ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

RICHARDSON SOLUTION FOR THE SPIN-ORBIT COUPLED FERMI GAS

M.Sc. THESIS

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Department of Physics Engineering

Physics Engineering Programme

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SPİN-MOMENTUM BAĞLI FERMİ GAZLARINDA RİCHARDSON ÇÖZÜMÜ

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To my parents...

FOREWORD

This study is conducted at Istanbul Technical University, Faculty of Science and Letters, under supervision of Assoc. Prof. A. Levent SUBAŞI. Furthermore I would like to thank my family for their endless spiritual and material support.

May 2015 Buğra TÜZEMEN

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ABBREVIATIONS

BCS	**:** Bardeen Cooper Schrieffer
BdG	**:** Bogoliubov de Gennes
SOC	**:** Spin-Orbit Coupling
PEP	**:** Pauli Exclusion Principle
CCS	**:** Charge Conjugation Symmetry
PHS	**:** Particle Hole Symmetry
SLS	**:** Sub Lattice Symmetry
TRS	**:** Time Reversal Symmetry

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RICHARDSON SOLUTION FOR THE SPIN-ORBIT COUPLED FERMI GAS

SUMMARY

Topological states of matter have been attracting increasing attention especially after the experimental discovery of topological insulators. The topological aspect of these phases of matter involves the topological excitations on the edges or surfaces which have zero energy. The topological aspect of the edge/surface states has signatures also in the bulk via the bulk-edge correspondence. Superconducting materials can also have topological states and the topological phases of insulators and superconductors can be classified in ten groups using fundamental symmetries of the Hamiltonian.

The model Hamiltonian considered in this thesis can support topological phases. In order to make this possible it involves an s-wave superconducting interaction between Fermions whose momentum is coupled to their spin. This Hamiltonian is recently realized for electrons in nano-wires. The nanowire has intrinsic spin-orbit coupling and is brought close to a conventional s-wave superconductor. The superconductivity is induced via the proximity effect. An alternative approach to realizing this Hamiltonian could be the ultra-cold atom systems. The arbitrary control of system parameters together with cooling techniques are promising for realizing superconducting systems. More recently, by using additional lasers Abelian and non-Abelian gauge fields could be included in the Hamiltonian. Therefore these systems present another realization for the spin-orbit coupled s-wave superconductor model.

The numerical approach to the solution of the model is an exact solution. This is an important feature because apart from mean field theory and other approaches, exact solution provides us to explore different proporties of the system. Obtaining the exact solution is possible because of the number of conserved quantities that commute with the Hamiltonian. For a given number of particles N, one obtains N nonlinear algebraic equations which have N complex roots. For particular interaction values, these equations harbor singularities. To get rid of these singularities, one must make necessary transformations and extrapolations.

The plan of the thesis is as follows. The first chapter will introduce the spin-momentum coupled superconducting Hamiltonian and identify its place in the topological classification of topological insulators and superconductors. The second chapter introduces the Richardson model and the equations for the numerical solution. The numerical solution for the equations avoiding the numerical singularities of the direct approach is presented and applied to the original Richardson problem in the third chapter. In the fourth chapter we show explicitly that the spin-orbit coupled superconducting Hamiltonian when written in the helicity basis takes the form of the Richardson Hamiltonian. The numerical solution is applied to this model in order to obtain total energy as a function of the interaction strength. In the last chapter we conclude with general remarks and future prospects.

SPİN-MOMENTUM BAĞLI FERMİ GAZLARINDA RİCHARDSON ÇÖZÜMÜ

ÖZET

Spin-momentum bağlı sistemler son zamanlarda deneysel imkanların artmasıyla teorik olarak da ilgi çekici bir konu haline gelmiştir. Bu tip sistemler, kuantum spin Hall etkisi, topolojik yalıtkanlar/süperiletkenler, yarıiletken heteroyapılar ve optik örgülerdeki aşırı soğuk atomik sistemler gibi örneklenebilir.

Maddenin topolojik halleri, özellikle topolojik yalıtkanların keşfinden sonra oldukça ilgi çeken bir araştırma konusu haline gelmiştir. Maddenin bu halinin topolojik oluşu, sistemin yüzeylerinde ve/veya kenarlarında oluşan sıfır enerjili uyarılmış durumlardan kaynaklanmaktadır. Bu durumlar dışarıdan gelen uyarma ve tedirgemelere karşı korunaklıdırlar. Yalıtkanlar gibi, süperiletken sistemleri de topolojik özellikler gösterebilir. Topolojik yalıtkan ve topolojik süperiletkenler, Hamiltonyenin temel simetri özelliklerini kullanarak on simetri grubu altında sınıflandırılabilirler. Uniter simetriler Hamiltonyende blok yapısı oluşturmasına rağmen topolojik sınıflandırmaya katkıda bulunmazlar. Anti-üniter simetriler ise topolojik sınıflandırmada önemli rol oynarlar.

Topolojik yalıtkanlara en bilinen örnek olarak tamsayılı kuantum spin Hall yalıtkanı verilebilir. İki boyutlu bu elektron sisteminde zaman tersinme simetrisi dışarıdan uygulanan manyetik alanın etkisiyle kırılır. Kuantum Hall yalıtkanını, vakuma karşı bir boyutlu bir sınırla sonlandırırsak, sıfır enerji kenar durumları meydana gelir. Bozulan zaman tersinme simetrisinin etkisiyle bu sınır durumları tek yönde ilerlemeye başlarlar. Bu sayede kenar durumları, dışarıdan gelecek etkilere karşı korunaklı olur.

Topolojik süperiletkenlere örnek olarak iki boyutlu kiral $p_x + ip_y$ tipi süperiletkenler verilebilir. Uyarılmış enerji aralıklarına sahip bu sistem, bir süperiletken olup zaman tersinme simetrisi bozunmuştur. İncelenen etkileşmeyen sistem, süperiletken durumdaki Bogoliubov de Gennes denklemlerinde de tanımlanan parçacığımsı uyarılmalardan oluşmaktadır. Bu 2 boyutlu kiral $p_x + ip_y$ durumu bir vakumla kesildiği zaman sınırlarda kiral modlar gözlemlenir. Tamsayılı kuantum spin Hall yalıtkanında olduğu gibi, bu sınır modları tek yönde ilerler ve dışarıdan gelecek uyarmalara karşı kapalıdırlar.

Topolojik süperiletkenlerin bir başka ilginç özelliği de Majorana fermiyonlarını barındırabilmeleridir. Majorana fermiyonları, kendisinin anti-parçacığı olan teorik parçacıklardır. Bugüne kadar yüksek enerji fiziği kapsamında gözlemlenmemelerine rağmen, yoğun madde fiziğinde kendisinin deşiği olan parçacığımsı uyarılmaları şeklinde karşılaşılmaktadır. Majorana fermiyonunun kendi kendisinin deşiği olması, bir elektron ve bir deşik durumlarının eşit şekilde üstüste gelmeleriyle oluşur. Bu durum, süperiletken sistemlerde Bogoliubov parçacığımsıları ile gerçekleşir.

Majorana fermiyonlarının teorik olarak ilgi çekmesinin en büyük sebebi abelyen olmayan değişim istatistikleri bulundurmalarıdır. Bunun önemli bir sonucu, Majorana fermiyonlarının topolojik süper bilgisayarlar için uygun adaylar olmasıdır.

Bu tezde üzerinde çalışılan model Hamiltonyen topolojik hallerin oluşumunu desteklemektedir. Bunu gerçekleştirmek için, Hamiltonyen, spin-momentum bağlı fermiyonların aralarında s-dalgası süperiletken etkileşimini içermektedir. Bu sistemlerde süperiletkenlik yakınlık etkisi ile elde edilebilir. Bu tip sistemlere alternatif bir yaklaşım ise, model Hamiltonyeni aşırı soğuk atom sistemleri ile gerçekleştirmetir. Sistemin değişkenleri üzerindeki yapay kontrol teknikleri, ileri soğutma teknikleri ile birleşince, süperiletken sistemleri oluşturmak için önemli bir teknik haline gelmiştir. Son zamanlarda ise bu tekniklere ek olarak lazer kullanımı ile Abelyen ve Abelyen olmayan alanlar ile de model Hamiltonyeni gerçeklemek mümkün hale gelmiştir.

