir.

Bu kısımdaki ikinci teoremde, $(PCHS)_n$ manifoldunun konharmonik eğrilik tensörü genelleştirilmiş rekürant ise, bu manifoldun skaler eğriliğinin

$$r = \frac{n(2-n)(n-1)\beta_i}{2(n+2)A_i - n\alpha_i}$$

ifadesine eşit olduğu ispatlanmıştır. Bunun sonucu olarak, skaler eğriliği sıfırdan farklı olan $(PCHS)_n$ manifoldunun konharmonik eğrilik tensörü rekürant ise, rekürans

vektör alanı ve 1-form A nın aralarındaki bağıntı $r(2(n+2)A_l - n\alpha_l) = 0$ denklemiyle bulunmuştur.

Bu kısımdaki üçüncü teoremde, $(PCHS)_n$ manifoldunun konharmonik eğrilik tensörü Codazzi tipinde ise, bu manifoldun skaler eğriliğinin sıfır olduğu bulunmuştur. Skaler eğriliği sabit olmayan $(PCHS)_n$, ilişkili vektör alanı tors-oluşturan vektör alanı olması durumunda, vektör alanları ϕ_k ve A_k kollinear olmaktadır.

Dördüncü bölümde, pseudo konharmonik simetrik uzay-zaman olarak adlandırılan uzay-zamanın bir özel durumu incelenmiştir. 4-boyutlu bir yarı-Riemannian pseudo konharmonik simetrik manifoldu bir g Lorentz metriği ile tanımlanabilir. Bu bölümde, akışkanın zaman benzeri hız vektör alanı U, $g(U,U) = -1$, olarak temel vektör alanına sahip mükemmel akışkanlı pseudo konharmonik simetrik uzay-zaman ele alınmıştır. Bir mükemmel akışkanlı pseudo konharmonik simetrik uzay-zaman için Ricci tensörü rekürant ise bu durumda enerji yoğunluğu ve izotropik basınç orantılıdır. Bu teoremden yararlanılarak, Ricci-rekürant olan bir mükemmel akışkanlı pseudo konharmonik simetrik uzay-zaman η-Einstein $(\eta = 2kp)$ uzay-zamana indirgenir. Bir mükemmel akışkanlı pseudo konharmonik simetrik uzay-zaman da kozmolojik sabit içermeyen Einstein denklemini sağlayan enerji momentum tensörü rekürant ise bu durumda ya bu manifoldun ilişkili vektör alanının diverjansı sıfırdır ya da bu manifold Einstein uzay-zamana indirgenir. Codazzi tipindeki konharmonik eğrilik tensö rünü kabul eden bir mükemmel akışkanlı pseudo konharmonik simetrik uzay-zaman, evrenimizin radrasyon dönemindeki durumunun denklemini temsil eder.

Son olarak beşinci bölümde,

$$H(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{n-2}[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + g(X,U)S(Y,Z) - g(Y,U)S(X,Z)]$$

koşulunu sağlayan pseudo konharmonik simetrik manifold için birkaç örnek sunulmuştur.

1. RIEMANNIAN MANIFOLDS

1.1 Basic Definitions

Suppose that M is an n-dimensional smooth manifold and G is a symmetric covariant tensor field of rank 2 on M. If $(U;u^i)$ is a local coordinate system on M, then the tensor field G can be expressed as

$$G = g_{ij}du^i \otimes du^j \tag{1.1}$$

on U, where $g_{ij} = g_{ji}$ is a smooth function on U. G provides a bilinear function on $T_p(M)$ at every point $p \in M$. Suppose $X = X^i \frac{\partial}{\partial u^i}$, $Y = Y^i \frac{\partial}{\partial u^i}$, then

$$G(X,Y) = g_{ij}X^iY^j. \tag{1.2}$$

We say that the tensor G is nondegenerate at the point p if whenever $X \in T_p(M)$ and

$$G(X,Y) = 0 \tag{1.3}$$

for all $Y \in T_p(M)$, it must be true that $X = 0$. This implies that G is nondegenerate at p if and only if the system of linear equations

$$g_{ij}(p)X^i = 0, \qquad 1 \leq j \leq m \tag{1.4}$$

has zero as its only solution, i.e., $det(g_{ij}(p)) \neq 0$.
If for all $X \in T_p(M)$ we have

$$G(X,X) \geq 0 \tag{1.5}$$

and the equality (1.3) holds only if $X = 0$, then we say G is positive definite at p. From linear algebra a necessary and sufficient condition for G to be positive definite is that the matrix (g_{ij}) be positive definite. Thus a positive definite tensor G is necessarily nondegenerate.

1.1.1 Riemannian manifold

If an n-dimensional smooth manifold M is given by G defined as a non-degenerate symmetric covariant tensor field of rank 2, then M is called a generalized Riemannian manifold, and G is called a fundamental tensor or metric tensor of M. If G is positive definite, then M is called a Riemannian manifold.

For a generalized Riemannian manifold M,

$$G(X,Y) = g_{ij}X^iY^j$$

specifies an inner product on the tangent space $T_p(M)$ at every point $p \in M$. For any $X,Y \in T_p(M)$, let

$$X \cdot Y = G(X,Y) = g_{ij}(p)X^iY^j. \tag{1.6}$$

When G is positive definite, it is meaningful to define the length of a tangent vector and the angle between two tangent vectors at the same point, i.e.,

$$|X| = \sqrt{g_{ij}X^iX^j} \tag{1.7}$$

$$cos\theta(X,Y) = \frac{X \cdot Y}{|X||Y|}. \tag{1.8}$$

Thus a Riemannian manifold is a differentiable manifold which has a positive definite inner product on the tangent space at every point. The inner product is required to be smooth. If X,Y are smooth tangent vector fields, then $X \cdot Y$ is a smooth function on M. The differential 2-form

$$ds^2 = g_{ij}du^idu^j \tag{1.9}$$

is independent of the choice of the local coordinate system u^i and is usually called the metric form or Riemannian metric. ds is precisely the length of an infinitesimal tangent vector, and is called the element of arc length.

Suppose $C : u^i = u^i(t), \quad t_o \leq t \leq t_1$, is a continuous and piecewise smooth parametrized curve on M. Then the arc length of C is defined to be

$$s = \int_{t_0}^{t_1} \sqrt{g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}dt. \tag{1.10}$$

Theorem 1.1 There exists a Riemannian metric on an n-dimensional smooth manifold M.

Let us assume that M is a generalized Riemannian manifold in the following discussion. When the local coordinate system is changed, the transformation formula for the components of a fundamental tensor G is

$$g'_{ij} = g_{kl}\frac{\partial u^k}{\partial u'^i}\frac{\partial u^l}{\partial u'^j}. \tag{1.11}$$

Since the matrix (g_{ij}) is nondegenerate, we may denote its inverse by (g^{ij}), i.e.,

$$g^{ik}g_{kj} = g_{jk}g^{ki} = \delta^i_j. \tag{1.12}$$

Then it is easy to show that the transformation formula for g^{ij} under a change of coordinates is given by

$$g'^{ij} = g^{kl}\frac{\partial u'^i}{\partial u^k}\frac{\partial u'^j}{\partial u^l}. \tag{1.13}$$

Hence (g^{ij}) is a symmetric contravariant tensor of rank 2.

1.1.2 A connection on a Riemannian manifold

[2] Let M be an n-dimensional differentiable manifold with tangent bundle TM. Let $\Gamma(TM)$ be denote the vector space of all vector fields on M. Let us denote the vector space of all tangent vectors at p by $T_p(M)$.

A connection on M is a map

$$\nabla : TM \times \Gamma(TM) \longrightarrow TM \tag{1.14}$$

such that, for $\xi \in T_p(M)$, $X, Y \in \Gamma(TM)$ and $f \in C^\infty(M)$,

$$\begin{aligned} &1.\nabla_\xi X \in T_p(M) \\ &2.\nabla_\xi(fX) = (\xi f)X + f(p)\nabla_\xi X \\ &3.\nabla_X Y \in \Gamma(TM). \end{aligned} \tag{1.15}$$

1.1.3 Affine connection

[3] Let M be an n-dimensional manifold. An affine connection is an operator

$$\nabla : C^\infty(TM) \times C^\infty(TM) \longrightarrow C^\infty(TM) \tag{1.16}$$

that assigns a new vector field $\nabla_X Y$ by two vector fields X and Y, that is linear in both variables. For any $a, b \in IR$ and any vector fields X, Y and Z,

$$\nabla_X(aY + bZ) = a\nabla_X Y + b\nabla_X Z \tag{1.17}$$

$$\nabla_{aX+bY} Z = a\nabla_X Z + b\nabla_Y Z \tag{1.18}$$

and ∇ satisfies the Leibnitz rule, that is, for any smooth function $f \in C^\infty(M, IR)$ and any two vector fields X and Y,

$$\nabla_X(fY) = df(X)Y + f\nabla_X Y \tag{1.19}$$

holds.

1.1.4 Torsion tensor

[3] Let X and Y be vector fields on a manifold M. Then the vector field

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{1.20}$$

defines a tensor field T of type (1,2). This tensor is called the torsion tensor, so that for any 1-form σ, we have

$$T(\sigma, X, Y) = \sigma(\tau(X,Y)). \tag{1.21}$$

The affine connection is called torsion-free (or symmetric) if the torsion vanishes, that is, for X, Y,

$$\nabla_X Y - \nabla_Y X = [X,Y] \tag{1.22}$$

holds.

1.1.5 Riemannian curvature tensor

[3] Let X, Y and Z be vector fields on a manifold M. Then the vector field

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]} Z \tag{1.23}$$

defines a tensor field R of type (1,3). This tensor is called the Riemannian curvature, so that for any 1-form σ

$$R(\sigma, Z, X, Y) = \sigma(R(X,Y)Z). \tag{1.24}$$

The affine connection is called flat if the curvature vanishes, that is, for any X, Y, Z

$$[\nabla_X, \nabla_Y]Z = \nabla_{[X,Y]} Z. \tag{1.25}$$

1.1.6 Levi-Civita connection

[3] Let (M,g) be a Riemannian manifold and ∇ be an affine connection on M. Then the connection ∇ is called compatible with the metric g if for any vector fields X,Y and Z, it satisfies the condition

$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \tag{1.26}$$

An affine connection that is torsion-free and compatible with the metric is called the Levi-Civita connection.

Theorem 1.2 (Fundamental Theorem of Riemannian Geometry) Suppose that M is an n-dimensional generalized Riemannian manifold. Then there exists a unique torsion-free and metric-compatible connection on M, called the Levi-Civita connection of M or the Riemannian connection of M.

Proof. Suppose that ∇ is a torsion-free and metric-compatible connection on M. Denote the connection matrix of ∇ under the local coordinates u^i by $w = (w_i^j)$, where

$$w_i^j = \Gamma_{ik}^j du^k. \tag{1.27}$$

Then we have

$$dg_{ij} = w_i^k g_{kj} + w_j^k g_{ki}, \tag{1.28}$$

$$\Gamma_{ik}^j = \Gamma_{ki}^j. \tag{1.29}$$

Denote

$$\Gamma_{ijk} = g_{lj}\Gamma_{ik}^l, \quad w_{ik} = g_{lk} w_i^l. \tag{1.30}$$

Then it follows from (1.28) and (1.29) that

$$\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ijk} + \Gamma_{jik}, \tag{1.31}$$

$$\Gamma_{ijk} = \Gamma_{kji} \tag{1.32}$$

or

$$g_{ij,k} = 0. \tag{1.33}$$

It implies that for a metric compatible connection, the metric tensor g_{ij} behaves like a constant under absolute differentiation.

Cycling the indices in (1.31), we get

$$\frac{\partial g_{ik}}{\partial u^j} = \Gamma_{ikj} + \Gamma_{kij}, \tag{1.34}$$

$$\frac{\partial g_{jk}}{\partial u^i} = \Gamma_{jki} + \Gamma_{kji}. \tag{1.35}$$

Then, we obtain

$$\Gamma_{ikj} = \frac{1}{2}(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}), \tag{1.36}$$

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}). \tag{1.37}$$

Thus, we see that the torsion-free and metric-compatible connection is determined uniquely by the metric tensor.

