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EXPLICIT SOLUTIONS OF THE CONFLUENT HYPERGEOMETRIC EQUATIN BY MEANS OF THE DIFFERINTEGRAL THEOREMS

(DİFERİNTEGRAL TEOREMLERİ YARDIMIYLA KONFLUENT HİPERGEOMETRİK DENKLEMİNİN AÇIK ÇÖZÜMLERİ)

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ABSTRACT

In fractional calculus, an area of applied mathematics, the differintegral is a combined differentiation/integration operator. Differintegral theory is used to solve some classes of differential equations and fractional differential equations. One of these equations is the confluent hypergeometric equation. In this paper, we intend to solve this equation by means of the differintegral theorems.

Keywords: Fractional calculus, Differintegral, Confluent hypergeometric equation, Differintegral theorems, Generalized Leibniz rule

ÖZ

Uygulamalı matematiğin bir alanı olan kesirli hesapta diferintegral, türev/integral operatörünün bir birleşimidir. Diferansiyel denklemlerin ve kesirli diferansiyel denklemlerin bazı sınıflarını çözmek için diferintegral teorisi kullanılmaktadır. Bu denklemlerden birisi konfluent hipergeometrik denklemidir. Bu makalede, diferintegral teoremleri yardımıyla bu denklemi çözmeyi hedefleriz.

Anahtar Kelimeler: *Kesirli hesap, Diferintegral, Konfluent hipergeometrik denklemi, Diferintegral teoremleri, Genelleştirilmiş Leibniz kuralı*

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1. INTRODUCTION

The widely investigated subject of fractional calculus *(that is, calculus of derivatives and integrals of any arbitrary real or complex order)* has gained considerable importance and popularity during the past three decades or so, due chiefly to its demonstrated applications in numerous seemingly diverse fields of science and engineering. We can mention that the fractional differential equations are playing an important role in fluid dynamics, traffic model with fractional derivative, measurement of viscoelastic material properties, modeling of viscoplasticity, control theory, relativity theory, economy, nuclear magnetic resonance, geometric mechanics, mechanics, optics, signal processing, robot technology, PID control systems, Schrödinger equation, heat transfer, filtration and so on.

Some of most obvious formulations based on the fundamental definitions of Riemann-Liouville fractional differentiation and fractional integration are, respectively,

$$ {}_aD_t^{\mu} f(t) = \frac{1}{\Gamma(k-\mu)} \frac{d^k}{dt^k} \int_a^t f(\tau)(t-\tau)^{k-\mu-1}\, d\tau \quad (k-1 \leq \mu < k), \tag{1} $$

and,

$$ {}_aD_t^{-\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_a^t f(\tau)(t-\tau)^{\mu-1}\, d\tau \quad (t > a, \mu > 0), \tag{2} $$

where $k \in \mathbb{N}$, $\mathbb{N}$ being the set of positive integers, Γ stands for Euler's function gamma [1-4].

Recently, by applying the Riemann-Liouville definitions of a differintegral (*that is, fractional derivative and fractional integral*) of order $\mu \in \mathbb{R}$, many authors have explicity obtained particular solutions of a number of families of homogeneous *(as well as non-homogeneous)* linear ordinary and partial differintegral equations (*see, for details,* [5]*; see also* [6,7]). An important example of Fuchsian differential equations is provided by the celebrated hypergeometric equation *(or, more precisely, the Gauss hypergeometric equation)*

$$ z(1-z)\frac{d^2u}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{du}{dz} - \alpha\beta u = 0, $$

whose study can be traced back to L. Euler, C.F. Gauss and E.E. Kummer. On the other hand, a special limit *(confluent)* case of the Gauss hypergeometric equation, in the form [8]

$$ \frac{d^2u}{dz^2} + \left(-\frac{1}{4} + \frac{\varkappa}{z} - \frac{\ell(\ell+1)}{z^2}\right)u = 0 \quad \left(\mu = \ell + \frac{1}{2}\right), $$

is refered to as the Whittaker equation whose systematic study was initiated by E.T. Whittaker.