Bu modelin Ortalama Alan Teorisi ya da Kesin Çaprazlama gibi yaklaşım kullanan yöntemler ile çözülmesi mümkün olmakla birlikte, kesin olarak çözümü de mümkündür. Kesin çözüm olması önemlidir çünkü Ortalama Alan Teorisi ya da Lanczos Yöntemi gibi yöntemlerden farklı olarak tüm etkileşim parametrelerinde de sonuç verip bize sistemin farklı özelliklerini inceleme fırsatı tanır.

Kesin çözümü elde etmek, Hamiltonyen ile arasında komütasyon bağıntıları bulunan sistemin korunumlu özelliklerinin sayısı ile mümkün olmaktadır. Genel BCS Hamiltonyeni'nin sonucu olan sistemde enerji seviyeleri eş aralıklı olurken, spin-momentum bağlı süperiletken sistemlerde enerji seviyeleri eş aralıklı değildir. N adet parçacık içeren fermiyon sistemi, spin ve momentum uzayında tersinme yozluk durumlarını içermektedir. Dalga fonksiyonu için kullanılan yaklaşım metodu ve uygun komütasyon bağıntıları ile sistemin enerjisi için N adet cebirsel, kompleks kökleri olan ve doğrusal olmayan denklem edilir.

Sistemin çözümü etkileşimin olmadığı limitten başlar ve etkileşim parametresi yavaş yavaş artırılarak sistemin özellikleri incelenir. Sistemdeki fermiyonlar arasındaki etkileşim parametresinin bazı değerleri için bu denklemler tekillik içerebilir. Bu tekilliklerden kurtulmak için denklemler üzerinde gerekli değişiklikler yapılmalıdır. Sistemi topyekün çözmek yerine, sistem limit boyutu sistemin yozluğu ile alakalı olacak şekilde dinamik kümelere ayrılır ve çözüm her kümenin kendi içinde ve diğer kümelerle olan çözümleri ayrı ayrı gerçekleştirilir. Tekillikleri güvenli bir şekilde ortadan kaldırmak için başvurulan bir diğer yöntem ise sona öngörüm yöntemidir. Bu yöntemin kullanılışı şu şekildedir: Eğer program çözüm bulamıyorsa ya da bulduğu çözüm, istenen kesinlikte değilse etkileşim parametresinde attığı adımı geri alıp daha küçük adımlarla ilerlemeyi dener. İstenen sayıda bu işlem tekrarlandıktan sonra hala sistem çözülemiyorsa, bu sefer program etkileşim parametresi için normalden daha büyük adım atarak ilerlemeye çalışır. Bu sayede tekilliklerin gerçekleştiği noktaların üzerine basmadan sistem en az hata ile çözülür.

Bu tezin içeriği şu şekildedir: İlk bölümde spin-momentum bağlı Hamiltonyen'in tanıtımı ve Hamiltonyenlerin simetrilerine göre sınıflandırılması incelenmektedir. İkinci bölüm, BCS Hamiltonyeni'nin Richardson yöntemi ile kesin olarak sayısal çözümünü elde etmek için oluşturulması gereken denklemleri içermektedir. Üçüncü bölüm, elde edilen denklemlerin tekilliklerini ortadan kaldırmak için yapılan dönüşümleri ve sona öngörüm yönteminin uygulanışını içerir. Farklı koşullar için sayısal çözüm örnekleri de bu bölümde verilmiştir. Dördüncü bölümde ise, Hamiltonyen spin-momentum bağlı hale getirilir. Bu Hamiltonyenin kesin çözümünün

olduğunu ispatlamak için, Hamiltonyen helisite bazına geçirilir. Uygun dönüşümler ve istenen komütasyon bağıntılarının gösterimi ile bu Hamiltonyen'in kesin çözümü olduğu ispatlanır. Sistemin nümerik çözümlerine örnekler de bu bölümdedir. Tez, son bölümde yapılan işin üzerine yorumlar ve gelecekteki planlar ile son bulmaktadır.

1. INTRODUCTION

Recent developments in experimental condensed matter physics have pulled some attention to the spin-orbit coupled systems which cover quantum spin Hall effects [1], topological insulators [2] and artificial systems [3]. Despite the studies provide vast theoretical insight, as all the studies operated in retrospect of the mean field theory, which is simply a good approximation for weakly interacting particles. More specific theoretical understanding of previously mentioned systems or strong pairing interactions are appreciated because it is remarkably difficult to provide exact solutions for many electron models.

1.1 Majorana Fermions

A degenerate, spin-orbit coupled Fermi gas with Zeeman field and spin-singlet pairing, can contain zero energy Majorana nodes. If the Zeeman field is beyond a critical value, these nodes obey non-Abelian exchange statistics. In solid state systems the same criteria can be realized when using a semiconductor nanowire with strong spin-orbit coupling, an s-wave superdoncutor and a magnetic field in a heterostructure.

Unlike Dirac fermions, Majorana fermions are their own anti-particle counterparts. Since there is no observed or theoretically envisioned elementary particles that are Majorana fermions, in regards to condensed matter systems one talks about Majorana modes, they are quasiparticle excitations instead of Majorana fermions. In condensed matter systems, a quasiparticle can be considered as its "own hole" which can be examined as a Majorana mode. What makes this interesting is that they are non-abelian anyons, which means, they have non-trivial exchange relations apart from ordinary fermions or bosons [4]. For they are their own hole, it would not be wrong to think Majorana modes as an electorn-hole superposition. It is known that such systems can be realized in superconducting systems by Bogoliubov quasiparticles which have both electron and hole components [5].

Apart from theoritical reasons, exploring their behaviour is important because their presence means a topological invariance [5].

1.2 Topology in Condensed Matter Systems

In a general sense, topology is on mathematical objects that can not change continuously. In mathematics, topology is concerned with continuous transformations of shapes into each other. These transformations may include bending, stretching, etc... But for example, tearing causes a discontinuity in transformation. Thus, if tearing is necessary to transform a shape into another, these two shapes will be in different topological groups.

In physics, topology found a use for itself in so many different areas. From cosmology to quantum field theories, topology became a key method to understand. In condensed matter physics, it basically means the presence or absence of particles with zero excitation energy. That is there is no energy requirement to excite a state from its ground state. Like the shapes transforming into each other, a physical topological system addresses whether the Hamiltonian of two quantum systems can be continuously transformed into each other. If two quantum systems can be continuously transformed into each other, that makes them topologically equivalent.

As a direct consequence of the Pauli exclusion principle, in many fermion systems, there is a reciprocal boundary namely the Fermi surface. In condensed matter systems, ground state can be set at the Fermi surface. That makes the states beyond the Fermi surface excited states and states below the occupied states.

There is a constraint on allowed energy states beyond the Fermi surface, this is the so called energy gap. This gap is the primal reason behind the classification into conductors, insulators, semi metals and semiconductors.

In the case of systems with an energy gap, two systems would be topologically equivalent if one's Hamiltonian can be transformed continuously into the other without changing the restriction of the energy gap. These transformations can only exist between two systems having the same number of occupied states, i.e., the same number of energy levels beneath zero. The reason that the occupied states can not freely move through zero, but it is favorable for excited states to decrease their energy if there

Table 1.1: Ten symmetry classes according to presence or absence of TRS, SLS and CCS

Label	TRS	CCS	SLS
A (Unitary)	0	0	0
AI (Orthogonal)	1	0	0
AII (Symplectic)	-1	0	0
AIII(Chiral Unitary)	0	0	1
BDI(Chiral Orthogonal)	-1	-1	1
CII(Chiral Symplectic)	1	1	1
D(BdG)	0	1	0
C(BdG)	-1	1	1
DIII(BdG)	0	-1	0
CI(BdG)	1	-1	1

is an empty negative state. So, in order to group two quantum system in the same topological group, the number of occupied states should not change under continuous transformation. Because of this condition, the number of occupied states is called a topological invariant. As a more general definition, topological invariance is a quantity for all topologically equivalent systems. If a topological invariant changes during a transformation, this process is called a topological phase transition.