Conversely, the Γ^k_{ij} indeed satisfy the transformation equation for the connection coefficients under a change of local coordinates. Hence they define an affine connection ∇ on M. Calculations also verify that they satisfy (1.31) and (1.32). Thus ∇ is a torsion-free and metric-compatible connection on M.

The existence and uniqueness of the Levi-Civita connection is the most important result in Riemannian Geometry. The Γ_{ikj} and Γ^k_{ij} are called Christoffel symbols of the first kind and second kind, respectively.

1.1.7 Covariant differentiation of tensor fields

[4] Let M be an n-dimensional manifold with the connection ∇. One extends the covariant differentiation to general (r,s)-tensor fields over M, namely, given the (r, s)-tensor field T, we define its covariant differential ∇T to be the (r, s + 1)-tensor field given by

$$\begin{aligned}(\nabla T)(\theta^1,...,\theta^r;X_0,X_1,...,X_r) =& (\nabla_{X_0} T)(\theta^1,...,\theta^r;X_1,...,X_s) \\ =& X_0(T(\theta^1,...,\theta^r;X_1,...,X_s)) \\ &- \sum_{j=1}^{r} T(\theta^1,...,\nabla_{X_0}\theta^j,...,\theta^r;X_1,...,X_s) \\ &- \sum_{k=1}^{s} T(\theta^1,...,\theta^r;X_1,...,\nabla_{X_0}X_k,...,X_s)\end{aligned} \tag{1.38}$$

where $\theta^1,...,\theta^r$ are 1-forms on M and $X_0,...,X_s$ are vector fields on M.

1.2 Some Properties of the Riemannian Curvature Tensor

[5] Let M be an n-dimensional manifold with the Levi-Civita connection ∇. The Riemann curvature tensor is a (1,3) tensor defined by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z \tag{1.39}$$

on vector fields X, Y, Z. Using the metric g we can change this to a (0,4)-tensor as follows

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

The Riemann curvature tensor satisfies the following rules:

1.$R(X,Y,Z,W) = -R(Y,X,Z,W) = R(Y,X,W,Z)$

2.$R(X,Y,Z,W) = R(Z,W,X,Y)$

3.$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0.$

1.2.1 Ricci tensor

[6] Let (M,g) be an n-dimensional Riemannian manifold and $e_1,...,e_n$ be an orthonormal basis of M.The Ricci tensor S is the covariant tensor field of type (0,2) defined as follows

$$S(X,Y) = \sum_{i=1}^{n} g(R(e_i,X)Y,e_i) \tag{1.40}$$

where $X,Y \in TM$ and R denotes the Riemannian curvature tensor.

1.2.2 The second bianchi identity

[7] Let ∇ be a linear connection and R be the Riemannian curvature tensor. The following cyclic summation is zero

$$(\nabla_W R)(X,Y)Z + (\nabla_X R)(Y,W)Z + (\nabla_Y R)(W,X)Z = 0. \tag{1.41}$$

This equation is known as the second Bianchi identity and also as the differential Bianchi identity.

1.2.3 Codazzi type tensor field

[8] Let (M,g) $(n \geq 3)$ be a smooth Riemannian manifold. A tensor T of type (0,2) on M is called a Codazzi type if it satisfies the Codazzi equation, that is,

$$(\nabla_X T)(Y,Z) = (\nabla_Y T)(X,Z) \tag{1.42}$$

for every tangent vectors X, Y, Z.

1.3 Some Special Vector Fields on Riemannian Manifolds

1.3.1 Torse-forming vector field

[9] A vector field ξ in a Riemannian manifold M is called torse-forming if it satisfies the following condition

$$\nabla_X \xi = \rho X + \phi(X)\xi \tag{1.43}$$

where $X \in TM$, $\phi(X)$ is a linear form and ρ is a function.
In local transcription, this reads

$$\xi^h_{,i} = \rho \delta^h_i + \xi^h \phi_i \tag{1.44}$$

where ξ^h and ϕ_i are the components of ξ and ϕ respectively, and δ^h_i is the Kronecker symbol. A torse-forming vector field ξ is called
i) Recurrent, if $\rho = 0$.
ii) Concircular, if the form ϕ_i is gradient covector, i.e., there is a function ψ such that $\phi = d\psi(X)$.
iii) Convergent, if it is concircular and $\rho = const.exp(\phi)$.
Therefore, by the aid of (1.44), recurrent vector fields are characterized by the following equation

$$\nabla_X \xi = \phi(X)\xi. \tag{1.45}$$

1.4 Some Special Riemannian Manifolds

In this section, we give the definitions of some special Riemannian manifolds to use in the next sections.

1.4.1 Generalized recurrent manifold

[10] A non-flat n-dimensional Riemannian manifold (M, g),$(n > 2)$ is called a generalized recurrent manifold if its curvature tensor R of type (0,4) satisfies the condition

$$(\nabla_X R)(Y, Z, U, W) = \alpha(X)R(Y, Z, U, W) + \beta(X)G(Y, Z, U, W) \tag{1.46}$$

where $G(Y,Z,U,W) = g(Y,W)g(Z,U) - g(Y,U)g(Z,W)$ and α, β are non-zero 1-forms.

1.4.2 Recurrent manifold

[11] A non-flat n-dimensional Riemannian manifold (M,g) is called recurrent manifold if its curvature tensor $R(X,Y,Z,U)$ satisfies the condition

$$(\nabla_W R)(X,Y,Z,U) = \alpha(W)R(X,Y,Z,U) \tag{1.47}$$

for a non-zero 1-form α.

1.4.3 Ricci-recurrent manifold

[12] A non-flat Riemannian manifold M is called a Ricci-recurrent manifold if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) \tag{1.48}$$

where ∇ is the Levi-Civita connection of the Riemannian metric g and A is a 1-form on M.

An n-dimensional Riemannian manifold (M,g) is said to be locally symmetric if the Riemannian curvature tensor R satisfies the condition $\nabla R = 0$ where ∇ denotes the covariant differentiation with respect to the metric tensor g [13]. This notion has been generalized in many ways by several authors introducing recurrent manifolds [14], 2-recurrent manifolds [15], Ricci recurrent manifolds [12], concircular recurrent manifolds [16, 17], weakly symmetric manifolds [18], weakly Ricci symmetric manifolds [19], conformally recurrent manifolds [20], projectively recurrent manifolds [21], generalized recurrent manifolds [22], generalized Ricci recurrent manifolds [23] etc.

2. CONFORMAL AND CONHARMONIC TRANSFORMATIONS

Conformal transformation of a Riemannian structure is an important object of study in differential geometry. Of considerable interest in a special type of conformal transformations, conharmonic transformations, which are conformal transformations are preserving the harmonicity property of smooth functions. This type of transformation was introduced by Ishii [24] in 1957 and is now studied from various points of view. It is well known that such transformation has an invariant tensor then this tensor is called as conharmonic curvature tensor. It is easy to verify that this tensor is an algebraic curvature tensor; that is, it possesses the classical symmetry properties of the Riemannian curvature tensor.

2.1 Conformal Transformation

[25] Let M be an n-dimensional Riemannian manifold with the metric tensor g and ρ a positive function on M. Then $(\bar{M},\bar{g})$ where

$$\bar{g} = e^{2\rho} g \tag{2.1}$$

is conformally diffeomorphic to (M,g) and the conformal diffeomorphism

$$\phi : (M,g) \longrightarrow (\bar{M},\bar{g})$$

is called a conformal transformation. When ρ is constant, ϕ is called a homothety. Let ∇ and $\bar{\nabla}$ be denote the operators of covariant differentiation with respect to g and $\bar{g}$, respectively. In this case, let $\left\{ \begin{matrix} h \\ ij \end{matrix} \right\}$ and $\left\{ \begin{matrix} \bar{h} \\ ij \end{matrix} \right\}$ be denote the Christoffel symbols formed by g and $\bar{g}$. Then we get ([24], [26], [27]),

$$\left\{ \begin{matrix} \bar{h} \\ ij \end{matrix} \right\} = \left\{ \begin{matrix} h \\ ij \end{matrix} \right\} + \rho_i \delta^h_j + \rho_j \delta^h_i - \rho^h g_{ji} \tag{2.2}$$

$$\bar{R}^h_{kji} = R^h_{kji} + \rho_{ki}\delta^h_j - \rho_{ji}\delta^h_k + g_{ki}\rho^h_j - g_{ji}\rho^h_k \tag{2.3}$$

$$\bar{S}_{ji} = S_{ji} - (n-2)\rho_{ji} - \rho^\alpha_\alpha g_{ji} \tag{2.4}$$

$$\bar{r} = e^{-2p}(r - 2(n-1)\rho^\alpha_\alpha) \tag{2.5}$$

where R^h_{kji}, S_{ji}, r and $\bar{R}^h_{kji}, \bar{S}_{ji}, \bar{r}$ are Riemannian Christoffel curvature tensor, Ricci curvature tensor and scalar curvature with respect to g and $\bar{g}$ respectively, and where we have put

$$\rho_{ji} = \nabla_j \rho_i - \rho_j \rho_i + \frac{1}{2}||\rho_\alpha||^2 g_{ji}, \tag{2.6}$$

$$\rho^k_k = g^{ji}\rho_{ji}, \tag{2.7}$$

$$\rho_i = \partial_i \rho. \tag{2.8}$$

2.2 Conformal Curvature Tensor

[28] Let M be an n-dimensional Riemannian manifold with the metric g and let $T(M)$ be the Lie algebra of differentiable vector fields in M. The Ricci operator Q of (M,g) is defined by $g(QX,Y) = S(X,Y)$, where S denotes the Ricci tensor of type (0,2) on M and $X,Y \in T(M)$. Weyl ([29], [30]) constructed a generalized curvature tensor on a Riemannian manifold which vanishes, whenever the metric is (locally) conformally equivalent to a flat metric. For dimensions $n \geq 3$, the Weyl (curvature) tensor or the Weyl conformal tensor, C is defined as the trace-free part of the Riemann curvature tensor,

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{2.9}$$

for $X,Y,Z \in T(M)$, where R and r denote the Riemannian curvature tensor and the scalar curvature of M, respectively. It is obvious from above that when $S(X,Y)Z = 0$, the Riemann curvature tensor reduces to the Weyl tensor. The Weyl tensor has all the algebraic properties of the Riemann curvature tensor,

$$C(X,Y,Z,W) = C(Z,W,X,Y)$$

$$C(X,Y,Z,W) + C(Y,Z,X,W) + C(Z,X,Y,W) = 0$$

and in addition to this, it is trace-free, i.e., $C^a_{abc} = 0$. It is also well known that the Weyl tensor is identically zero in three dimensional manifold.

A space is called flat if its Riemann curvature tensor vanishes, $R(X,Y,Z,W) = 0$. A space is called conformally flat if there exists a conformal transformation $\bar{g} = e^{2\rho}g$ such that the Riemann curvature tensor in the space with metric $\bar{g}$ vanishes, i.e., $C(X,Y,Z,W) = 0$.

[31] Conformal transformations, or mappings, have many important properties and uses. One property relevant to image transformation is the preservation of local shape (except sometimes at isolated points). This example uses a $2-D$ conformal transformation to warp an image. The mapping from output to input, $g : R^2 \to R^2$, is defined in terms of a complex analytic function $G : C \to C$, where

$$G(z) = \frac{(z + \frac{1}{z})}{2}.$$

We define g via a direct correspondence between each point $(x,y) \in R^2$ (the Euclidean plane) and the point $z = x + iy \in C$ (the complex plane),

$$g(x,y) = (Re(w), Im(w)) = (u,v)$$

where

$$w = u + iv = G(x + iy).$$

This conformal mapping is important in fluid mechanics because it transforms lines of flow around a circular disk (or cylinder, if we add a third dimension) to straight lines.

[32] Conformal mapping has various applications in the field of medical physics.For example conformal mapping is applied to brain surface mapping. This is based on the fact that any genus zero (The genus of a connected, orientable surface is an integer representing the maximum number of cuttings along closed simple curves without rendering the resultant manifold disconnected; a sphere,disk or annuls have genus zero) surface can be mapped conformally onto the sphere and any local portion thereof onto a disk.