Other classes of non-Fuchsian differential equations which we shall consider in this investigation include the so-called Fukuhara equation [9]

$$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}-(1-z+z^2)u=0,$$

the Tricomi equation [10]

$$\frac{d^2u}{dz^2}+\left(\alpha+\frac{\beta}{z}\right)\frac{du}{dz}+\left(\gamma+\frac{\delta}{z}+\frac{\varepsilon}{z^2}\right)u=0,$$

and the Bessel equation [11]

$$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}-(z^2-v^2)u=0.$$

Moreover, in [12], Inc obtained the particular solutions of the confluent hypergeometric differential equation by using the nabla fractional calculus operator which is an important operator in discrete fractional calculus. Virchenko's study [13] is devoted to further development of important case of Wright's hypergeometric function and its applications to the generalization of $\Gamma-$, $B-$, $\psi-$, $\zeta-$, Volterra functions. In [14], Srivastava and Saxena expressed some Volterra-type fractional integro-differential equations with a multivariable confluent hypergeometric function as their kernel. And, Campos solved the extended confluent hypergeometric differential equation in [15].

In this paper, we also obtained the fractional solutions of the confluent hypergeometric equation by using the differintegral theorems. The most important advantage of these theorems is applicaple to the singular equations.

2. MATERIALS AND METHODS

2.1. Definition If the function $f(z)$ is analytic (*regular*) inside and on C, where $C=\{C^-,C^+\}$, C^- is a contour along the cut joining the points z and $-\infty+i\mathrm{Im}(z)$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point at $-\infty$, and C^+ is a contour along the cut joining the points z and $\infty+i\mathrm{Im}(z)$, which starts from the point at ∞, encircles the point z once counter-clockwise, and returns to the point at ∞,

$$\begin{aligned} f_\mu(z)&=[f(z)]_\mu=\frac{\Gamma(\mu+1)}{2\pi i}\int_C\frac{f(\tau)d\tau}{(\tau-z)^{\mu+1}} \quad (\mu\notin\mathbb{Z}^-),\\ f_{-k}(z)&=\lim_{\mu\to-k}f_\mu(z) \quad (k\in\mathbb{Z}^+), \end{aligned} \tag{3}$$

where $\tau\neq z$,

$$\begin{aligned} &-\pi\le\arg(\tau-z)\le\pi \quad \text{for } C^-,\\ &0\le\arg(\tau-z)\le2\pi \quad \text{for } C^+. \end{aligned} \tag{4}$$

In that case, $f_\mu(z)$ $(\mu>0)$ is the fractional derivative of $f(z)$ of order μ and $f_\mu(z)$ $(\mu<0)$ is the fractional integral of $f(z)$ of order $-\mu$, confirmed (*in each case*) that

$$\left|f_\mu(z)\right|<\infty \quad (\mu\in\mathbb{R}). \tag{5}$$

[4].

2.2. Lemma *(Linearity)* Let $f(z)$ and $g(z)$ be analytic and single-valued functions. If f_μ and g_μ exist, then

$$\begin{aligned} &\textbf{(i)} \quad [c_1 f(z)]_\mu = c_1[f(z)]_\mu, \\ &\textbf{(ii)} \quad [c_1 f(z) + c_2 g(z)]_\mu = c_1[f(z)]_\mu + c_2[g(z)]_\mu, \end{aligned} \tag{6}$$

where c_1 and c_2 are constants and $\mu \in \mathbb{R}$, $z \in \mathbb{C}$.

2.3. Lemma *(Index law)* Let $f(z)$ be an analytic and single-valued function. If $(f_\nu)_\mu$ and $(f_\mu)_\nu$ exist, then

$$\{[f(z)]_\nu\}_\mu = [f(z)]_{\nu+\mu} = \{[f(z)]_\mu\}_\nu, \tag{7}$$

where $\nu, \mu \in \mathbb{R}$, $z \in \mathbb{C}$ and $\left|\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+1)\Gamma(\mu+1)}\right| < \infty$.

2.4. Lemma *(Generalized Leibniz rule)* Let $f(z)$ and $g(z)$ be single-valued and analytic functions. If f_μ and g_μ exist, then

$$(f.g)_\mu = \sum_{k=0}^{\infty} \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-k)\Gamma(k+1)} f_{\mu-k}.g_k, \tag{8}$$

where $\mu \in \mathbb{R}$, $z \in \mathbb{C}$ and $\left|\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-k)\Gamma(k+1)}\right| < \infty$.