1.3 Symmetries

To classify the topological systems, symmetry properties are important tools. In most general sense, concept of symmetry is concerned with whether the properties of a system changes or are conserved during a transformation. Consider a gapped Hamiltonian; its topological properties are stable under a translational symmetry. Although unitary symmetries makes the Hamiltonian diagonal, this criteria says that there are three main symmetries which considered as an indicator of topological features [6]. So, the classification can be done with time-reversal symmetry (TRS), sub-lattice symmetry (SLS) and charge conjugation symmetry (CCS). Note that CCS is equivalent to particle-hole symmetry.

1.3.1 Time-Reversal Symmetry (TRS)

Time-reversal symmetry is related with the conservation of energy. In quantum mechanics, TRS is represented by an anti-unitary operator. It can be formed as $\mathscr{T} = UK$ where U is an unitary operator and K is the complex conjugation operator. Let us check if our model Hamiltonian is invariant under TRS:

The model Hamiltonian is:

$$H = \sum_{j,m} \varepsilon_j c^{\dagger}_{jm} c_{jm} - g \sum_{j,j',m,m\prime} c^{\dagger}_{jm} c^{\dagger}_{j\bar{m}} c_{j'\bar{m}'} c_{j'm'} \tag{1.1}$$

TRS means that, $\mathscr{T} m \to \bar{m}$ and $\mathscr{T} j \to -j$. So the above Hamiltonian will have the following form:

$$H = \sum_{j,m} \varepsilon_{-j} c^{\dagger}_{-j\bar{m}} c_{-j\bar{m}} - g \sum_{-j,-j',\bar{m},\bar{m}'} c^{\dagger}_{-j\bar{m}} c^{\dagger}_{-jm} c_{-j'm'} c_{-j'\bar{m}'} \tag{1.2}$$

For there is a summation over j and m values, we can safely swap $-j$ values with j values and m with $\bar{m}$. And with the use of necessary commutation relations, it can be seen that above Hamiltonian is the same with Eq. 1.1. So, that means the model Hamiltonian commutes with the TRS operator, and is invariant under TRS. As an example, for fermionic systems, the TRS operator has the form of:

$$\mathscr{T} = i\sigma_y K \tag{1.3}$$

Where K is complex conjugation operator. Its effect on Hamiltonian is:

$$H = \sigma_y H^* \sigma_y \tag{1.4}$$

As a result of its definition it has a property of $\mathscr{T}^2 = -1$.

1.3.2 Charge Conjugation Symmetry (CCS)

While CCS means the conservation of charge and it has a place in most of the branches of physics, in many-body systems it also means particle-hole symmetry, that is, the symmetry between the number of electrons and holes. To examine the PHS, the Hamiltonian (H) must be in the form of BdG Hamiltonian (H_{BdG}) which has the following structure:

$$H_{BdG} = \begin{pmatrix} T & \Delta \\ -\Delta^* & -T^* \end{pmatrix} \tag{1.5}$$

To obtain that structure, H must be written in the form of:

$$H = \frac{1}{2} C^{\dagger} H_{BdG} C \tag{1.6}$$

where $C = (c_1, .., c_n, {c_1}^{\dagger}, .., {c_n}^{\dagger})^T$.

Derivation of the form of H_{BdG} can be found at App. A3. PHS has the following property [7]:

$$H_{BdG} = -\tau_x H_{BdG} \tau_x$$

which is consistant with our model Hamiltonian.

1.3.3 Sub-Lattice Symmetry (SLS)

If the system consists two groups of particles in a way that its Hamiltonian has zero-valued matrix elements within each group and nonzero matrix elements between them, the Hamiltonian can be brought to the following form:

$$H = \begin{pmatrix} 0 & T \\ T^{\dagger} & 0 \end{pmatrix} \tag{1.7}$$

Where T's are the hopping terms between two groups. An example for this type of system can be given as the hexagonal carbon lattice of graphene. SLS can be examined by using $\sigma_z$which equals to 1 for one sublattice and -1 for the other [8]. The effect of SLS on Hamiltonian is:

$$-H = \sigma_z H \sigma_z \tag{1.8}$$

However SLS can also be considered as the absolute value of the product of TRS and PHS [6]. Also it should be noted that SLS can be examined while both TRS and PHS are absent.

2. THE RICHARDSON MODEL

2.1 Exactly Solvable Models

Exactly solvable models constitutes a way to investigate the properties of the systems which have non-linear terms without using such methods as mean field theory and perturbation theory. A major approach to deal with these type of systems is the Bethe Ansatz. It deals with the one-dimensional many-body quantum Hamiltonians in terms of finding the exact solution. This approach was first used by Hans Bethe in 1931 on solving the one-dimensional anti-ferromagnetic Heisenberg Model [9]. Since then it has been used to study Bose gas and Hubbard model.

Exact solution of a pairing-force Hamiltonian was introduced by Richardson in the 1960s [10]. He solved the eigenvalue problem for fermions and bosons. The problem provides n coupled non-linear equations for n pairs. The system is exactly solvable because it involves a complete set of commutation relations [11]. The result for the fermion case harbors some singularities for some critical values of pairing parameter. These singularities makes the problem hard to treat numerically. Richardson avoided these singularities by introducing particular transformations to the system, but it is possible to avoid these singularities for more general forms of the Hamiltonian.

2.2 Pairing Force Hamiltonian

The exactly solvable pairing Hamiltonian has the following structure:

$$H = \sum_{j,m} \varepsilon_j a^{\dagger}_{jm} a_{jm} - g \sum_{j,j',m,m\prime} a^{\dagger}_{jm} a^{\dagger}_{j\bar{m}} a_{j'\bar{m}'} a_{j'm'} \tag{2.1}$$

The system consists doubly degenerate, equispaced, single particle levels. The energy of the levels is given by:

$$\varepsilon_i = n \qquad n = 1, 2, ..., 2N \tag{2.2}$$

where N is the number of particles of one degeneracy level. This pairing system can be described with reduced BCS Hamiltonian:

$$H = \sum_{j,\sigma} \varepsilon_j c^{\dagger}_{j\sigma} c_{j\sigma} - g \sum_{i,j} c^{\dagger}_{i+} c^{\dagger}_{i-} c_{j-} c_{j+} \tag{2.3}$$

Time-reversed states of this hamiltonian are $|j,\sigma\rangle$ where j=1,2,....2N and σ is spin.

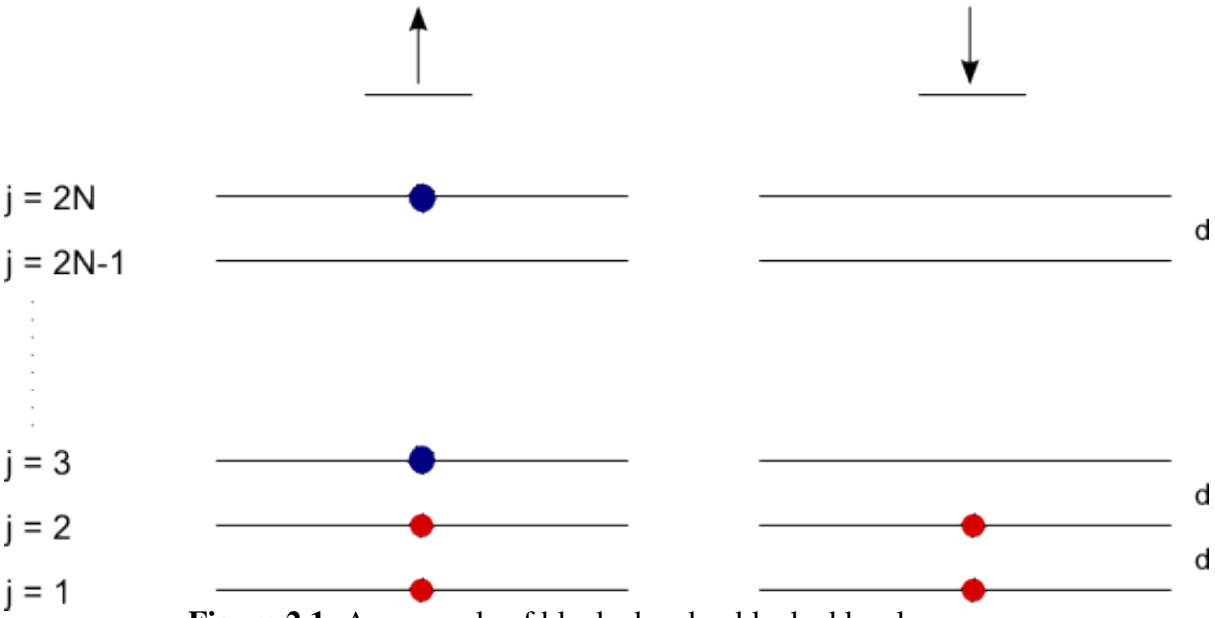


Figure 2.1: An example of blocked and unblocked levels

Let's define the single particle levels as blocked levels and number of blocked levels (and particles) as 'b' which constitutes the set B. Since they are not paired, they do not contribute to the pair scattering term of the Hamiltonian. The rest of the levels are either empty or filled with pairs and they form the set U as in 'unblocked'. Therefore the total number of particles,