[33] There are different aspects of conformal mapping that can be used for practical applications though the essence remains the same: it preserves the angle and shape locally and mappings of harmonic potentials remains harmonic. These properties of conformal mapping make it advantageous in complex situations , specifically electromagnetic potential problems for general systems.

[34] Conformal transformations and conformal vector fields are important concepts in both Riemannian and pseudo-Riemannian geometry. Conformally flat spaces have been characterized by Cotton, Finzi and Schouten in the early 20th century.
In General Relativity conformal transformations are important since they preserve the causal structure up to time orientation and light-like geodesics up to parametrization.

Already in the early days of Einstein' s relativity theory, Kasner studied the question whether two fields both obeying Einstein' s equations of gravitation can ever have the same light rays. Motivated by this question about light rays, Kasner [35] proved the following: When a conformal representation of an Einstein manifold on a flat space is possible, the manifold is isometric to a flat space. In modern terminology, this is the statement that a vacuum spacetime which is locally conformally flat must be flat.

[36] Brinkmann investigated conformal transformations between two Einstein spaces as well as conditions for a space to be conformal to an Einstein space. For global conformal geometry, the conformal development map was introduced by Kuiper [37] in 1949. Using this concept he proved that a compact and simply connected Riemannian manifold which is locally conformally flat must be globally conformally equivalent with the standard sphere.

2.3 Harmonic Function

[25] A harmonic function f is a function whose Laplacian

$$\Delta f = g^{ji}\nabla_j\nabla_i f$$

vanishes. It is easily seen that

$$\bar{\Delta} f = \bar{g}^{ji}\bar{\nabla}_j\bar{\nabla}_i f = e^{-2\rho}g^{ji}\nabla_j\nabla_i f + (n-2)e^{-2\rho}\rho^h(\partial_h f).$$

Therefore, we can state the following proposition.

Proposition 2.1 [25] Let f be a harmonic function on an n-dimensional manifold (M,g) where $n > 2$. Then f is a harmonic function on $(\bar{M},\bar{g})$ if and only if

$$\rho^h(\partial_h f) = 0.$$

A skew-symmetric tensor w which satisfies

$$Rotw = 0 \quad and \quad Divw = 0 \tag{2.10}$$

is called a harmonic tensor,that is

$$dw = 0 \quad and \quad \delta w = 0 \tag{2.11}$$

where d and δ are the operators of exterior differential and codifferential with respect to ∇.

2.4 Conharmonic Transformation

[25] From Proposition 2.1 , we see that a harmonic function is not transformed into a harmonic function by the conformal transformation in general. In [24], the condition has been found upon ρ in order that the function defined by

$$\bar{f} = e^{2\alpha\rho} f \tag{2.12}$$

may become a harmonic function to the Riemannian metric $\bar{g}$ for a suitable constant α. Then by use of (2.12), we obtain ([24]),

$$\bar{\Delta} f = e^{2(\alpha-1)\rho}[\Delta f + 2\alpha(\Delta\rho)f + (4\alpha+n-2)\rho^i(\partial_i f) + 2\alpha(2\alpha+n-2)||\rho_i||^2 f]. \tag{2.13}$$

From (2.13), we have, [24], the following proposition

Proposition 2.2 [24] Let f be a harmonic function on (M,g) and $\alpha = \frac{2-n}{4}$. Then the function $\bar{f}$ defined by (2.12) is harmonic for $\bar{g}$ if and only if

$$\Delta\rho + \frac{(n-2)}{2}||\rho_k||^2 = 0. \tag{2.14}$$

2.5 Conharmonic Curvature Tensor

In this point of view, Y.Ishi [24] called a conharmonic transformation which is the conformal transformation $\phi : (M,g) \longrightarrow (\bar{M},\bar{g})$ satisfying (2.14). If we put

$$H(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{n-2}[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + g(X,U)S(Y,Z) - g(Y,U)S(X,Z)] \tag{2.15}$$

where R and S denote the Riemannian curvature tensor of type (0,4) defined by $R(X,Y,Z,U) = g(R(X,Y)Z,U)$ and the Ricci tensor of type (0,2), respectively. The curvature tensor defined by the equation (2.15) is known as the conharmonic curvature tensor. A manifold whose the conharmonic curvature tensor vanishes at every point of the manifold is called conharmonically flat. Thus, this tensor represents the deviation of the manifold from conharmonic flatness. It satisfies all the symmetric properties of the Riemannian curvature tensor.

By means of the equations (2.3)-(2.7) and (2.15), we have the following proposition.

Proposition 2.3 [25] Let $\phi : (M,g) \longrightarrow (\bar{M},\bar{g})$ be a conformal transformation. Then the following conditions are equivalent :

(1) ϕ is a conharmonic transformation.

(2) $\rho_k^k = 0$.

(3) $r = e^{2\rho}\bar{r}$.

(4) $\bar{g}_{ij}\bar{r} = g_{ij}r$.

(5) $\bar{S}_{ij} = S_{ij} - (n-2)\rho_{ij}$

(6) H_{kji}^h is invariant under ϕ.

If we consider (2.3) and (2.7), then we can easily get that the following proposition holds.

Proposition 2.4 [25] Let $\phi : (M,g) \longrightarrow (\bar{M},\bar{g})$ be a conformal transformation with $dimM > 2$. The Riemannian Christoffel curvature tensor is invariant under the transformation ϕ if and only if $\rho_{ij} = 0$.

There are many physical applications of the conharmonic curvature tensor. For example, Abdussattar, [38], showed that sufficient condition for a space-time to be conharmonic to a flat space-time is that the conharmonic curvature tensor vanishes identically. A conharmonically flat space-time is either empty in which case it is flat or filled with a distribution represented by the energy momentum tensor T possesing the algebraic structure of an electromagnetic field and conformal to a flat space-time [38]. Also he described the gravitational field due to a distribution of pure radiation in presence of disordered radiation by means of spherically symmetric conharmonically flat space-time. Conharmonic curvature tensor have been studied by S. A. Siddiqui and Z. Ahsan [39], C. Ozgur [40], Ghosh, De and Taleshian [41] and many others.

3. PSEUDO CONHARMONICALLY SYMMETRIC MANIFOLDS

3.1 Pseudo Symmetric Manifolds

As is well known, symmetric spaces play an important role in differential geometry. Riemannian symmetric spaces are the most beautiful and the most important Riemannian manifolds. This class of spaces contains many prominent examples which are of great importance for various branches of mathematics, like compact Lie groups, Grassmannians and bounded symmetric domains. Any symmetric space has its own special geometry; euclidean, elliptic and hyperbolic geometry are only the very first examples.

The study of Riemannian symmetric spaces was initiated in the late twenties by E. Cartan [13], who, in particular, obtained a classification of those spaces. Let (M,g) be an n-dimensional semi-Riemannian manifold, i.e. a manifold M with the metric tensor g with arbitrary signature and let ∇ be the Levi- Civita connection of (M,g).

A semi-Riemannian manifold is called locally symmetric [13] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M,g). This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [42]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature.

During the last five decades, the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent such as conformally symmetric manifolds by M. C. Chaki and B. Gupta [43], recurrent manifolds introduced by A. G. Walker [14], conformally recurrent manifolds by T. Adati and T. Miyazawa [20], pseudo symmetric manifolds introduced by M. C. Chaki [44], weakly symmetric manifolds by L. Tamassy and T. Q. Binh [18], projective symmetric manifolds by G. Soos [45], pseudo conformally symmetric spaces by U. C. De and H. A. Biswas [46], almost pseudo conformally symmetric spaces by U. C. De and A. K. Gazi [47], weakly conformally symmetric spaces by U. C. De and S. Bandyopadhyay

[48], pseudosymmetric manifolds by R. Deszcz [49], etc.

In 1967, R. N. Sen and M. C. Chaki [50] studied certain curvature restrictions on a certain kind of conformally flat space of class one and they obtained the following expression of the covariant derivative of the curvature tensor :

$$R^h_{ijk,l} = 2\lambda_l R^h_{ijk} + \lambda_i R^h_{ljk} + \lambda_j R^h_{ilk} + \lambda_k R^h_{ijl} + \lambda^h R_{lijk} \tag{3.1}$$

where R^h_{ijk} are the components of the curvature tensor R, $R_{lijk} = g_{hl}R^h_{ijk}$, λ_i is a non-zero covariant vector and $'$,$'$ denotes the covariant differentiation with respect to the metric tensor g_{ij}.

Later in 1987, M. C. Chaki [44] called a manifold whose curvature tensor satisfies (3.1), as a pseudo symmetric manifold.

In the index free notation, this can be stated as follows: A non-flat semi-Riemannian manifold (M,g), $n \geq 2$ is said to be a pseudo symmetric manifold [44] if its curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)W =& 2A(X)R(Y,Z)W + A(Y)R(X,Z)W + A(Z)R(Y,X)W \\ &+ A(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho\end{aligned} \tag{3.2}$$

where A is a non-zero 1-form, ρ is a vector field defined by

$$g(X,\rho) = A(X) \tag{3.3}$$

for all vector fields X and ∇ denotes the operator of the covariant differentiation with respect to the metric tensor g. The 1-form A is called the associated 1-form of the manifold. If $A = 0$, then the manifold reduces to a symmetric manifold in the sense of E. Cartan. An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$.

This is to be noted that the notion of pseudo symmetric manifold studied in particular by R. Deszcz ([49], [51], [52], [53]) is different from that of M. C. Chaki [44].

In 1988, M. C. Chaki [54] introduced the notion of a pseudo Ricci symmetric manifold. A non-flat Riemannian manifold is called pseudo Ricci symmetric and denoted by $(PRS)_n$ if the Ricci tensor S of type $(0,2)$ of the manifold is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y) \tag{3.4}$$

where ∇ denotes the Levi-Civita connection and A is a non-zero 1-form. The name "Pseudo Ricci symmetric" is chosen, because if the 1-form A in (3.4) is taken as zero, then the equation assumes the form $(\nabla_X S)(Y,Z) = 0$ and the manifold thus becomes Ricci symmetric.

Pseudo Ricci symmetric manifold is a particular case of a weakly Ricci symmetric manifold introduced by Tamassy and Binh [55]. It is known [54] that in a $(PRS)_n$ if the scalar curvature r is constant then $r = 0$.

A concrete example of a $(PRS)_n$ is given by Ozen and Altay [56]. Also, Sen and Chaki [50] studied hypersurfaces of a conformally flat space of class-one and obtained in a natural way the notion of the pseudo Ricci symmetric manifold. On the other hand, Chaki [54] proved the existence of a pseudo Ricci symmetric manifold by considering a linear connection on a Riemannian manifold.

3.2 Pseudo Conharmonically Symmetric Manifolds

Let us consider the conformal transformation be a conharmonic one as the equation (2.14). Thus, from the equation (2.14), a rank-four tensor H that remains invariant under the conharmonic transformation of a Riemannian manifold (M,g) is found as

$$H(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{n-2}[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + g(X,U)S(Y,Z) - g(Y,U)S(X,Z)] \quad \textbf{(3.5)}$$

where R and S denote the Riemannian curvature tensor of type (0,4) defined by $R(X,Y,Z,U) = g(R(X,Y)Z,U)$ and the Ricci tensor of type (0,2), respectively. The curvature tensor defined by (3.5) is known as the conharmonic curvature tensor.

A manifold whose conharmonic curvature tensor vanishes at every point of the manifold is called conharmonically flat. Thus, this tensor represents the deviation of the manifold from conharmonic flatness.

This thesis deals with an n-dimensional pseudo conharmonically symmetric Riemannian manifold (M,g) (non-conharmonically flat) whose the conharmonic curvature tensor H satisfies the condition

$$(\nabla_X H)(Y,Z,U,V) = 2A(X)H(Y,Z,U,V) + A(Y)H(X,Z,U,V) + A(Z)H(Y,X,U,V) + A(U)H(Y,Z,X,V) + A(V)H(Y,Z,U,X) \quad \textbf{(3.6)}$$

where A is a non-zero 1-form.

Such a manifold is called a pseudo conharmonically symmetric manifold and denoted by $(PCHS)_n$. Since the conformal curvature tensor vanishes identically for $n = 3$, we assume that $n > 3$ throughout this thesis.