2.5. Property For a constant λ,

$$\left(\mathrm{e}^{\lambda z}\right)_\nu = \lambda^\nu \mathrm{e}^{\lambda z} \quad (\lambda \neq 0, \nu \in \mathbb{R}, \mathrm{z} \in \mathbb{C}). \tag{9}$$

2.6. Property For a constant λ,

$$\left(\mathrm{e}^{-\lambda z}\right)_\nu = \mathrm{e}^{-i\pi\nu} \lambda^\nu \mathrm{e}^{-\lambda z} \quad (\lambda \neq 0, \nu \in \mathbb{R}, z \in \mathbb{C}). \tag{10}$$

2.7. Property For a constant λ,

$$\left(z^\lambda\right)_\nu = \mathrm{e}^{-i\pi\nu} z^{\lambda-\nu} \frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)} \quad \left(\nu \in \mathbb{R}, z \in \mathbb{C}, \left|\frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)}\right| < \infty\right). \tag{11}$$

2.8. Property

$$\Gamma(z+1) = z\Gamma(z+1) = z!, \tag{12}$$

and,

$$\Gamma(\nu-k)=(-1)^k\frac{\Gamma(\nu)\Gamma(1-\nu)}{\Gamma(k+1-\nu)}, \tag{13}$$

where $k\in\mathbb{Z}_0^+$ and $\nu\in\mathbb{R}$.

2.9. Theorem Let $\mathcal{P}(z;\mathcal{p})$ and $\mathcal{Q}(z;\mathcal{q})$ be polynomials in z of degrees $\mathcal{p}$ and $\mathcal{q}$, respectively, defined by

$$\mathcal{P}(z;\mathcal{p})=\sum_{k=0}^{\mathcal{p}}a_k z^{\mathcal{p}-k}=a_0\prod_{j=1}^{\mathcal{p}}(z-z_j)\quad(a_0\neq 0,\mathcal{p}\in\mathbb{N}), \tag{14}$$

and,

$$\mathcal{Q}(z;\mathcal{q})=\sum_{k=0}^{\mathcal{q}}b_k z^{\mathcal{q}-k}\quad(b_0\neq 0,\mathcal{q}\in\mathbb{N}). \tag{15}$$

Suppose also that $f_{-\mu}\neq 0$ exists for a given function f.

Then the nonhomogeneous linear ordinary fractional differintegral equation

$$\mathcal{P}(z;\mathcal{p})\varphi_\nu(z)+\left[\sum_{k=1}^{\mathcal{p}}\binom{\mu}{k}\mathcal{P}_k(z;\mathcal{p})+\sum_{k=1}^{\mathcal{q}}\binom{\mu}{k-1}\mathcal{Q}_{k-1}(z;\mathcal{q})\right]\varphi_{\nu-k}(z)$$
$$+\binom{\mu}{\mathcal{q}}\mathcal{q}!\,b_0\varphi_{\nu-\mathcal{q}-1}(z)=f(z)\quad(\mathcal{p},\mathcal{q}\in\mathbb{N},\nu,\mu\in\mathbb{R}), \tag{16}$$

has a particular solution of the form

$$\varphi(z)=\left\{\left[\frac{f_{-\mu}(z)}{\mathcal{P}(z;\mathcal{p})}\mathrm{e}^{\mathcal{H}(z;\mathcal{p},\mathcal{q})}\right]_{-1}\mathrm{e}^{-\mathcal{H}(z;\mathcal{p},\mathcal{q})}\right\}_{\mu-\nu+1}\quad\left(z\in\mathbb{C}\setminus\{z_1,\dots,z_{\mathcal{p}}\}\right), \tag{17}$$

where for suitable condition,

$$\mathcal{H}(z;\mathcal{p},\mathcal{q})=\int^{z}\frac{\mathcal{Q}(\xi;\mathcal{q})}{\mathcal{P}(\xi;\mathcal{p})}d\xi\qquad\left(z\in\mathbb{C}\setminus\{z_1,\dots,z_{\mathcal{p}}\}\right), \tag{18}$$

confirmed that the second component of (17) exists. Moreover, the homogeneous linear ordinary fractional differintegral equation

$$\mathcal{P}(z;\mathcal{p})\varphi_\nu(z)+\left[\sum_{k=1}^{\mathcal{p}}\binom{\mu}{k}\mathcal{P}_k(z;\mathcal{p})+\sum_{k=1}^{\mathcal{q}}\binom{\mu}{k-1}\mathcal{Q}_{k-1}(z;\mathcal{q})\right]\varphi_{\nu-k}(z)$$

$$+\binom{\mu}{q} q!\, b_0 \varphi_{\nu-q-1}(z) = 0 \quad (p, q \in \mathbb{N}, \nu, \mu \in \mathbb{R}), \quad (19)$$

has solutions of the form

$$\varphi(z) = K\left[\mathrm{e}^{-\mathcal{H}(z;p,q)}\right]_{\mu-\nu+1}, \quad (20)$$

where $\mathcal{H}(z; p, q)$ is given by (18), it being confirmed that the second component of (20) exist and K is an arbitrary constant [16].