$$N = 2n + b \tag{2.4}$$

For an eigenstate of this hamiltonian we start with the following ansatz:

$$|n,B\rangle = \prod_{i\in B} c^{\dagger}_{i\sigma} |\psi_n\rangle_u \tag{2.5}$$

Here, $c^{\dagger}_{i\sigma}$ creates a fermion from set B and $|\psi_n\rangle_u$ defines unblocked levels which are given as:

$$|\psi_n\rangle_u = \sum_{j_1,j_2,\ldots,j_n}^{U} \psi(j_1,j_2,\ldots,j_n) \prod_{\nu=1}^{n} b^{\dagger}_{j_\nu} |0\rangle \tag{2.6}$$

$b_j^\dagger$ creates an electron pair, $b_j^\dagger = c_{j+}^\dagger c_{j-}^\dagger$. Also j_ν labels the νth pair. An example can be given as:

	Ground State	1^{st} Excited State
j_1	1	1
j_2	2	2
$\vdots$	$\vdots$	$\vdots$
j_{n-1}	$n-1$	$n-1$
j_n	n	$n+1$

For singly occupied levels, the energy levels are:

$$\varepsilon_b = \sum_{i \in B} \varepsilon_i \tag{2.7}$$

For unblocked states, the hamiltonian can be rewritten in terms of pair operators:

$$H_u = \sum_{i,j}^{u} (2\varepsilon_j \delta_{ij} - g) b_i^\dagger b_j \tag{2.8}$$

Schrodinger equation for unblocked states:

$$H_u \left|\psi_n\right\rangle_u = \varepsilon_n \left|\psi_n\right\rangle_u \tag{2.9}$$

And the number of pairs:

$$\sum_{j}^{u} b_j^\dagger b_j \left|\psi_n\right\rangle_u = n \left|\psi_n\right\rangle_u \tag{2.10}$$

Diagonalizing this hamiltonian will take a bit more work than the usual since the pair operators obey the hard-core boson relations.

Before defining hard-core boson relations, let us look at the basic bosonic and fermionic commutation relations respectively.

$$\left[a_i^\dagger, a_j\right] = a_i^\dagger a_j - a_j a_i^\dagger = \delta_{ij}, \qquad \left[a_i^\dagger, a_j^\dagger\right] = \left[a_i, a_j\right] = 0 \tag{2.11}$$

$$\left\{c_i^\dagger, c_j\right\} = c_i^\dagger c_j + c_j c_i^\dagger = \delta_{ij}, \qquad \left\{c_i^\dagger, c_j^\dagger\right\} = \left\{c_i, c_j\right\} = 0 \tag{2.12}$$

In this particular case, while single particle states are obeying general fermionic anti-commutation rules, one can not treat the paired particles in the same way. One shall treat them as hard-core bosons since they can not occupy the same quantum state but the system has a symmetric wavefunction.

Hard-core boson relations are commutation relations of fermionic pair operators. First relation is trivial:

$$b_j^{\dagger 2} = 0 \tag{2.13}$$

Second relation is where the hard-core boson properties are used.

$$\left[b_j, b_{j'}^\dagger\right] = \left[c_{j-}c_{j+}, c_{j'+}^\dagger c_{j'-}^\dagger\right]$$

$$\begin{aligned}\left[b_j, b_{j'}^\dagger\right] &= -c_{j-}c_{j'+}^\dagger\left\{c_{j'-}^\dagger, c_{j+}\right\} + c_{j-}\left\{c_{j'+}^\dagger, c_{j+}\right\}c_{j'-}^\dagger \\ &\quad - c_{j'+}^\dagger\left\{c_{j'-}^\dagger, c_{j-}\right\}c_{j+} + \left\{c_{j'+}^\dagger, c_{j-}\right\}c_{j'-}^\dagger c_{j+}\end{aligned} \tag{2.14}$$

$$c_{j-}c_{j'-}^\dagger\delta_{jj'} - c_{j'+}^\dagger c_{j+}\delta_{jj'} = \delta_{jj'}(1 - c_{j-}^\dagger c_{j-} - c_{j+}^\dagger c_{j+})$$

$$c_{j-}^\dagger c_{j-} + c_{j+}^\dagger c_{j+} = \left\{\begin{array}{c} 2 \\ 0 \end{array}\right\}$$

$c_{j-}^\dagger c_{j-}$ operator is called as the number operator. It counts how many fermions at the particular state. As a result of PEP, the number operator can result to either one or zero for a particular state.

$$b_j^\dagger b_j = \left\{\begin{array}{c} 1 \\ 0 \end{array}\right\}$$

$$\left[b_j, b_{j'}^\dagger\right] = \delta_{jj'}(1 - 2b_j^\dagger b_j) \tag{2.15}$$

Third relation is more straightforward with the use of Eq. 2.15.

$$\left[b_j^\dagger b_j, b_{j'}^\dagger\right] = b_j^\dagger\left[b_j, b_{j'}^\dagger\right] + \left[b_j^\dagger, b_{j'}^\dagger\right]b_j = \delta_{jj'}b_j^\dagger(1 - b_j^\dagger b_j) \tag{2.16}$$

$$\left[b_j^\dagger b_j, b_{j'}^\dagger\right] = \delta_{jj'}b_j^\dagger \tag{2.17}$$

2.3 One Pair Cooper Problem

Before investigating The Richardson Model, let us first examine the one pair Cooper problem using above definitions. The Hamiltonian is given by Eq. 2.8 and one pair states can be constructed as:

$$|\psi_{pair}\rangle = \sum_j C_j b_j^\dagger |0\rangle \tag{2.18}$$

Here $|0\rangle$ corresponds to the vacuum state, and C_j's are coefficients of each eigenstate.

$$\langle 0| b_j H |\psi_{pair}\rangle = \langle 0| b_j E_{pair} |\psi_{pair}\rangle \tag{2.19}$$

$$2\varepsilon_j C_j - g\sum_{j'} C_{j'} = E_{pair} C_j \quad , \qquad g\sum_{j'} C_{j'} = C \tag{2.20}$$

$$C_j = \frac{C}{2\varepsilon_j - E_{pair}} \tag{2.21}$$

$$\frac{1}{g} = \sum_j \frac{1}{2\varepsilon_j - E_{pair}} \tag{2.22}$$

And with normalization, we can find C as:

$$C = \frac{1}{\sum_j \frac{1}{2\varepsilon_j - E_{pair}}} \tag{2.23}$$

Finally the wavefunction of a pair is:

$$|\psi_{pair}\rangle = C\sum_j \frac{b_j^\dagger}{2\varepsilon_j - E_{pair}} |0\rangle \tag{2.24}$$

Above equation states that there are as many eigenstates as the number of energy levels.