L denotes the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S of type (0,2), that is

$$g(LX,Y) = S(X,Y). \tag{3.7}$$

Let e_i, $(1 \leq i \leq n)$ be an orthonormal basis of the tangent space at any point of the manifold. From (3.5), we have

$$\begin{aligned} \overline{H}(X,Y) &= \sum_{i=1}^{n} H(X,e_i,e_i,Y) = \sum_{i=1}^{n} H(e_i,X,Y,e_i) \\ &= -\frac{r}{n-2} g(X,Y) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \sum_{i=1}^{n} H(e_i,e_i,X,Y) &= \sum_{i=1}^{n} H(X,Y,e_i,e_i) \\ &= 0 \end{aligned} \tag{3.9}$$

where r is the scalar curvature of the manifold. Also, from (3.5) it follows that [57]

$$\begin{aligned} &H(X,Y,Z,U) = -H(Y,X,Z,U) \\ &H(X,Y,Z,U) = -H(X,Y,U,Z) \\ &H(X,Y,Z,U) = H(Z,U,X,Y) \\ &H(X,Y,Z,U) + H(X,Z,U,Y) + H(X,U,Y,Z) = 0. \end{aligned} \tag{3.10}$$

We assume that our manifold is $(PCHS)_n$. Thus, the relation (3.6) holds.

Proposition 3.1 [57] In a Riemannian manifold V_n $(n > 3)$, the conharmonic curvature tensor satisfies the second Bianchi Identity, i.e., the following relation

$$H_{hijk,l} + H_{hikl,j} + H_{hilj,k} = 0 \tag{3.11}$$

holds if and only if the Ricci tensor is of Codazzi type.

Theorem 3.1 In a pseudo conharmonically symmetric Riemannian manifold, the conharmonic curvature tensor satisfies the second Bianchi Identity, i.e.,

$$H_{hijk,l} + H_{hikl,j} + H_{hilj,k} = 0. \tag{3.12}$$

Proof. Permutating l,j,k in (3.6), we obtain

$$\begin{aligned} H_{hijk,l} &= 2A_lH_{hijk} + A_hH_{lijk} + A_iH_{hljk} + A_jH_{hilk} + A_kH_{hijl} \\ H_{hikl,j} &= 2A_jH_{hikl} + A_hH_{jikl} + A_iH_{hjkl} + A_kH_{hijl} + A_lH_{hikj} \\ H_{hikl,j} &= 2A_jH_{hikl} + A_hH_{jikl} + A_iH_{hjkl} + A_kH_{hijl} + A_lH_{hikj} \end{aligned} \tag{3.13}$$

and adding these three equations, we find

$$\begin{aligned} H_{hijk,l} + H_{hikl,j} + H_{hiljk} =& 2A_lH_{hijk} + A_hH_{lijk} + A_iH_{hljk} + A_jH_{hilk} + A_kH_{hijl} \\ &+ 2A_jH_{hikl} + A_hH_{jikl} + A_iH_{hjkl} + A_kH_{hijl} + A_lH_{hikj} \\ &+ 2A_jH_{hikl} + A_hH_{jikl} + A_iH_{hjkl} + A_kH_{hijl} + A_lH_{hikj}. \end{aligned} \tag{3.14}$$

Arraying (3.14), we get

$$\begin{aligned} H_{hijk,l} + H_{hikl,j} + H_{hiljk} =& A_l(2H_{hijk} + H_{hikj} + H_{hikj}) \\ &+ A_h(H_{lijk} + H_{jikl} + H_{kilj}) \\ &+ A_i(H_{hljk} + H_{hjkl} + H_{hklj}) \\ &+ A_j(H_{hilk} + 2H_{hikl} + H_{hilk}) \\ &+ A_k(H_{hijl} + H_{hijl} + 2H_{hilj}). \end{aligned} \tag{3.15}$$

Thus, from (3.10), (3.15) reduces to

$$H_{hijk,l} + H_{hikl,j} + H_{hiljk} = 0 \tag{3.16}$$

i.e. the conharmonic curvature tensor of $(PCHS)_n$ satisfies the second Bianchi Identity.

Theorem 3.2 A pseudo conharmonically symmetric manifold with non-zero scalar curvature is of closed associated 1-form.

Proof. Multiplying (3.6) by g^{hk}, we find

$$\overline{H}_{ij,l} = 2A_l\overline{H}_{ij} + A^hH_{lijh} + A_i\overline{H}_{lj} + A_j\overline{H}_{il} + A^hH_{hijl} \tag{3.17}$$

where $\overline{H}_{ij} = H^h_{ijh}$. Again multiplying (3.17) by g^{ij}, we obtain

$$\begin{aligned} h_{,l} &= 2A_lh + A^h\overline{H}_{lh} + A^i\overline{H}_{li} + A^i\overline{H}_{il} + A^h\overline{H}_{hl} \\ &= 2A_lh + 4A^h\overline{H}_{lh} \end{aligned} \tag{3.18}$$

where $h = \overline{H}_{ij}g^{ij}$. If we take the covariant derivative of (3.18), we find

$$h_{,lk} = 2A_{l,k}h + 2A_l h_{,k} + 4A^h_{,k}\overline{H}_{lh} + 4A^h\overline{H}_{lh,k}. \tag{3.19}$$

Changing l and k in (3.19), we obtain

$$h_{,kl} = 2A_{k,l}h + 2A_k h_{,l} + 4A^h_{,l}\overline{H}_{kh} + 4A^h\overline{H}_{kh,l} \tag{3.20}$$

and subtracting (3.19) from (3.20), we get

$$\begin{aligned} &2A_{l,k}h - 2A_{k,l}h + 2A_l h_{,k} - 2A_k h_{,l} \\ &+ 4A^h_{,k}\overline{H}_{lh} - 4A^h_{,l}\overline{H}_{kh} + 4A^h\overline{H}_{lh,k} - 4A^h\overline{H}_{kh,l} = 0. \end{aligned} \tag{3.21}$$

Putting (3.17) and (3.18) in (3.21), we obtain

$$\begin{aligned} &2(A_{l,k} - A_{k,l})h + 2A_l(2A_k h + 4A^h\overline{H}_{kh}) - 2A_k(2A_l h + 4A^h\overline{H}_{lh}) \\ &+ 4A^h_{,k}\overline{H}_{lh} - 4A^h_{,l}\overline{H}_{kh} \\ &+ 4A^h(2A_k\overline{H}_{lh} + A^m H_{klhm} + A_l\overline{H}_{kh} + A_h\overline{H}_{lk} + A^m H_{mlhk}) \\ &- 4A^h(2A_l\overline{H}_{kh} + A^m H_{lkhm} + A_k\overline{H}_{lh} + A_h\overline{H}_{kl} + A^m H_{mkhl}) = 0. \end{aligned} \tag{3.22}$$

From (3.10), we find

$$\begin{aligned} &2(A_{l,k} - A_{k,l})h + 4A^h_{,k}\overline{H}_{lh} - 4A^h_{,l}\overline{H}_{kh} + 4A^h A^m(H_{klhm} + H_{mlhk} - H_{lkhm} - H_{mkhl}) \\ &+ 4A^h A_l\overline{H}_{kh} - 4A^h A_k\overline{H}_{lh} = 0. \end{aligned} \tag{3.23}$$

In this case, from the second Bianchi Identity,

$$H_{mlhk} + H_{mklh} + H_{mhkl} = 0$$

$$H_{mlhk} - H_{mkhl} = -H_{mhkl} = H_{klhm}$$

$$H_{klhm} + H_{mlhk} - H_{lkhm} - H_{mkhl} = 3H_{klhm}. \tag{3.24}$$

Thus, by using (3.24), it can be obtained that

$$2(A_{l,k} - A_{k,l})h + 4A^h_{,k}\overline{H}_{lh} - 4A^h_{,l}\overline{H}_{kh} + 12A^h A^m H_{klhm} + 4A^h A_l\overline{H}_{kh} - 4A^h A_k\overline{H}_{lh} = 0 \tag{3.25}$$

and by using (3.10), it is found that $A^h A^m H_{klhm} = 0$. In this case, we can easily see from (3.25) that

$$2(A_{l,k} - A_{k,l})h + 4A^h_{,k}\overline{H}_{lh} - 4A^h_{,l}\overline{H}_{kh} + 4A^h A_l\overline{H}_{kh} - 4A^h A_k\overline{H}_{lh} = 0. \tag{3.26}$$

Putting $h = -\frac{n}{n-2}r$ and $\overline{H}_{ij} = -\frac{r}{n-2}g_{ij}$ in (3.26)

$$2(A_{l,k} - A_{k,l})(-\frac{n}{n-2}r) + 4A^h_{,k}(-\frac{r}{n-2}g_{lh}) - 4A^h_{,l}(-\frac{r}{n-2}g_{kh}) + 4A^hA_l(-\frac{r}{n-2}g_{kh}) - 4A^hA_k(-\frac{r}{n-2}g_{lh}) = 0. \tag{3.27}$$

In this case, the equation (3.27) can be reduces to

$$r[2n(A_{l,k} - A_{k,l}) + 4(A^h_{,k}g_{lh} - A^h_{,l}g_{kh}) + 4(A^hA_lg_{kh} - A^hA_kg_{lh})] = 0. \tag{3.28}$$

And, we get from (3.28),

$$r[2n(A_{l,k} - A_{k,l}) + 4(A_{l,k} - A_{k,l}) + 4(A_kA_l - A_lA_k)] = 0. \tag{3.29}$$

Finally,

$$(2n+4)(A_{l,k} - A_{k,l})r = 0. \tag{3.30}$$

If our manifold admits non-zero scalar curvature then it follows from (3.30) that

$$A_{l,k} - A_{k,l} = 0. \tag{3.31}$$

By the aid of (3.31), we can say that the associated 1-form of this manifold is closed. Thus, the proof is completed.

Theorem 3.3 A Riemannian manifold admits a divergence-free conharmonic curvature tensor is of constant scalar curvature.

Proof. From the second Bianchi Identity, we have

$$R^h_{ijk,h} = S_{ij,k} - S_{ik,j} \tag{3.32}$$

and then

$$S^h_{k,h} = \frac{1}{2}r_{,k} \tag{3.33}$$

where r is the scalar curvature and S_{ij} is the Ricci tensor of this manifold. Thus from (3.5),

$$H^h_{ijk,l} = R^h_{ijk,l} - \frac{1}{n-2}(g^{hm}g_{ij}S_{mk,l} - \delta^h_jS_{ik,l} + \delta^h_kS_{ij,l} - g^{hm}g_{ik}S_{mj,l}). \tag{3.34}$$

Contracting on h and l in (3.34), we find

$$H^h_{ijk,h} = R^h_{ijk,h} - \frac{1}{n-2}(g_{ij}S^h_{k,h} - S_{ik,j} + S_{ij,k} + g_{ik}S^h_{j,h}). \tag{3.35}$$

By using (3.32) and (3.33), (3.35) reduces to

$$\begin{aligned} H^h_{ijk,h} =& S_{ij,k} - S_{ik,j} - \frac{1}{n-2}(\frac{1}{2}g_{ij}r_{,k} - S_{ik,j} + S_{ij,k} + \frac{1}{2}g_{ik}r_{,j}) \\ =& \frac{n-3}{n-2}(S_{ij,k} - S_{ik,j}) - \frac{1}{2(n-2)}(g_{ij}r_{,k} - g_{ik}r_{,j}). \end{aligned} \tag{3.36}$$

If we assume that the conharmonic curvature tensor of this manifold is divergence-free then we find from (3.36)

$$(n-3)(S_{ij,k} - S_{ik,j}) - \frac{1}{2}(g_{ij}r_{,k} - g_{ik}r_{,j}) = 0. \tag{3.37}$$

Multiplying (3.37) by g^{ij} and putting (3.33) in (3.37), we obtain

$$(n-3)(r_{,k} - S^j_{k,j}) - \frac{1}{2}(n-1)r_{,k} = 0 \tag{3.38}$$

and then

$$(n-3)(r_{,k} - \frac{1}{2}r_{,k}) - \frac{1}{2}(n-1)r_{,k} = 0. \tag{3.39}$$

In this case,

$$-2r_{,k} = 0. \tag{3.40}$$

Thus, we finally get that the scalar curvature of this manifold r is constant.
This completes the proof.