3. MAIN RESULTS

The hypergeometric equation

$$x(1-x)\frac{d^2\varphi(x)}{dx^2} + [c - (a+b+1)x]\frac{d\varphi(x)}{dx} - ab\varphi(x) = 0, \quad (21)$$

has three regular singular points at $x = 0,1$ and ∞ *(a, b and c are parameters)*. By setting $x = z/b$ and taking the limit as $b \to \infty$, we can merge the singularities at b and infinity. This gives us the confluent equation as

$$z\frac{d^2\varphi}{dz^2} + (c - z)\frac{d\varphi}{dz} - a\varphi = 0, \quad (22)$$

solutions of which are the confluent hypergeometric functions, which are shown as $M(a, c; z)$.

The confluent hypergeometric equation has a regular singular point at $z = 0$ and an essential singularity at infinity. Bessel functions, $J_n(z)$, and the Laguerre polynomials, $L_n(z)$, can be written in terms of the solutions of the confluent hypergeometric equation as

$$J_n(z) = \frac{\mathrm{e}^{-iz}}{n!}\left(\frac{z}{2}\right)^n M\left(n + \frac{1}{2}, 2n + 1; 2iz\right),$$
$$L_n(z) = M(-n, 1; z).$$

Linearly independent solutions of Eq. (22) are given as

$$\varphi_1(z) = M(a, c; z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots,$$
$$(c \neq 0, -1, -2, \ldots),$$

and,

$$\varphi_2(z) = z^{1-c} M(a + 1 - c, 2 - c; z) \quad (c \neq 2,3,4, \ldots).$$

Integral representation of the confluent hypergeometric functions, which are also shown as ${}_1F_1(a, b; z)$, can be given as

$$M(a,c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 \mathrm{e}^{zt}t^{a-1}(1-t)^{c-a-1}dt \quad (a,c\in\mathbb{R},c>a>0).$$

[17].

Now, for Eq. (22), we use the transformation as

$$\varphi(z)=z^{-c/2}\mathrm{e}^{z/2}u(z) \quad \left[u(z)=z^{c/2}\mathrm{e}^{-z/2}\varphi(z)\right]. \tag{23}$$

So, we can write

$$\frac{d\varphi}{dz}=z^{-\frac{c}{2}-1}\mathrm{e}^{z/2}\left[z\frac{du}{dz}+\frac{1}{2}(z-c)u\right], \tag{24}$$

and,

$$\frac{d^2\varphi}{dz^2}=z^{-\frac{c}{2}-2}\mathrm{e}^{z/2}\left\{z^2\frac{d^2u}{dz^2}+z(z-c)\frac{du}{dz}+\frac{1}{4}[(z-c)^2+2c]u\right\}. \tag{25}$$

By substituting (23), (24) and (25) into (22), we have

$$\frac{d^2u}{dz^2}+\left(-\frac{1}{4}+\frac{\frac{c}{2}-a}{z}+\frac{2c-c^2}{4z^2}\right)u=0. \tag{26}$$

After, we can write Eq. (26) as follows

$$\frac{d^2u}{dz^2}+\left[-\frac{1}{4}+\frac{\frac{c}{2}-a}{z}+\frac{\frac{1}{4}-\left(\frac{c-1}{2}\right)^2}{z^2}\right]u=0. \tag{27}$$

By using Theorem (2.9), we have [18]

$$\mu=2,\quad p=q=1,\quad a_0=h\neq 0,\quad a_1=0,\quad b_0=s\neq 0,\quad b_1=t, \tag{28}$$

so that

$$\mathcal{P}(z;1)=hz,\quad \mathcal{P}_1(z;1)=h, \tag{29}$$

and,

$$\mathcal{Q}(z;1)=sz+t,\quad \mathcal{Q}_1(z;1)=s. \tag{30}$$

After, by using Eq. (18), we obtain

$$\mathcal{H}(z;1,1)=\int^{z}\frac{\mathcal{Q}(\xi;1)}{\mathcal{P}(\xi;1)}d\xi=\ln\left[(hz)^{t/h}\mathrm{e}^{sz/h}\right]. \tag{31}$$