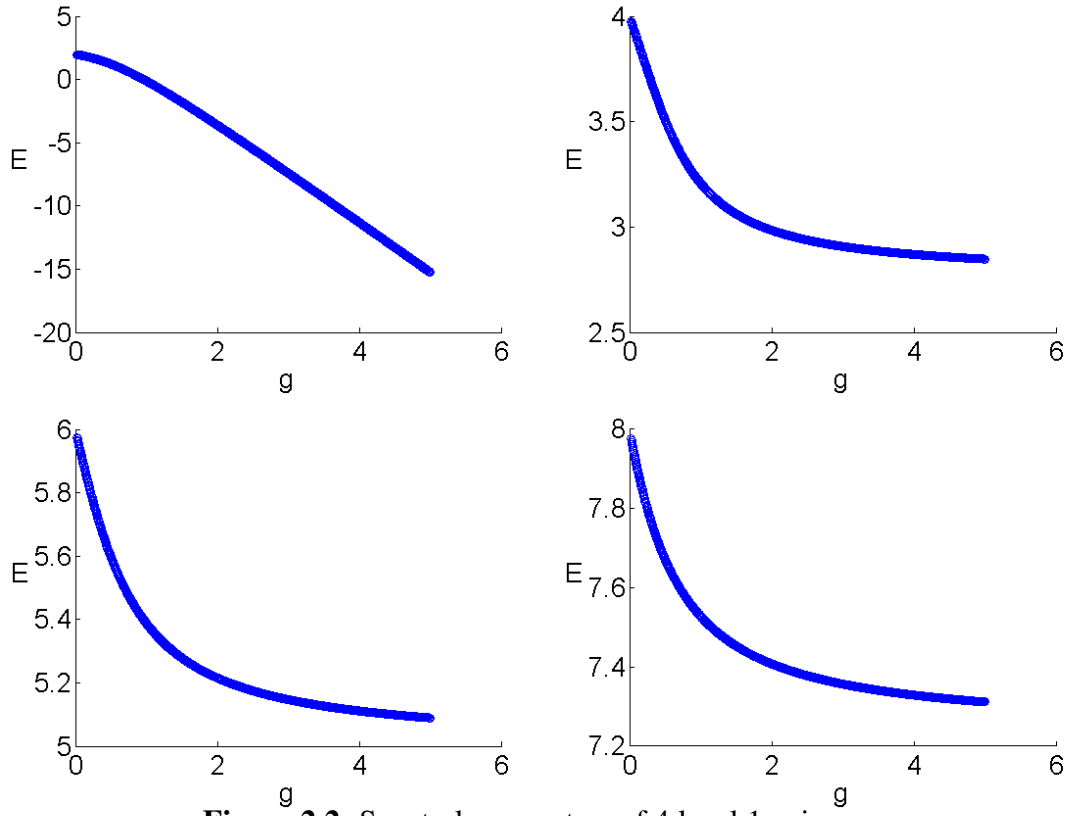


Figure 2.2: Spectral parameters of 4 level 1 pairs vs. g

2.4 Equations of The Richardson Model

Now that hard-core boson properties are investigated, we can start to work on Eq. 2.8. In order to work on that, an ansatz must be taken for $|\psi_n\rangle_u$:

$$|\psi_n\rangle_u = \prod_{\nu=1}^{n} B_{J_\nu}^\dagger |0\rangle \tag{2.25}$$

$$B_{J_\nu}^\dagger = \sum_j^u \frac{b_j^\dagger}{(2\varepsilon_j - E_{J_\nu})} \tag{2.26}$$

To check that Eq. 2.25 weather is an eigenstate of H_u or not, we have to apply Schrödinger equation,

$$H_u |\psi_n\rangle_u = \xi_n |\psi_n\rangle_u \tag{2.27}$$

where $\xi_n = \sum_\nu^n E_{J_\nu}$.

To hop H_u over pair creation operators for in order to use the identity of $H_u|0\rangle = 0$, only thing will be remain is the commutator of H_u and $\prod_{\nu=1}^n B_{J_\nu}^\dagger$.

$$H_u \prod_{\nu=1}^{n} B_{J_\nu}^{\dagger} |0\rangle = \left[H_u, \prod_{\nu=1}^{n} B_{J_\nu}^{\dagger} \right] |0\rangle + \prod_{\nu=1}^{n} B_{J_\nu}^{\dagger} H_u |0\rangle = \left[H_u, \prod_{\nu=1}^{n} B_{J_\nu}^{\dagger} \right] |0\rangle$$

$$\left[H_u, \prod_{\nu=1}^{n} B_{J_\nu}^{\dagger} \right] = \sum_{\nu=1}^{n} \left\{ \left(\prod_{\eta=1}^{\nu-1} B_{J_\eta}^{\dagger} \right) \left[H_u, B_{J_\nu}^{\dagger} \right] \left(\prod_{\mu=\nu+1}^{n} B_{J_\mu}^{\dagger} \right) \right\} \tag{2.28}$$

To evaluate this commutation we have to reshape H_u as following:

$$H_u = \sum_{j}^{u} 2\varepsilon_j b_j^{\dagger} b_j - g B_0^{\dagger} B_0, \qquad where \qquad B_0 = \sum_{j}^{u} b_j^{\dagger} \tag{2.29}$$

And with use of following commutations:

$$\bullet \left[b_j^{\dagger} b_j, B_{J_\nu}^{\dagger} \right] = \frac{b_j^{\dagger}}{(2\varepsilon_j - E_{J_\nu})} \tag{2.30}$$

$$\bullet \left[B_0, B_{J_\nu}^{\dagger} \right] = \sum_{j}^{u} \frac{(1 - 2b_j^{\dagger} b_j)}{(2\varepsilon_j - E_{J_\nu})} \tag{2.31}$$

$$\left[H_u, B_{J_\nu}^{\dagger} \right] = \left[\sum_{j}^{u} 2\varepsilon_j b_j^{\dagger} b_j - g B_0^{\dagger} B_0, B_{J_\nu}^{\dagger} \right]$$

$$= \left[\sum_{j}^{u} 2\varepsilon_j b_j^{\dagger} b_j, B_{J_\nu}^{\dagger} \right] - g \left[B_0^{\dagger} B_0, B_{J_\nu}^{\dagger} \right]$$

$$= \sum_{j}^{u} \frac{2\varepsilon_j b_j^{\dagger}}{(2\varepsilon_j - E_{J_\nu})} - g B_0^{\dagger} \sum_{j}^{u} \frac{(1 - 2b_j^{\dagger} b_j)}{(2\varepsilon_j - E_{J_\nu})} - \sum_{j}^{u} b_j^{\dagger} + B_0^{\dagger}$$

$$\bullet \left[H_u, B_{J_\nu}^{\dagger} \right] = E_{J_\nu} B_{J_\nu}^{\dagger} + B_0^{\dagger} \left(1 - g \sum_{j}^{u} \frac{(1 - 2b_j^{\dagger} b_j)}{(2\varepsilon_j - E_{J_\nu})} \right) \tag{2.32}$$

Now we can calculate Eq. 2.27 via Eq. 2.28.