Theorem 3.4 A $(PCHS)_n$ admits a divergence-free conharmonic curvature tensor is of zero scalar curvature tensor.

Proof. Assuming that the conharmonic curvature tensor of $(PCHS)_n$ is divergence-free, from (3.6), we get

$$2A_hH^h_{ijk} + A^hH_{hijk} - A_j\overline{H}_{ik} + A_k\overline{H}_{ij} = 0. \tag{3.41}$$

Multiplying (3.41) by g^{ij}, we obtain

$$2A^h\overline{H}_{hk} + A^h\overline{H}_{hk} - A^i\overline{H}_{ik} + A_kh = 0 \tag{3.42}$$

and

$$2A^h\overline{H}_{hk} = -A_kh. \tag{3.43}$$

It follows from (3.5), (3.43) reduces to

$$2A^h(-\frac{r}{n-2}g_{hk}) = -A_k(-\frac{n}{n-2}r). \tag{3.44}$$

Thus,

$$rA_k = 0. \tag{3.45}$$

Since A_k is not zero for $(PCHS)_n$, we get that r must be zero. The proof is completed.

Theorem 3.5 If $(PCHS)_n$ is recurrent then either the scalar curvature of this manifold is zero or the recurrence vector field and the associated 1-form are related by

$$\lambda_l = \frac{2(n+2)}{n}A_l.$$

Proof. By taking the covariant derivative of (3.5), we get

$$H_{hijk,l} = R_{hijk,l} - \frac{1}{n-2}(g_{ij}S_{hk,l} - g_{hj}S_{ik,l} + g_{hk}S_{ij,l} - g_{ik}S_{hj,l}). \tag{3.46}$$

Comparing (3.46) with (3.6), using (3.5) and assuming our manifold is recurrent, i.e,

$$R_{hijk,l} = \lambda_l R_{hijk} \tag{3.47}$$

and

$$S_{ij,l} = \lambda_l S_{ij}, \quad r_{,l} = \lambda_l r \tag{3.48}$$

are satisfied. By using (3.6), (3.47) and (3.48), we obtain

$$\begin{aligned}
&\lambda_l R_{hijk} - \frac{1}{n-2}(g_{ij}\lambda_l S_{hk} - g_{hj}\lambda_l S_{ik} + g_{hk}\lambda_l S_{ij} - g_{ik}\lambda_l S_{hj}) \\
&= 2A_l(R_{hijk} - \frac{1}{n-2}(g_{ij}S_{hk} - g_{hj}S_{ik} + g_{hk}S_{ij} - g_{ik}S_{hj})) \\
&+ A_h(R_{lijk} - \frac{1}{n-2}(g_{ij}S_{lk} - g_{lj}S_{ik} + g_{lk}S_{ij} - g_{ik}S_{lj})) \\
&+ A_i(R_{hljk} - \frac{1}{n-2}(g_{lj}S_{hk} - g_{hj}S_{lk} + g_{hk}S_{lj} - g_{lk}S_{hj})) \\
&+ A_j(R_{hilk} - \frac{1}{n-2}(g_{il}S_{hk} - g_{hl}S_{ik} + g_{hk}S_{il} - g_{ik}S_{hl})) \\
&+ A_k(R_{hijl} - \frac{1}{n-2}(g_{ij}S_{hl} - g_{hj}S_{il} + g_{hl}S_{ij} - g_{il}S_{hj})).
\end{aligned} \tag{3.49}$$

Arraying the equation (3.49), we find

$$
\begin{aligned}
&(\lambda_l - 2A_l)R_{hijk} - A_hR_{lijk} - A_iR_{hljk} - A_jR_{hilk} + A_kR_{hijl} \\
&+ \frac{1}{n-2}[S_{hk}(2A_lg_{ij} + A_ig_{lj} + A_jg_{il} - \lambda_lg_{ij}) \\
&- S_{ik}(2A_lg_{hj} + A_hg_{lj} + A_jg_{hl} - \lambda_lg_{hj}) \\
&+ S_{ij}(2A_lg_{hk} + A_hg_{lk} + A_kg_{hl} - \lambda_lg_{hk}) \\
&- S_{hj}(2A_lg_{ik} + A_ig_{lk} + A_kg_{il} - \lambda_lg_{ik}) \\
&+ S_{lk}(A_hg_{ij} - A_ig_{hj}) - S_{lj}(A_hg_{ik} - A_ig_{hk}) \\
&+ S_{il}(A_jg_{hk} - A_kg_{hj}) - S_{hl}(A_jg_{ik} - A_kg_{ij})] = 0. \qquad \textbf{(3.50)}
\end{aligned}
$$

Multiplying (3.50) by g^{hk} and g^{ij}, we get

$$r(-n(\lambda_l - 2A_l) + 4A_l) = 0. \qquad \textbf{(3.51)}$$

It follows from (3.51) that either r is zero or $\lambda_l = \frac{2(n+2)}{n}A_l$. Thus, this completes the proof.

3.3 Some Properties of Pseudo Conharmonically Symmetric Riemannian Manifolds

Theorem 3.6 If a $(PCHS)_n$ admits generalized recurrent curvature tensor then the scalar curvature of this manifold is in the following form

$$r = \frac{n^2(n-1)\beta_i}{(2n+4)A_i - n\alpha_i} \quad (A_i \neq \frac{n}{2n+4}\alpha_i)$$

where β and α are the recurrence vector fields and A is the associated 1-form of this manifold.

Proof. If we assume that $(PCHS)_n$ admits generalized recurrent curvature tensor then from (1.46), we also have

$$S_{ij,l} = \alpha_lS_{ij} + (n-1)\beta_lg_{ij}. \qquad \textbf{(3.52)}$$

By taking the covariant derivative of (3.5) and putting (1.46) and (3.52) in the last equation, we find

$$H_{hijk,l} = R_{hijk,l} - \frac{1}{n-2}(g_{ij}S_{hk,l} - g_{hj}S_{ik,l} + g_{hk}S_{ij,l} - g_{ik}S_{hj,l}). \qquad \textbf{(3.53)}$$

Using (1.46) and (3.52), we obtain

$$\begin{aligned} H_{hijk,l} =& \alpha_l R_{hijk} + \beta_l (g_{hk}g_{ij} - g_{hj}g_{ik}) \\ & - \frac{1}{n-2}[g_{ij}(\alpha_l S_{hk} + (n-1)\beta_l g_{hk}) - g_{hj}(\alpha_l S_{ik} + (n-1)\beta_l g_{ik}) \\ & + g_{hk}(\alpha_l S_{ij} + (n-1)\beta_l g_{ij}) - g_{ik}(\alpha_l S_{hj} + (n-1)\beta_l g_{hj})]. \end{aligned} \tag{3.54}$$

Thus, we get

$$\begin{aligned} H_{hijk,l} =& \alpha_l R_{hijk} + \beta_l (g_{hk}g_{ij} - g_{hj}g_{ik}) \\ & - \frac{1}{n-2}[\alpha_l (g_{ij}S_{hk} - g_{hj}S_{ik} + g_{hk}S_{ij} - g_{ik}S_{hj})] \\ & - \frac{n-1}{n-2}\beta_l [2(g_{hk}g_{ij} - g_{hj}g_{ik})]. \end{aligned} \tag{3.55}$$

In this case,

$$\begin{aligned} H_{hijk,l} =& \alpha_l [R_{hijk} - \frac{1}{n-2}(g_{ij}S_{hk} - g_{hj}S_{ik} + g_{hk}S_{ij} - g_{ik}S_{hj})] \\ & - \frac{n}{n-2}\beta_l (g_{hk}g_{ij} - g_{hj}g_{ik}). \end{aligned} \tag{3.56}$$

By putting (3.5) in (3.56), we obtain

$$H_{hijk,l} = \alpha_l H_{hijk} - \frac{n}{n-2}\beta_l (g_{hk}g_{ij} - g_{hj}g_{ik}). \tag{3.57}$$

Contracting on h and k in (3.57), we find

$$\overline{H}_{ij,l} = \alpha_l \overline{H}_{ij} - \frac{n}{n-2}\beta_l (n-1) g_{ij}. \tag{3.58}$$

After that, contracting on i and j in (3.58), it can be obtained that

$$h_{,l} = \alpha_l h - \frac{n^2(n-1)}{n-2}\beta_l. \tag{3.59}$$

Using (3.5), we get

$$-\frac{n}{n-2}r_{,l} = \alpha_l (-\frac{n}{n-2}r) - \frac{n^2(n-1)}{n-2}\beta_l \tag{3.60}$$

$r_{,l} = \lambda_l r$ and putting $\lambda_l = \frac{2n+4}{n}A_l$ in (3.60), it can be found that

$$n(\frac{2n+4}{n})A_l r - n\alpha_l r = n^2(n-1)\beta_l. \tag{3.61}$$

Then, (3.61) reduces to

$$[(2n+4)A_l - n\alpha_l]r = n^2(n-1)\beta_l. \tag{3.62}$$

Finally, we get from (3.62)

$$r = \frac{n^2(n-1)\beta_l}{(2n+4)A_l - n\alpha_l}$$

where $A_l \neq \frac{n}{2n+4}\alpha_l$. Thus, the proof is completed.

Theorem 3.7 If the conharmonic curvature tensor of a $(PCHS)_n$ is generalized recurrent then the scalar curvature of this manifold is in the following form

$$r = \frac{n(2-n)(n-1)\beta_i}{2(n+2)A_i - n\alpha_i}$$

where $\alpha_i \neq \frac{2(n+2)}{n}A_i$. And if $\alpha_i > \frac{2(n+2)}{n}A_i$ then this manifold is of positive scalar curvature or if $\alpha_i < \frac{2(n+2)}{n}A_i$ then it is of negative scalar curvature where β_i have positive values.

Proof. Now, we assume that the conharmonic curvature tensor of $(PCHS)_n$ is generalized recurrent, from (1.46), we can write

$$H_{hijk,l} = \alpha_l H_{hijk} + \beta_l(g_{hk}g_{ij} - g_{hj}g_{ik}). \tag{3.63}$$

Comparing the equation (3.63) with (3.6), we get

$$\alpha_l H_{hijk} + \beta_l(g_{hk}g_{ij} - g_{hj}g_{ik}) = 2A_l H_{hijk} + A_h H_{lijk} + A_i H_{hljk} + A_j H_{hilk} + A_k H_{hijl}. \tag{3.64}$$

Contracting on h and k in (3.64)

$$\alpha_l \overline{H}_{ij} + \beta_l(n-1)g_{ij} = 2A_l \overline{H}_{ij} + A^h H_{lijh} + A_i \overline{H}_{lj} + A_j \overline{H}_{il} + A_h H_{hijl}. \tag{3.65}$$

Again, contracting on i and j in (3.65),

$$\alpha_l h + \beta_l n(n-1) = 2A_l h + 4A^h \overline{H}_{lh}. \tag{3.66}$$

Using (3.5), we get

$$\alpha_l(-\frac{n}{n-2}r) + \beta_l n(n-1) = 2A_l(-\frac{n}{n-2}r) + 4A^h(-\frac{r}{n-2}g_{lh}). \tag{3.67}$$

And then

$$[(2n+4)A_l - n\alpha_l]r = n(2-n)(n-1)\beta_l. \tag{3.68}$$

Finally, we obtain from (3.68)

$$r = \frac{n(2-n)(n-1)\beta_i}{2(n+2)A_i - n\alpha_i} \tag{3.69}$$

where $\alpha_i \neq \frac{2(n+2)}{n}A_i$. Assuming that β_i have positive values. Thus, if $\alpha_i > \frac{2(n+2)}{n}A_i$, from (3.69) then the scalar curvature is positive or if $\alpha_i < \frac{2(n+2)}{n}A_i$ then the scalar curvature is negative. Thus, the proof is completed.