3.1. Theorem Let $\left|f_{\mu}(z)\right|<\infty$ and $f_{-\mu}\neq 0$. The nonhomogeneous second order linear ordinary differential equation as

$$hz\frac{d^{2}\varphi}{dz^{2}}+(sz+\mu h+t)\frac{d\varphi}{dz}+\mu s\varphi(z)=f(z)\quad(h\neq 0,\mu\in\mathbb{R}), \tag{32}$$

has a solution as follows

$$\varphi(z)=\left\{\left[f_{-\mu}(z)(hz)^{(t-h)/h}\mathrm{e}^{sz/h}\right]_{-1}(hz)^{-t/h}\mathrm{e}^{-sz/h}\right\}_{\mu-1}. \tag{33}$$

Furthermore, the homogeneous second order linear ordinary differential equation as

$$hz\frac{d^{2}\varphi}{dz^{2}}+(sz+\mu h+t)\frac{d\varphi}{dz}+\mu s\varphi(z)=0\quad(h\neq 0,\mu\in\mathbb{R}), \tag{34}$$

has a solution as follows

$$\varphi(z)=K\left[(hz)^{-t/h}\mathrm{e}^{-sz/h}\right]_{\mu-1}, \tag{35}$$

where K is an arbitrary constant [18].

Now, by using Theorem (3.1), we set

$$h=1,\quad s=-1,\quad t=c-a,\quad \mu=a. \tag{36}$$

So, we obtain the equation as

$$z\frac{d^{2}\varphi}{dz^{2}}+(c-z)\frac{d\varphi}{dz}-a\varphi(z)=0. \tag{37}$$

After, we find the solution of Eq. (37) as follows

$$\varphi(z)=K[z^{a-c}\mathrm{e}^{z}]_{a-1}. \tag{38}$$

Finally, we have the solution of Eq. (27) as

$$u(z)=Kz^{c/2}\mathrm{e}^{-z/2}[z^{a-c}\mathrm{e}^{z}]_{a-1}. \tag{39}$$

3.2. Example Let $a=3$ and $c=1$ for Eq. (38) and Eq. (39). So, we obtain

$$\varphi(z)=K(z^{2}\mathrm{e}^{z})_{2}, \tag{40}$$

and,

$$u(z) = Kz^{1/2}\mathrm{e}^{-z/2}(z^2\mathrm{e}^z)_2. \tag{41}$$

By using Eq. (1), we have

$$(z^2\mathrm{e}^z)_2 = \frac{1}{\Gamma(1)}\frac{d^3}{dz^3}\int_0^z \tau^2\mathrm{e}^\tau\, d\tau = \mathrm{e}^z(z^2+4z+2). \tag{42}$$

After, by substituting (42) into (40) and (41), we find the solutions as

$$\varphi(z) = K\mathrm{e}^z(z^2+4z+2), \tag{43}$$

and,

$$u(z) = Kz^{1/2}\mathrm{e}^{z/2}(z^2+4z+2). \tag{44}$$

3.3. Theorem Let $|(z^{a-c})_k| < \infty$ $(k \in \mathbb{Z}^+ \cup \{0\})$, $z \neq 0$, and $\left|\frac{1}{z}\right| < 1$. The solution of (38) can be written as follows

$$\varphi(z) = Kz^{a-c}\mathrm{e}^z\ {}_2F_0\left[1-a, c-a; \frac{1}{z}\right], \tag{45}$$

where ${}_2F_0$ is the Gauss hypergeometric function.

Proof. By means of (8), we have

$$\varphi(z) = K\sum_{k=0}^{\infty}\frac{\Gamma(a)}{\Gamma(a-k)\Gamma(k+1)}(z^{a-c})_k(\mathrm{e}^z)_{a-1-k}. \tag{46}$$

By using (9), (11), (12) and (13), we can rewrite the Eq. (46) as follows

$$\varphi(z) = K\sum_{k=0}^{\infty}\frac{\Gamma(k+1-a)}{(-1)^k\Gamma(1-a)}\frac{1}{k!}(-1)^k z^{a-c-k}\frac{\Gamma(k+c-a)}{\Gamma(c-a)}\mathrm{e}^z,$$

$$= Kz^{a-c}\mathrm{e}^z\sum_{k=0}^{\infty}[1-a]_k\,[c-a]_k\frac{1}{k!}\left(\frac{1}{z}\right)^k,$$

$$= Kz^{a-c}\mathrm{e}^z\ {}_2F_0\left[1-a, c-a; \frac{1}{z}\right]. \tag{47}$$

4. CONCLUSION

In this paper, we used the differintegral theorems for the confluent hypergeometric equation. We also obtained hypergeometric forms of the fractional solutions. Solutions of the singular equations can be obtained by means of these theorems.

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