$$
\begin{aligned}
H_u|\psi_n\rangle_u &= \sum_{\nu=1}^{n}\left\{\left(\prod_{\eta=1}^{\nu-1}B^\dagger_{J_\eta}\right)E_{J_\nu}B^\dagger_{J_\nu}\left(\prod_{\mu=\nu+1}^{n}B^\dagger_{J_\mu}\right)\right\}|0\rangle \\
&\quad+\sum_{\nu=1}^{n}\left\{\left(\prod_{\eta=1}^{\nu-1}B^\dagger_{J_\eta}\right)B^\dagger_0\Big(1-g\sum_j^u\frac{(1-2b^\dagger_jb_j)}{(2\varepsilon_j-E_{J_\nu})}\Big)\left(\prod_{\mu=\nu+1}^{n}B^\dagger_{J_\mu}\right)\right\}|0\rangle \\
&= \xi_n|\psi_n\rangle_u+\sum_{\nu=1}^{n}\left\{\left(\prod_{\eta=1}^{\nu-1}B^\dagger_{J_\eta}\right)B^\dagger_0\left(\prod_{\mu=\nu+1}^{n}B^\dagger_{J_\mu}\right)\right\}|0\rangle \\
&\quad-\sum_{\nu=1}^{n}\left\{\left(\prod_{\eta=1}^{\nu-1}B^\dagger_{J_\eta}\right)B^\dagger_0g\sum_j^u\frac{(1-2b^\dagger_jb_j)}{(2\varepsilon_j-E_{J_\nu})}\left(\prod_{\mu=\nu+1}^{n}B^\dagger_{J_\mu}\right)\right\}|0\rangle \\
&= \sum_{\nu=1}^{n}\left(B^\dagger_0\prod_{\eta=1(\neq\nu)}^{n}B^\dagger_{J_\eta}\right)|0\rangle \\
&\quad-\sum_{\nu=1}^{n}\left(\prod_{\eta=1}^{\nu-1}B^\dagger_{J_\eta}\right)\frac{B^\dagger_0g}{(2\varepsilon_j-E_{J_\nu})}\left(\prod_{\mu=\nu+1}^{n}B^\dagger_{J_\mu}\right)|0\rangle \\
&\quad+\sum_{\nu=1}^{n}\left(\prod_{\eta=1}^{\nu-1}B^\dagger_{J_\eta}\right)\sum_j^u\frac{2gB^\dagger_0b^\dagger_jb_j}{(2\varepsilon_j-E_{J_\nu})}\left(\prod_{\mu=\nu+1}^{n}B^\dagger_{J_\mu}\right)|0\rangle+\xi_n|\psi_n\rangle_u \\
&= \xi_n|\psi_n\rangle_u+\sum_{\nu=1}^{n}\left(1-\sum_j^u\frac{g}{(2\varepsilon_j-E_{J_\nu})}\right)B^\dagger_0\left(\prod_{\eta=1(\neq\nu)}^{n}B^\dagger_{J_\eta}\right)|0\rangle \\
&\quad+\sum_{\nu=1}^{n}\left(\prod_{\eta=1}^{\nu-1}B^\dagger_{J_\eta}\right)\left[\sum_j^u\frac{2gB^\dagger_0b^\dagger_jb_j}{(2\varepsilon_j-E_{J_\nu})}\right]\left(\prod_{\mu=\nu+1}^{n}B^\dagger_{J_\mu}\right)|0\rangle \qquad \textbf{(2.33)}
\end{aligned}
$$

In order to continue, following commutation has to be evaluated:

$$
\begin{aligned}
\left[\sum_j^u\frac{2gB^\dagger_0b^\dagger_jb_j}{(2\varepsilon_j-E_{J_\nu})},B^\dagger_{J_\mu}\right] &\to \left[B^\dagger_0\sum_j^ub^\dagger_jb_j,B^\dagger_{J_\mu}\right] \\
\left[B^\dagger_0\sum_j^ub^\dagger_jb_j,B^\dagger_{J_\mu}\right]=B^\dagger_0\left[\sum_j^ub^\dagger_jb_j,B^\dagger_{J_\mu}\right] &= B^\dagger_0\sum_j^u\frac{b^\dagger_j}{(2\varepsilon_j-E_{J_\nu})} \\
\left[\sum_j^u\frac{2gB^\dagger_0b^\dagger_jb_j}{(2\varepsilon_j-E_{J_\nu})},B^\dagger_{J_\mu}\right] &= \sum_j^u\frac{2gB^\dagger_0}{(2\varepsilon_j-E_{J_\nu})}\frac{b^\dagger_j}{(2\varepsilon_j-E_{J_\mu})} \qquad \textbf{(2.34)}
\end{aligned}
$$

$$
\sum_j^u\frac{b^\dagger_j}{(2\varepsilon_j-E_{J_\nu})(2\varepsilon_j-E_{J_\mu})}=\sum_j^u\left(\frac{b^\dagger_j}{(2\varepsilon_j-E_{J_\nu})}-\frac{b^\dagger_j}{(2\varepsilon_j-E_{J_\mu})}\right)\frac{1}{E_{J_\nu}-E_{J_\mu}} \qquad \textbf{(2.35)}
$$

- $$\left[\sum_j^u\frac{2gB^\dagger_0b^\dagger_jb_j}{(2\varepsilon_j-E_{J_\nu})},B^\dagger_{J_\mu}\right]=2gB^\dagger_0\frac{B^\dagger_{J_\nu}-B^\dagger_{J_\mu}}{E_{J_\nu}-E_{J_\mu}} \qquad \textbf{(2.36)}$$

Now back to Eq. 2.33 and define the last(third) term as α

$$\begin{aligned}\alpha &= \sum_{\nu=1}^{n}\left(\prod_{\eta=1}^{\nu-1}B^{\dagger}_{J_\eta}\right)\sum_{\mu=\nu+1}^{n}\left[\prod_{\eta'=\nu+1}^{\mu-1}B^{\dagger}_{J'_\eta}\left(2gB^{\dagger}_0\frac{B^{\dagger}_{J_\nu}-B^{\dagger}_{J_\mu}}{E_{J_\nu}-E_{J_\mu}}\right)\prod_{\mu'=\mu+1}^{n}B^{\dagger}_{J'_\mu}\right]|0\rangle \\ &= \sum_{\mu=1}^{n}\left[\sum_{\nu=1}^{\mu-1}\frac{2g}{E_{J_\nu}-E_{J_\mu}}\right]B^{\dagger}_0\left(\prod_{\eta=1(\neq\mu)}^{n}B^{\dagger}_{J_\eta}\right)|0\rangle-\sum_{\nu=1}^{n}\left[\sum_{\mu=\nu+1}^{n}\frac{2g}{E_{J_\nu}-E_{J_\mu}}\right]B^{\dagger}_0\left(\prod_{\eta=1(\neq\nu)}^{n}B^{\dagger}_{J_\eta}\right)|0\rangle\end{aligned}$$

Collect dummy indices into one:

$$\alpha=\sum_{\nu=1}^{n}\left[\sum_{\mu=1(\neq\nu)}^{n}\frac{2g}{E_{J_\nu}-E_{J_\mu}}\right]B^{\dagger}_0\left(\prod_{\eta=1(\neq\nu)}^{n}B^{\dagger}_{J_\eta}\right)|0\rangle \tag{2.37}$$

And let's go back to Eq. 2.33 for the last time:

$$\begin{aligned}H_u|\psi_n\rangle_u &= \xi_n|\psi_n\rangle_u+\sum_{\nu=1}^{n}\left(1-\sum_{j}^{u}\frac{g}{(2\varepsilon_j-E_{J_\nu})}\right)B^{\dagger}_0\left(\prod_{\eta=1(\neq\nu)}^{n}B^{\dagger}_{J_\eta}\right)|0\rangle \\ &\quad+\sum_{\nu=1}^{n}\left[\sum_{\mu=1(\neq\nu)}^{n}\frac{2g}{E_{J_\nu}-E_{J_\mu}}\right]B^{\dagger}_0\left(\prod_{\eta=1(\neq\nu)}^{n}B^{\dagger}_{J_\eta}\right)|0\rangle\end{aligned} \tag{2.38}$$

So $(H_u-\xi_n)|\psi_n\rangle_u=0$ if only

$$1-\sum_{j}^{u}\frac{g}{(2\varepsilon_j-E_{J_\nu})}+\sum_{\mu=1(\neq\nu)}^{n}\frac{2g}{E_{J_\mu}-E_{J_\nu}}=0 \tag{2.39}$$

Above equation consists n nonlinear, coupled set of equations. While the first two terms correspond to the one pair system, the last term contains the many body interactions. It holds n unknowns which corresponds to the pair energies. Pair energies can be either real or complex conjugate pairs.