Corollary 3.1 If the conharmonic curvature tensor of $(PCHS)_n$ with non-zero scalar curvature is recurrent then the recurrence vector field and the associated 1-form of this manifold are related by the following form

$$\alpha_i = \frac{2(n+2)}{n}A_i.$$

Proof. If we assume that the conharmonic curvature tensor of $(PCHS)_n$ is recurrent, from Theorem 3.7, for $\beta_i = 0$, we get

$$r(2(n+2)A_l - n\alpha_l) = 0. \tag{3.70}$$

Considering that $(PCHS)_n$ admits non-zero scalar curvature. By the aid of (3.70), we find that

$$\alpha_l = \frac{2(n+2)}{n}A_l.$$

Thus, the proof is completed.

Theorem 3.8 If the conharmonic curvature tensor of a $(PCHS)_n$ is Codazzi type then this manifold must be of zero scalar curvature.

Proof. Let us assume that the conharmonic curvature tensor of $(PCHS)_n$ be Codazzi type. Then we have

$$\overline{H}_{ij,l} - \overline{H}_{il,j} = 0 \tag{3.71}$$

where $\overline{H}$ is the form as in (3.8). Contracting (3.6) on h and k,

$$\overline{H}_{ij,l} = 2A_l\overline{H}_{ij} + A^h H_{lijh} + A_i\overline{H}_{lj} + A_j\overline{H}_{il} + A^h H_{hijl}. \tag{3.72}$$

By changing j and l in (3.72), we get

$$\overline{H}_{il,j} = 2A_j\overline{H}_{il} + A^h H_{jilh} + A_i\overline{H}_{jl} + A_l\overline{H}_{ij} + A^h H_{hilj}. \tag{3.73}$$

Subtracting two equations (3.72) and (3.73), we obtain

$$A_l\overline{H}_{ij} + A^h H_{lijh} - A_j\overline{H}_{il} + 2A^h H_{hijl} - A^h H_{jilh} = 0. \tag{3.74}$$

Contracting on i and j in (3.74),

$$A_l h + 2A^h \overline{H}_{hl} = 0. \tag{3.75}$$

Using (3.5) in (3.75), it can be seen that

$$A_l(-\frac{n}{n-2}r) + 2A^h(-\frac{r}{n-2}g_{hl}) = 0. \tag{3.76}$$

From (3.76), we finally get

$$(n+2)A_l r = 0. \tag{3.77}$$

Since $A_l \neq 0$, the scalar curvature of $(PCHS)_n$ must be zero. This completes the proof.

Theorem 3.9 In a $(PCHS)_n$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P if this manifold admits pseudo symmetric curvature tensor whose the associated 1-form is the same with the 1-form of $(PCHS)_n$ where

$$g(X,P) = A(X)$$

for every vector field P and non-zero 1-form A.

Proof. We assume that the curvature tensor of $(PCHS)_n$ is pseudo symmetric. Then we have the equation (3.1). By using (3.1), we get

$$r_{,m} = 2A_m r + 4A^l S_{ml}. \tag{3.78}$$

By taking the covariant derivative of (3.5), we get

$$h_{,m} = 2A_m h + 4A^l \overline{H}_{ml}. \tag{3.79}$$

Moreover, by taking the covariant derivative of (3.8) and putting it in (3.79),we get

$$-\frac{n}{n-2}r_{,m} = 2A_m h + 4A^l \overline{H}_{ml}. \tag{3.80}$$

Then putting (3.78) and (3.8) in (3.80), we find

$$-\frac{n}{n-2}2A_m r + 4A^l S_{ml} = -\frac{2nr}{n-2}A_m - \frac{4r}{n-2}A^l g_{ml} \tag{3.81}$$

and

$$4nA^l S_{ml} = 4rA_m. \tag{3.82}$$

In this case,

$$A^l S_{ml} = \frac{1}{n} r A_m. \tag{3.83}$$

Thus, the proof is completed.

Theorem 3.10 If a $(PCHS)_n$ with non-constant scalar curvature admits the associated vector field as a torse-forming vector field related by the vector field ϕ_k then the vector fields ϕ_k and A_k must be collinear.

Proof. Contracting on h and k in (3.6)

$$\overline{H}_{ij,l} = 2A_l\overline{H}_{ij} + A^h H_{lijh} + A_i\overline{H}_{lj} + A_j\overline{H}_{il} + A^h H_{hijl} \tag{3.84}$$

and contracting again on i and j in (3.84), we obtain

$$h_{,l} = 2A_l h + 4A^h\overline{H}_{lh}. \tag{3.85}$$

Then, using (3.5), we get

$$r_{,l} = \frac{2(n+2)}{n} A_l r. \tag{3.86}$$

By taking the covariant derivative of (3.86), we find

$$r_{,lk} = \frac{2(n+2)}{n}(A_{l,k} r + A_l r_{,k}). \tag{3.87}$$

Substituting (3.86) in (3.87), it can be easily seen that

$$r_{,lk} = \frac{2(n+2)}{n}[A_{l,k} + \frac{2(n+2)}{n} A_l A_k] r. \tag{3.88}$$

If we take A_k as a torse-forming vector field given by

$$A_{l,k} = \rho g_{lk} + \phi_k A_l \tag{3.89}$$

where ρ is a scalar function, ϕ is a linear form then by putting (3.89) in (3.88), we obtain

$$r_{,lk} = \frac{2(n+2)}{n}[\rho g_{lk} + \phi_k A_l + \frac{2(n+2)}{n} A_l A_k] r. \tag{3.90}$$

By changing l and k in (3.90), it is seen that

$$r_{,kl} = \frac{2(n+2)}{n}[\rho g_{kl} + \phi_l A_k + \frac{2(n+2)}{n} A_k A_l] r. \tag{3.91}$$

Subtracting the equation(3.90) from (3.91), we get

$$\phi_k A_l - \phi_l A_k = 0. \tag{3.92}$$

In this case, it can be said that ϕ_k and A_k are collinear. Thus, the proof is completed.

4. PSEUDO CONHARMONICALLY SYMMETRIC SPACETIMES

This section deals with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study, the spacetime of general relativity is a connected four-dimensional semi-Riemannian manifold (M^4, g) with Lorentz metric g with signature $(-,+,+,+)$. The geometry of the Lorentz metric begins with the study of the casual character of vectors of the manifold. It is due to this casuality that the Lorentz manifold becomes a convenient choice for the study of general relativity.

Here, we consider a special type of spacetime which is called pseudo conharmonically symmetric spacetime. A semi-Riemannian four-dimensional pseudo conharmonically symmetric manifold may similarly be defined by taking a Lorentz metric g with signature $(-,+,+,+)$.

In this section, we consider a perfect fluid pseudo conharmonically symmetric spacetime having the basic vector field as the timelike velocity vector field U of the fluid, that is

$$g(U,U) = -1. \tag{4.1}$$

For a perfect fluid spacetime with the generator U as the flow vector field of the flow and k as the gravitational constant, we have the Einstein's equation without cosmological constant as

$$S_{ij} - \frac{r}{2} g_{ij} = kT_{ij} \tag{4.2}$$

where k is the gravitational constant, T is the energy momentum tensor of type (0,2) given by

$$T_{ij} = (\sigma + p)A_i A_j + p g_{ij} \tag{4.3}$$

with σ and p as the energy density and isotropic pressure of the fluid, respectively, and A as the associated 1-form.

By the aid of (4.2) and (4.3), we can write

$$S_{ij} = (\frac{r}{2} + kp)g_{ij} + k(\sigma + p)A_iA_j. \tag{4.4}$$

Taking a frame field and contracting (4.4) over i and j, we obtain

$$r = k(\sigma - 3p). \tag{4.5}$$

Firstly, we give some properties of $(PCHS)_n$ to use the other theorems.

Theorem 4.1 The scalar curvature of a $(PCHS)_n$ is of the following form

$$r_{,l} = (\frac{2n+4}{n})A_l r$$

where A_l is the associated vector field and r is the scalar curvature of $(PCHS)_n$.

Proof. We assume that our manifold is $(PCHS)_n$. From (3.6), we have

$$H_{ijkl,m} = 2A_mH_{ijkl} + A_iH_{mjkl} + A_jH_{imkl} + A_kH_{ijml} + A_lH_{ijkm} \tag{4.6}$$

where A_l is the associated vector field of $(PCHS)_n$.

Multiplying (4.6) by $g^{il}g^{jk}$, we get

$$h_{,m} = 2A_mh + 4A^l\overline{H}_{ml}. \tag{4.7}$$

By taking the covariant derivative of (3.8), we find

$$h_{,m} = -\frac{n}{n-2}r_{,m} \tag{4.8}$$

and using (4.7), we obtain

$$r_{,l} = (\frac{2n+4}{n})A_l r. \tag{4.9}$$

Thus, the proof is completed.

Corollary 4.1 In a $(PCHS)_n$, if the scalar curvature r is constant then r must be zero.

Proof. In a $(PCHS)_n$, we have (4.9). If we assume that r is constant, from (4.9), we get $A_l r = 0$. Since $A_l \neq 0$ in a $(PCHS)_n$, thus, r must be zero.

Theorem 4.2 For a perfect fluid of $(PCHS)_4$ spacetime, if the Ricci tensor is recurrent then the energy density and isotropic pressure are proportional.

Proof. In a perfect fluid of $(PCHS)_4$ spacetime, we have (4.4). By taking the covariant derivative of (4.4), we get

$$S_{ij,l} = (\frac{r_{,l}}{2} + kp_{,l})g_{ij} + k(\sigma_{,l} + p_{,l})A_iA_j + k(\sigma + p)(A_{i,l}A_j + A_iA_{j,l}). \tag{4.10}$$

If our spacetime is Ricci-recurrent then, from Theorem 3.5, we get

$$S_{ij,l} = 3A_lS_{ij}. \tag{4.11}$$

Comparing (4.11) with (4.10) and using (4.4), we can find that

$$\begin{aligned} 3A_l[(\frac{r}{2} + kp)g_{ij} + k(\sigma + p)A_iA_j] &= (\frac{3A_lr}{2} + kp_{,l})g_{ij} \\ &+ k(\sigma_{,l} + p_{,l})A_iA_j + k(\sigma + p)(A_{i,l}A_j + A_iA_{j,l}). \end{aligned} \tag{4.12}$$

And then

$$(p_{,l} - 3A_lp)g_{ij} + (\sigma_{,l} + p_{,l} - 3A_l(\sigma + p))A_iA_j + (\sigma + p)(A_{i,l}A_j + A_iA_{j,l}) = 0. \tag{4.13}$$

Multiplying (4.13) by A^iA^j, we get

$$\sigma_{,l} = 3A_l\sigma. \tag{4.14}$$

From Theorem 4.1, for a $(PCHS)_4$ spacetime, we have

$$r_{,l} = 3A_lr. \tag{4.15}$$

By taking the covariant derivative of (4.5), we find

$$r_{,l} = k(\sigma_{,l} - 3p_{,l}). \tag{4.16}$$

In this case,by using (4.15), we obtain

$$3A_lr = k(\sigma_{,l} - 3p_{,l}). \tag{4.17}$$

Then using (4.5) and (4.14), we get

$$3A_lk(\sigma - 3p) = k(3A_l\sigma - 3p_{,l}) \tag{4.18}$$

which yields to

$$p_{,l} = 3A_lp. \tag{4.19}$$

Comparing (4.14) and (4.19), we finally get

$$\sigma = cp. \tag{4.20}$$

Thus, this completes the proof.

Theorem 4.3 A perfect fluid of $(PCHS)_4$ spacetime which is Ricci-recurrent reduces to η-Einstein spacetime with the condition $\eta = 2kp$.

Proof. In a perfect fluid of $(PCHS)_4$ spacetime, if the Ricci tensor is recurrent then the condition (4.20) holds. By taking $c = 1$ in (4.20), we find

$$\sigma = p \tag{4.21}$$

which yields by virtue of (4.5) and (4.21) to

$$r = -2kp. \tag{4.22}$$

Comparing (4.4), (4.21) and (4.22), we find

$$S_{ij} = (-\frac{2kp}{2} + kp)g_{ij} + 2kpA_iA_j. \tag{4.23}$$

Finally, it can be obtained that

$$S_{ij} = 2kpA_iA_j. \tag{4.24}$$

Thus, the equation (4.24) shows that this spacetime reduces to η-Einstein spacetime with the condition $\eta = 2kp$. This completes the proof.

Theorem 4.4 In a perfect fluid of $(PCHS)_4$ spacetime, if the energy momentum tensor obeying Einstein's equation without cosmological constant is recurrent then either the associated vector field of this manifold is divergence-free or this manifold reduces to an Einstein spacetime.

Proof. By taking the covariant derivative of (4.3), we find

$$T_{ij,l} = (\sigma_{,l} + p_{,l})A_iA_j + (\sigma + p)(A_{i,l}A_j + A_iA_{j,l}) + p_{,l}g_{ij}. \tag{4.25}$$

We assume that our spacetime is Ricci-recurrent.

By taking the covariant derivative of (4.2) and using (4.11), we find

$$S_{ij,l} - \frac{r_{,l}}{2}g_{ij} = kT_{ij,l}. \tag{4.26}$$

Thus,

$$3A_l S_{ij} - \frac{r_{,l}}{2} g_{ij} = kT_{ij,l}. \tag{4.27}$$

By using the equation (4.2) and (4.15), we obtain

$$3A_l(kT_{ij} + \frac{r}{2} g_{ij}) - \frac{3A_l r}{2} g_{ij} = kT_{ij,l}. \tag{4.28}$$

In this case,

$$T_{ij,l} = 3A_l T_{ij}. \tag{4.29}$$

Putting (4.29) in (4.25), we get

$$3A_l T_{ij} = (\sigma_{,l} + p_{,l})A_i A_j + (\sigma + p)(A_{i,l}A_j + A_i A_{j,l}) + p_{,l} g_{ij}. \tag{4.30}$$

Changing the indices j and l in (4.30), we find

$$3A_j T_{il} = (\sigma_{,j} + p_{,j})A_i A_l + (\sigma + p)(A_{i,j}A_l + A_i A_{l,j}) + p_{,j} g_{il}. \tag{4.31}$$

Subtracting the equations (4.30) and (4.31) and using (4.3), we obtain

$$\begin{aligned} 3A_l[(\sigma + p)A_i A_j + p g_{ij}] - 3A_j[(\sigma + p)A_i A_l + p g_{il}] &= p_{,l} g_{ij} - p_{,j} g_{il} \\ &+ (\sigma_{,l} + p_{,l})A_i A_j - (\sigma_{,j} + p_{,j})A_i A_l \\ &+ (\sigma + p)[(A_{i,l}A_j + A_i A_{j,l}) - (A_{i,j}A_l + A_i A_{l,j})]. \end{aligned} \tag{4.32}$$

Contracting on i and j in (4.32), we get

$$\begin{aligned} 3A_l[-(\sigma + p) + 4p] + 3A_l(\sigma + p) - 3A_l p &= 4p_{,l} - p_{,l} \\ &- (\sigma_{,l} + p_{,l}) - (\sigma_{,j} + p_{,j})A^j A_l - (\sigma + p)(A^j_{,j}A_l + A^j A_{l,j}). \end{aligned} \tag{4.33}$$

In this case, arraying the equation (4.33), it is obtained that

$$9A_l p = 2p_{,l} - \sigma_{,l} - (\sigma_{,j} + p_{,j})A^j A_l - (\sigma + p)(A^j_{,j}A_l + A^j A_{l,j}). \tag{4.34}$$

Multiplying (4.34) by A^l, we find

$$-9p = 3A^l p_{,l} + (\sigma + p)A^l_{,l}. \tag{4.35}$$

Since our spacetime is Ricci-recurrent, from Theorem 4.2, we get from (4.19) and (4.35)

$$-9p = 3A^l(3A_l p) + (\sigma + p)A^l_{,l}. \tag{4.36}$$

Thus

$$(\sigma + p)A^{l}_{,l} = 0. \tag{4.37}$$

If we assume that $\sigma = -p$ then, from (4.5) and (4.4) it can be obtained that

$$r = k(\sigma - 3p) = -4kp. \tag{4.38}$$

With the help of (4.4) and (4.38), we get

$$S_{ij} = (-\frac{4kp}{2} + kp)g_{ij} + k(-p + p)A_iA_j. \tag{4.39}$$

Finally, we obtain

$$S_{ij} = -kpg_{ij}. \tag{4.40}$$

The equation (4.40) gives us, this spacetime reduces to an Einstein spacetime. Thus, we must say that $\sigma \neq -p$. In this case, the vector field generated by the 1-form A is divergence-free. This completes the proof.

Theorem 4.5 A perfect fluid $(PCHS)_4$ spacetime admits Codazzi type conharmonic curvature tensor represents the equation of state in the radiation era in the evoluation of our universe.

Proof. In a perfect fluid for $(PCHS)_4$ spacetime, if the conharmonic curvature tensor is Codazzi type then from Theorem 3.8, we can say that the scalar curvature must be zero. In this case, the equation (4.5) reduces to

$$\sigma = 3p.$$

This corresponds the equation of state in the radiation era in the evoluation of our universe. The radiation is the era before the present matter dominated era. Thus, the proof is completed.

5. SOME EXAMPLES OF PSEUDO CONHARMONICALLY SYMMETRIC MANIFOLDS

In this section we will give some examples for $(PCHS)_n$ satisfying the conditions (3.5) and (3.6).

Example 5.1 We define a Riemannian metric on R^n $(n \geq 4)$ by the formula,

$$ds^2 = \varphi(dx^1)^2 + k_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1dx^n \tag{5.1}$$

where $[k_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constant and φ is a function of $x^1, x^2, ..., x^{n-1}$ and independent of x^n. Let each Latin index runs over $1, 2, .., n$ and each Greek index runs over $2, 3, .., (n-1)$.

In the metric considered, the only non-vanishing components of Christoffel symbols, the curvature tensor and the Ricci tensor are, according to [58]

$$\begin{gathered}\Gamma^\beta_{11} = -\frac{1}{2}k^{\alpha\beta}\varphi_{,\alpha}, \quad \Gamma^n_{11} = \frac{1}{2}\varphi_{,1} \quad \Gamma^n_{1\alpha} = \frac{1}{2}\varphi_{,\alpha} \\ R_{1\alpha\beta1} = \frac{1}{2}\varphi_{,\alpha\beta}, \quad S_{11} = \frac{1}{2}k^{\alpha\beta}\varphi_{,\alpha\beta}\end{gathered} \tag{5.2}$$

where "," denotes the partial differentiation with respect to the coordinates and $k^{\alpha\beta}$ are the elements of the matrix inverse to $[k_{\alpha\beta}]$.

We consider $k_{\alpha\beta}$ as the kronecker symbol $\delta_{\alpha\beta}$ and φ as, [10]

$$\varphi = (M_{\alpha\beta} + \delta_{\alpha\beta})x^\alpha x^\beta e^{(x^1)^2} \tag{5.3}$$

where $M_{\alpha\beta}$ are constant and satisfy the relations

$$\begin{gathered}M_{\alpha\beta} = 0 \quad for \quad \alpha \neq \beta \\ M_{\alpha\beta} \neq 0 \quad for \quad \alpha = \beta \\ \sum_{\alpha=2}^{n-1} M_{\alpha\alpha} = 0.\end{gathered} \tag{5.4}$$

In this case, we have the following relations

$$\begin{gathered}\varphi_{,\alpha\beta} = 2(M_{\alpha\beta} + \delta_{\alpha\beta})e^{(x^1)^2} \\ \delta_{\alpha\beta}\delta^{\alpha\beta} = n-2 \quad and \quad \delta^{\alpha\beta}M_{\alpha\beta} = \sum_{\alpha=2}^{n-1} M_{\alpha\alpha} = 0.\end{gathered} \tag{5.5}$$

Thus, from (5.3) and (5.5), we have

$$\delta^{\alpha\beta}\varphi_{,\alpha\beta} = 2(n-2)e^{(x^1)^2}. \tag{5.6}$$

By using (5.2), we find the only non-zero components for R_{hijk} and S_{ij} as

$$\begin{aligned} R_{1\alpha\alpha1} &= \frac{1}{2}\varphi_{,\alpha\alpha} = (1+M_{\alpha\alpha})e^{(x^1)^2} \\ S_{11} &= \frac{1}{2}\varphi_{,\alpha\beta}\delta^{\alpha\beta} = (n-2)e^{(x^1)^2}. \end{aligned} \tag{5.7}$$

Hence, the only non-zero components of the conharmonic curvature tensor H_{hijk} are

$$\begin{aligned} H_{1\alpha\alpha1} &= R_{1\alpha\alpha1} - \frac{1}{n-2}(g_{\alpha\alpha}S_{11}) \\ &= (1+M_{\alpha\alpha})e^{(x^1)^2} - \frac{1}{n-2}(n-2)e^{(x^1)^2} \\ &= M_{\alpha\alpha}e^{(x^1)^2} \end{aligned} \tag{5.8}$$

which never vanish. From (5.8), the only non-zero components of the derivative of H_{hijk} are

$$\begin{aligned} H_{1\alpha\alpha1,1} &= 2x^1 M_{\alpha\alpha}e^{(x^1)^2} \\ &= 2x^1 H_{1\alpha\alpha1}. \end{aligned} \tag{5.9}$$

Now, we assume that the associated 1-form of this manifold is

$$A_i(x) = \begin{cases} \frac{x^1}{2}, & \text{for } i = 1 \\ 0, & \text{otherwise} \end{cases} \tag{5.10}$$

at any point $x \in V_n$.

To verify the relation (3.6) it is sufficient to prove that the equation

$$H_{1\alpha\alpha1,1} = 4A_1 H_{1\alpha\alpha1}. \tag{5.11}$$

By the aid of (5.9) and (5.10), we can easily see that (5.11) is satisfied. The other components of each term of (3.6) vanish identically and the relation (3.6) holds trivially.

Under our assumptions (5.1), (5.3) and (5.4), this manifold is a $(PCHS)_n$.

Example 5.2 In this section we will give the second example for $(PCHS)_n$ satisfying the conditions (3.5) and (3.6).

We define a Riemannian metric on R^n $(n \geq 4)$ by the formula,

$$ds^2 = \varphi(dx^1)^2 + k_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n \tag{5.12}$$

where $[k_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constant and φ is a function of $x^1, x^2, ..., x^{n-1}$ and independent of x^n. Let each Latin index runs over $1, 2, .., n$ and each Greek index runs over $2, 3, .., (n-1)$.

In the metric considered, the only non-vanishing components of Christoffel symbols, the curvature tensor and the Ricci tensor are, according to [58]

$$\begin{gathered}\Gamma^\beta_{11} = -\frac{1}{2}k^{\alpha\beta}\varphi_{,\alpha}, \quad \Gamma^n_{11} = \frac{1}{2}\varphi_{,1} \quad \Gamma^n_{1\alpha} = \frac{1}{2}\varphi_{,\alpha} \\ R_{1\alpha\beta1} = \frac{1}{2}\varphi_{,\alpha\beta}, \quad S_{11} = \frac{1}{2}k^{\alpha\beta}\varphi_{,\alpha\beta}\end{gathered} \tag{5.13}$$

where "," denotes the partial diferentiation with respect to the coordinates and $k^{\alpha\beta}$ are the elements of the matrix inverse to $[k_{\alpha\beta}]$.