3. SOLVING THE RICHARDSON EQUATIONS

3.1 Dealing With Singularities

Before solving The Richardson equations, a possible singularity must be taken care of in the case of $2\varepsilon_j = E_{J_j}$ as well as $E_{J_\mu} = E_{J_\nu}$. These equations become unstable at the vicinities of the singularities. The way to treat these equations starts with an approximate suggestion for non-interacting or weak-interaction limits. Then, as g increases adiabatically, the solution evolves to desired point. To get rid of the singularities, one must separate the system into clusters around single-particle levels [12]. By this, Eq. 2.39 can be considered as sum of two systems. One solution comes from the cluster itself and the other solution comes from the interaction of the chosen cluster with other clusters. Rewrite Eq. 2.39 as:

$$\frac{1}{2g} + \sum_j \frac{d_j}{(2\varepsilon_j - E_{J_\nu})} + \sum_{\mu=1(\neq\nu)}^{n} \frac{1}{E_{J_\mu} - E_{J_\nu}} = 0 \tag{3.1}$$

where

$$d_j = \frac{\nu_j - \Omega_j}{2} \tag{3.2}$$

$$\nu_j : \textit{seniority} \tag{3.3}$$

$$\Omega j : \textit{pair degeneracy} \tag{3.4}$$

To solve this equation one can treat the clusters which formed around different single-particle levels seperately.

$$F_k(E_{J_\nu}) + \frac{d_k}{2\varepsilon_k - E_{J_\nu}} + \sum_{\mu\neq\nu}^{n} \frac{1}{E_{J_\mu} - E_{J_\nu}} = 0 \tag{3.5}$$

with

$$F_k(E_{J_\nu}) = \frac{1}{2g} + \sum_{j,j\neq k} \frac{d_j}{(2\varepsilon_j - E_{J_\nu})} + \sum_{\mu=1(\neq\nu)}^{n} \frac{1}{E_{J_\mu} - E_{J_\nu}} = 0 \tag{3.6}$$

To get rid of the numerical singularities, we define two new variables.

$$S_p = \sum_\alpha (2\varepsilon_k - E_{J_\nu})^p = \sum_\alpha \xi_\alpha^p \tag{3.7}$$

$$R_p = \sum_{\alpha} (2\varepsilon_k - E_{J_\nu})^p F_k(E_{J_\nu}) \tag{3.8}$$

Multiply Eq. 3.5 with ξ_α^p and rewrite $\frac{1}{E_{J_\mu} - E_{J_\nu}}$ as $\frac{1}{(2\varepsilon_k - E_{J_\nu}) - (2\varepsilon_k - E_{J_\mu})}$ then the equation becomes:

$$\boxed{d_k S_{p-1} + \sum_{\alpha,\beta,\alpha\neq\beta} \frac{\xi_\alpha^p}{\xi_\alpha - \xi_\beta} + R_p = 0} \tag{3.9}$$

$$\frac{\xi_\alpha^p}{\xi_\alpha - \xi_\beta} = \frac{(\xi_\alpha - \xi_\beta + \xi_\beta)^p}{\xi_\alpha - \xi_\beta} \tag{3.10}$$

$$= \sum_{n=0}^{p} (\xi_\alpha - \xi_\beta)^{n-1} \xi_\beta^{p-n} \binom{p}{n} \tag{3.11}$$

Sort $n = 0$ term out, which is $\frac{\xi_\beta^p}{\xi_\alpha - \xi_\beta}$. And the second term of Eq. 3.9 is:

$$\sum_{\alpha,\beta}{}' \frac{\xi_\alpha^p}{\xi_\alpha - \xi_\beta} = \sum_{\alpha,\beta}{}' \frac{\xi_\beta^p}{\xi_\alpha - \xi_\beta} + \sum_{\alpha,\beta}{}' \sum_{n=1}^{p} (\xi_\alpha - \xi_\beta)^{n-1} \xi_\beta^{p-n} \binom{p}{n} \tag{3.12}$$

Change the indices of the first term in right hand side and pass it to the other side of the equation:

$$2\sum_{\alpha,\beta}{}' \frac{\xi_\alpha^p}{\xi_\alpha - \xi_\beta} = \sum_{\alpha,\beta}{}' \sum_{n=1}^{p} (\xi_\alpha - \xi_\beta)^{n-1} \xi_\beta^{p-n} \binom{p}{n} \tag{3.13}$$

$$= \sum_{\alpha,\beta}{}' \sum_{n=0}^{p-1} (\xi_\alpha - \xi_\beta)^{n} \xi_\beta^{p-n-1} \binom{p}{n+1} \tag{3.14}$$

$$= \sum_{\alpha,\beta}{}' \sum_{n=0}^{p-1} \sum_{m=0}^{n} (-1)^{n-m} \xi_\alpha^m \xi_\beta^{p-1-m} \binom{p}{n+1} \binom{n}{m} \tag{3.15}$$

$$= \sum_{\alpha,\beta} \sum_{n=0}^{p-1} \sum_{m=0}^{n} (-1)^{n-m} S_m S_{p-1-m} \binom{p}{n+1} \binom{n}{m}$$

$$- S_{p-1} \sum_{\alpha,\beta,\alpha=\beta} \sum_{n=0}^{p-1} \sum_{m=0}^{n} (-1)^{n-m} \binom{p}{n+1} \binom{n}{m} \tag{3.16}$$

Notice that the conditional sum on Eq. 3.13 turns into two different sums, one of is for all α's and β's and the other is for $\alpha = \beta$. To check the $\alpha = \beta$ term of the Eq. 3.16 we take advantage of some combinatorics identities which can be found in App. A. 1:

for $n = 0$:

$$S_{p-1} \sum_{n=0}^{p-1} \sum_{m=0}^{n} (-1)^{n-m} \binom{p}{n+1} \binom{n}{m} = pS_{p-1} \tag{3.17}$$

for $n \neq 0$:

$$S_{p-1} \sum_{n=0}^{p-1} \binom{p}{n+1} \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} = 0 \tag{3.18}$$

The second term of Eq. 3.16 yields to zero. The first term of Eq. 3.16, for $m = 0$

$$\begin{aligned} \sum_{n=0}^{p-1} \sum_{m=0}^{n} S_m S_{p-1-m} (-1)^{n-m} \binom{p}{n+1} \binom{n}{m} &= \sum_{n=0}^{p-1} S_0 S_{p-1} (-1)^n \binom{p}{n+1} \\ &= N_k S_{p-1} \sum_{n=0}^{p-1} (-1)^n \binom{p}{n+1} \\ &= N_k S_{p-1} \end{aligned} \tag{3.19}$$

and for $m \neq 0$

$$\sum_{n=1}^{p-1} \sum_{m=1}^{n} S_m S_{p-1-m} (-1)^{n-m} \binom{p}{n+1} \binom{n}{m} = \sum_{m=1}^{p-1} S_m S_{p-1-m} \sum_{n=m}^{p-1} (-1)^n \binom{p}{n+1} \binom{n}{m} \tag{3.20}$$

Make the change $k = 1 + m$ and check the term $n = p-1$, $k = p$:

$$-\sum_{n=1}^{p-1} \sum_{k=2}^{n+1} S_{k-1} S_{p-k} \binom{p}{n+1} \binom{n}{k-1} (-1)^{n-k} = N_k S_{p-1} \tag{3.21}$$

And the remaining terms of Eq. 3.20 yields into:

$$\begin{aligned} & \sum_{k=2}^{p-1} S_{k-1} S_{p-k} \binom{p-1}{k-1} (-1)^{p-k} - \sum_{n=1}^{p-2} \sum_{k=2}^{n+1} S_{k-1} S_{p-k} \binom{n}{k-1} \binom{p}{n+1} (-1)^{n-k} \\ = & \sum_{k=2}^{p-1} S_{k-1} S_{p-k} \binom{p-1}{k-1} (-1)^{p-k} - \sum_{k=2}^{p-1} \sum_{n=k-1}^{p-2} (-1)^n S_{k-1} S_{p-k} \binom{p}{n+1} \binom{n}{k-1} \end{aligned}$$

$$(-1)^{-k} \left[\binom{p-1}{k-1} (-1) p - \sum_{n=k-1}^{p-2} (-1)^n \binom{p}{n+1} \binom{n}{k-1} \right] = 1$$

To sum all up with the use of Eq. 3.16, Eq. 3.17, Eq. 3.19, Eq. 3.21 :

$$2 \sum_{\alpha,\beta}{}' \frac{\xi_\alpha^p}{\xi_\alpha - \xi_\beta} = -pS_{p-1} + 2N_k S_{p-1} + \sum_{k=2}^{p-1} S_{k-1} S_{p-k} \tag{3.22}$$

And go back to Eq. 3.9 in order to obtain the desired identity.

$$(d_k + N_k - \frac{p}{2}) S_{p-1} + \frac{1}{2} \sum_{k=2}^{p-1} S_{k-1} S_{p-k} + R_p = 0 \tag{3.23}$$

3.2 Exact Solutions of The Richardson Model

With above transformations, the problem is switched to S variables from E's. Also with the use of definition of S, it is easy to switch between two variables by constructing a polynomial equation which has roots that corresponds to $2\varepsilon_k - E_{J_\nu}$ values. To solve the system, the first thing to do is defining the initial conditions of the system. For Richardson Model, single-particle levels are equally spaced and two-fold degenerate. The energy levels are $\varepsilon_i = 2i, i = 1,2,....,N_l$ and the degeneracy has its place in the equations as the effective degeneracy. For fermions, it is $d_j = \frac{\nu_j - \Omega_j}{2}$, where ν_j is the seniority and Ω_j is the pair degeneracy. It is easy to observe that for fermions $d_j \leq 0$ because of the Pauli principle.