We consider $k_{\alpha\beta}$ as the kronecker symbol $\delta_{\alpha\beta}$ and φ as,

$$\varphi = M_{\alpha\beta}x^\alpha x^\beta e^{(x^1)^2} \tag{5.14}$$

where $M_{\alpha\beta}$ are constants and satisfy the relations

$$M_{\alpha\beta} \neq 0 \quad for \quad \alpha, \beta = 2, ..., (n-1)$$

$$\sum_{\alpha=2}^{n-1} M_{\alpha\alpha} = 0. \tag{5.15}$$

In this case, we have the following relations

$$\varphi_{,\alpha\beta} = 2M_{\alpha\beta}e^{(x^1)^2}$$

$$\delta^{\alpha\beta}M_{\alpha\beta} = \sum_{\alpha=2}^{n-1} M_{\alpha\alpha} = 0. \tag{5.16}$$

Thus, from (5.14) and (5.16), we have

$$\delta^{\alpha\beta}\varphi_{,\alpha\beta} = 0. \tag{5.17}$$

By using (5.13), we find the only non-zero components for R_{hijk} and S_{ij} as

$$\begin{aligned}R_{1\alpha\beta1} &= \frac{1}{2}\varphi_{,\alpha\beta} = M_{\alpha\beta}e^{(x^1)^2} \\ S_{11} &= \frac{1}{2}\varphi_{,\alpha\beta}\delta^{\alpha\beta} = 0.\end{aligned} \tag{5.18}$$

Hence, the only non-zero components of the conharmonic curvature tensor H_{hijk} are

$$\begin{aligned} H_{1\alpha\alpha 1} &= R_{1\alpha\alpha 1} - \frac{1}{n-2}(g_{\alpha\alpha}S_{11}) \\ &= M_{\alpha\alpha}e^{(x^1)^2} \end{aligned} \tag{5.19}$$

which never vanish. From (5.19), the only non-zero components of the derivative of H_{hijk} are

$$\begin{aligned} H_{1\alpha\beta 1,1} &= 2x^1 M_{\alpha\beta}e^{(x^1)^2} \\ &= 2x^1 H_{1\alpha\beta 1}. \end{aligned} \tag{5.20}$$

Let us consider the associated 1-form as

$$A_i(x) = \begin{cases} \frac{x^1}{2}, & \text{for } i = 1 \\ 0, & \text{otherwise} \end{cases} \tag{5.21}$$

at any point $x \in V_n$.

To verify the relation (3.6) it is sufficient to prove that the equation

$$H_{1\alpha\beta 1,1} = 4A_1 H_{1\alpha\beta 1}. \tag{5.22}$$

By the aid of (5.20) and (5.21), we can easily see that (5.22) is satisfied. The other components of each term of (3.6) vanish identically and the relation (3.6) holds trivially.

Under our assumptions (5.12), (5.14) and (5.15), this manifold is a $(PCHS)_n$.

REFERENCES

[1] **Chern S.S., Chen W.H. and Lam K.S.** (1999). Lectures on Differential Geometry Series on University Mathematics. Vol.1, pp. 133-143.

[2] **Burstall F.E.** (1999) Basic Riemannian Geometry, L.M.S. Lect. Note Series, Department of Mathematical Sciences University of Bath.

[3] **Avramidi I.** (2010) Lecture Notes, Introduction to Differential Geometry, Department of Mathematics New Mexico Institute of Mining and Technology.

[4] **Chavel I.** (2006) Riemannian Geometry, A Modern Introduction, Second Edition, Cambridge University Press.

[5] **Petersen P.** (2006) Riemannian Geometry, Second Edition, Springer.

[6] **Kobayashi S. and Nomizu K.** (1963) Foundations of Differential Geometry. Interscience Publishers, New York London. Vol.1.

[7] **Url-1** <diffgeom.subwiki.org/wiki/Second-Bianchi-identity>, 05.09.2013.

[8] **Catino G., Mantegazza C. and Mazzieri L.** (2012) A Note on Codazzi Tensors.

[9] **Yano K.** (1944). On Torse Forming Directions in Riemannian Spaces, Proc. Imp. Acad., Tokyo. Vol.20, pp. 340-345.

[10] **De U. C. and De A.** (2012). On Almost Pseudo-Conformally Symmetric Ricci-Recurrent Manifolds with Applications to Relativity, Czechoslavak Mathematical Journal, Vol.62(137), pp. 1055-1072.

[11] **Ruse H.S.** (1949). Three-Dimensional Spaces of Recurrent Curvature, Proc. Lond. Math. Soc., Vol.50, pp. 438-446.

[12] **Patterson E. M.** (1952). Some Theorems on Ricci-Recurrent Spaces, J. Lond. Math. Soc., Vol.27, pp. 287-295.

[13] **Cartan E.** (1926). Sur Une Classe Remarquable D espaces de Riemann, Bull. Soc. Math., France, Vol.54, pp. 214-264.

[14] **Walker A.G.** (1950). On Ruse' s Spaces of Recurrent Curvature, Proc. Lond. Math. Soc., Vol.52, pp. 36-64.

[15] **Lichnerowicz A.** (1952). Courbure Nombres de Betti et Espaces Symmetriques, Proc. of the Intern. Cong. of Math., Vol.2, pp. 216-223.

[16] **Miyazawa T.** (1996). On Riemannian Space Admitting Some Recurrent Tensor, TRU Math. J., Vol.2, pp. 11-18.

[17] **Yano K.** (1940). Concircular Geometry 1, Proc. Imp. Acad., Tokyo, Vol.16, pp. 195-200.

[18] **Tamassy L. and Binh T.Q.** (1989). On Weakly Symmetric and Weakly Projective Symmetric Riemannian Manifolds, Colloq. Math. Soc. Janos Bolyai, Vol.56, pp. 663-670.

[19] **Tamassy L. and Binh T.Q.** (1993). On Weak Symmetries of Einstein and Sasakian Manifolds, Tensor(N.S.), Vol.53, pp. 140-148.

[20] **Adati T. and Miyazawa T.** (1967). On Riemannian Space with Recurrent Conformal Curvature, Tensor(N.S.), Vol.18, pp. 348-354.

[21] **Adati T. and Miyazawa T.** (1977). On Projective Transformations of Projective Recurrent Spaces, Tensor(N.S.), Vol.31, pp. 49-54.

[22] **Arslan K., De U.C., Murathan C. and Yildiz A.** (2009). On Generalized Recurrent Riemannian Manifolds, Acta Math. Hungar., Vol.123, pp. 27-39.

[23] **De U.C., Guha N. and Kamilya D.** (1995). On Generalized Ricci-Recurrent Manifolds, Tensor(N.S.), Vol.56, pp. 312-317.

[24] **Ishi Y.** (1957). On Conharmonic Transformation, Tensor(N.S.), Vol.7, pp. 73-80.

[25] **Kim B.H., Kim I.B. and Lee S.M.** (1997). Conharmonic Transformation and Critical Riemannian Metrics, Comm. Korean Math. Soc., Vol.12(2), pp. 347-354.

[26] **Chen B. Y.** (1973). Geometry of Submanifold, Marcel Dekker, New York.

[27] **Tachibana S.** (1963). Exercises on Riemannian Geometry(Japanese), Asakura, Tokyo.

[28] **De U.C. and Biswas S.** (2006). A note on ξ-Conformally Flat Contact Manifolds, Bull. Malays. Math. Sci. Soc. (2), Vol. 29(1), pp. 51-57.

[29] **Weyl H.** (1918). Reine Infinitesimalgeometrie, Math. Z., Vol. 2(3-4), pp. 384-411.

[30] **Weyl H.** (1921). Zur Infinitesimalgeometrie, Einordnung der projektiven und der konformen Auffassung, Gottingen Nachrichten, pp. 99-112.

[31] **Strang G.** (1986). Introduction to Applied Mathematics, Wellesley-Cambridge Press, Wellesley, M.A., pp. 340-341.

[32] **Gu X., Wang Y., Chan T.F., Thompson P.M. and Yau S.T.** (2004). Genus Zero Surface Conformal Mapping and Its Application to Brain Surface Mapping, IEEE Transaction on Medical Imaging, Vol. 23(8), pp. 949-958.

[33] **Ganguli S.** (2008). Conformal Mapping and its Applications, Department of Physics, University of Tennessee, Knoxville, TN 37996.

[34] **Kuhnel W. and Rademacher H.B.** (1995) Conformal Transformations of Pseudo-Riemannian Manifolds, Erwin-Schrodinger Institute for Mathematical Physics in Vienna.

[35] **Kasner E.** (1921). Einstein' s Theory of Gravitation: Determination of the Field by Light Signals, Amer. J. Math., Vol. 43, pp. 20-28.

[36] **Brinkmann H.W.** (1925). Einstein Spaces Which are Mapped Conformally on Each Other, Math. Ann., Vol. 94, pp. 119-145.

[37] **Kuiper N.H.** (1949). On Conformally-Flat Spaces in the Large, Ann. Math. (2), Vol. 50, pp. 916-924.

[38] **Abdussattar D.B.** (1996). On Conharmonic Transformations in General Relativity, Bull. Calcutta Math. Soc., Vol. 6, pp. 465-700.

[39] **Siddiqui S. A. and Ahsan Z.** (2010). Conharmonic Curvature Tensor and the Spacetime of General Relativity, Diff. Geo.-Dynamical Systems, Balkan Society of Geometers, Vol. 12, pp. 213-220.

[40] **Ozgur C.** (2003). On ϕ-Conformally Flat Lorentzian Para-Sasakian Manifolds, Radovi Matematicki, Vol. 12, pp. 99-106.

[41] **Ghosh S., De U. C. and Taleshian A.** (2011). Conharmonic Curvature Tensor on N(K)-Contact Metric Manifolds, International Scholarly Research Network, ISRN Geometry, 10.5402/423798.

[42] **O' Neill B.** (1983). Semi-Riemannian Geometry with Applications to the Relativity, Academic Press, New York-London.

[43] **Chaki M.C. and Gupta B.** (1963). On Conformally Symmetric Spaces, Indian J. Math., Vol. 5, pp. 113-295.

[44] **Chaki M.C.** (1987). On Pseudo Symmetric Manifolds, Analele Stiint Ale Univ. 'Al I Cuza' Din Iasi, Vol. 38, pp. 53-58.

[45] **Soos G.** (1958). Uber die geod*ä*tischen Abbildungen von Riemannschen R*ä*umen auf projektivsymmetrische Riemannsche R*ä*ume, Acta Math. Acad. Sci. Hungar, Vol. 9, pp. 359-361.

[46] **De U. C. and Biswas H. A.** (1993). On Pseudo-Conformally Symmetric Manifolds, Bull. Calcutta Math. Soc., Vol. 85(5), pp. 479-486.

[47] **De U. C. and Gazi A. K.** (2009). On Almost Pseudo Conformally Symmetric Manifolds, Demonstratio Math., Vol. 42(4), pp. 861-878.

[48] **De U. C. and Bandyopadhyay S.** (2000). On Weakly Conformally Symmetric Spaces, Publ. Math. Debrecen, Vol. 57(1-2), pp. 71-78.

[49] **Deszcz R.** (1992). On Pseudosymmetric Spaces, Bull. Soc. Math. Belg. Ser. A, Vol. 44(1), pp. 1-34.

[50] **Sen R.N. and Chaki M.C.** (1967). On Curvature Restrictions of a Certain Kind of Conformally-Flat Riemannian Space of Class One, Proc. Nat. Inst. Sci. India Part A, Vol. 33, pp. 100-102.

[51] **Belkhelfa M., Deszcz R., Glogowska M., Hotlos M., Kowalczyk D. and Verstraelen L.** (2002). On Some Type of Curvature Conditions, Banach Center Publ., Inst. Math. Polish Acad. Sci., Vol. 57, pp. 179-194.

[52] **Deszcz R. and Grycak W.**(1987). On Some Classes of Warped Product Manifolds, Bull. Inst. Math. Acad. Sinica, Vol. 19, pp. 311-322.

[53] **Deszcz R., Haesen S. and Verstraelen L.**(2008). On Natural Symmetries, Topics in Differential Geometry, Ed. Rom. Acad. Bucharest, pp. 249-308.

[54] **Chaki M.C.**(1988). On Pseudo Ricci Symmetric Manifolds, Bulgar. J. Phys., Vol. 15, pp. 526-531.

[55] **Tamassy L. and Binh T. Q.**(1993). On Weak Symmetries of Einstein and Sasakian Manifolds, International Conference on Differential Geometry and its Applications(Bucharest,1992), Tensor (N.S.) 53, Commemoration Vol. 1, pp. 140-148.

[56] **Ozen F. and Altay S.**(2002). On Weakly and Pseudo-Symmetric Riemannian Spaces, Indian J. Pure Appl. Math., Vol. 33, pp. 1477-1488.

[57] **Shaikh A.A. and Hui S.K.**(2008). On Weakly Conharmonically Symmetric Manifolds, Tensor(N.S), Vol. 70, pp. 119-134.

[58] **Roter W.**(1974). On Conformally Symmetric Ricci-Recurent Spaces, Colloquium Mathematicum, Vol. 31, pp. 87-96.

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