After setting the levels, the pairs are placed just below the levels. For example, if we are looking for a ground state solution and, there are 8 levels, 4 pairs can be placed in the vicinity of the first 4 levels. Once the pairs are introduced to the system, they have to be separated into clusters. Each cluster is associated with one single-particle level. Therefore, the cluster size must be limited to a value that gives the number of pairs that converges to an energy level. This number is given by $n_k = -2d_k + 1$. When the setting of clusters and associating them with energy levels are done, initial S and R variables can be constructed accordingly.

Starting at non-interaction limit, the system must be solved gradually to the desired coupling constant. While increasing the coupling constant, at critical points, singularities occur. This is where the collapse of two pairs into complex conjugate pairs happens. To avoid this problem, an extrapolation can be made. For S values are always real and change smoothly, the extrapolation has to be done on them and to make the program safer, extrapolation can be spread on whole iterations. Crossing a single-particle level means that the cluster information needs to be updated. Cluster change results in a change of sign on S values which results to one has to be careful about extrapolation while crossing a single-particle level. Nevertheless, in general, a linear extrapolation can be used safely.

$$S = S'' + \frac{(S'' - S')(g - g'')}{g'' - g'} \tag{3.24}$$

As it can be seen from the Fig. 3.1 and Fig. 3.3, where the real parts of the pair energies collapse, complex conjugated parts of the single-pair energies emerge. If their energies

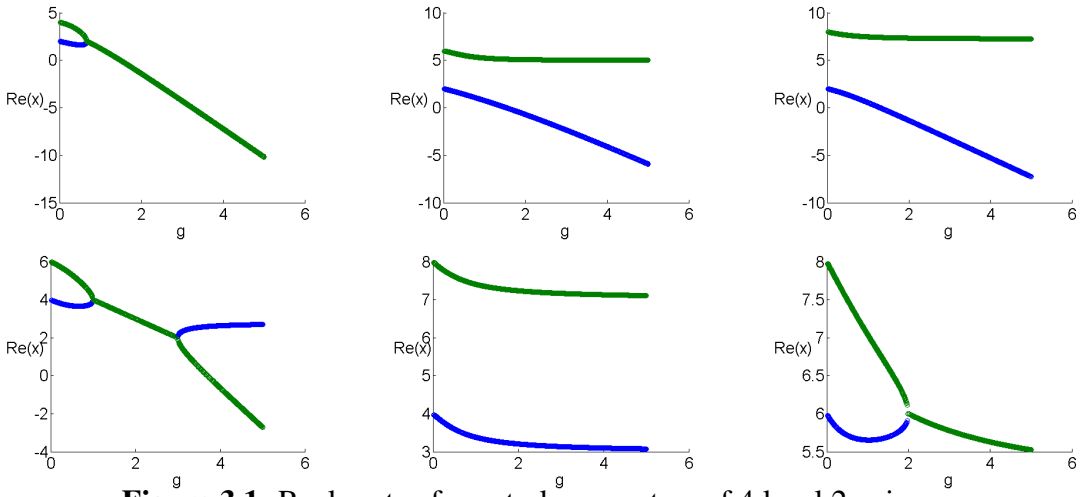


Figure 3.1: Real parts of spectral parameters of 4 level 2 pairs vs. g

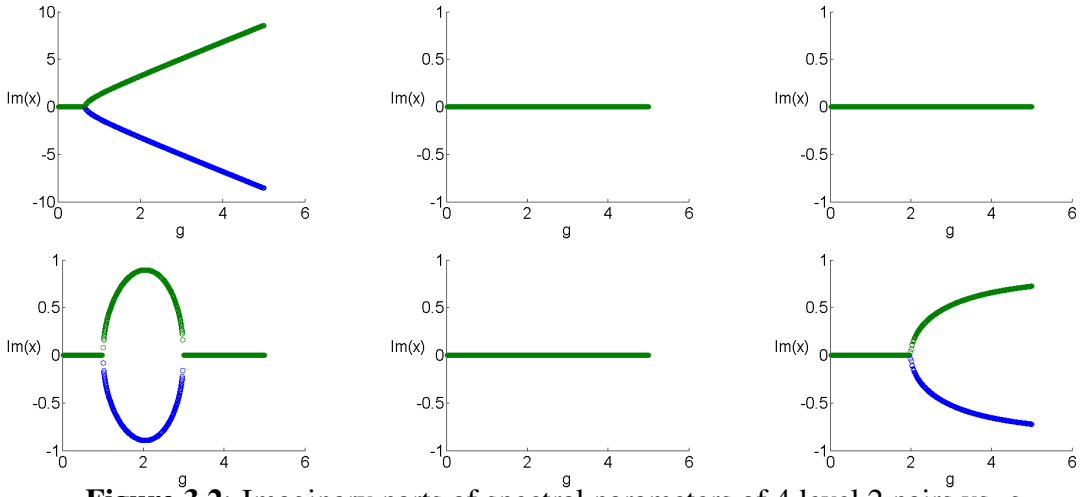


Figure 3.2: Imaginary parts of spectral parameters of 4 level 2 pairs vs. g

is in the vicinity of lowest single-particle level, collapsed pairs tends to go to $-\infty$ as g goes to ∞. But as the energy values gets larger, these pairs asymptotically sit on a single-particle level. Also, as an interesting remark, it is possible for collapsed pairs to split up while passing through a single-particle level.

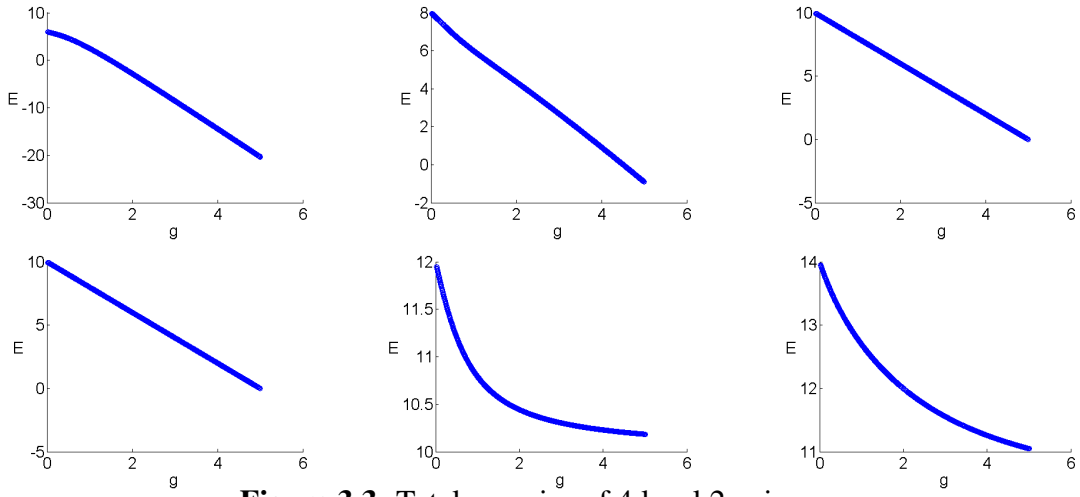


Figure 3.3: Total energies of 4 level 2 pairs vs. g

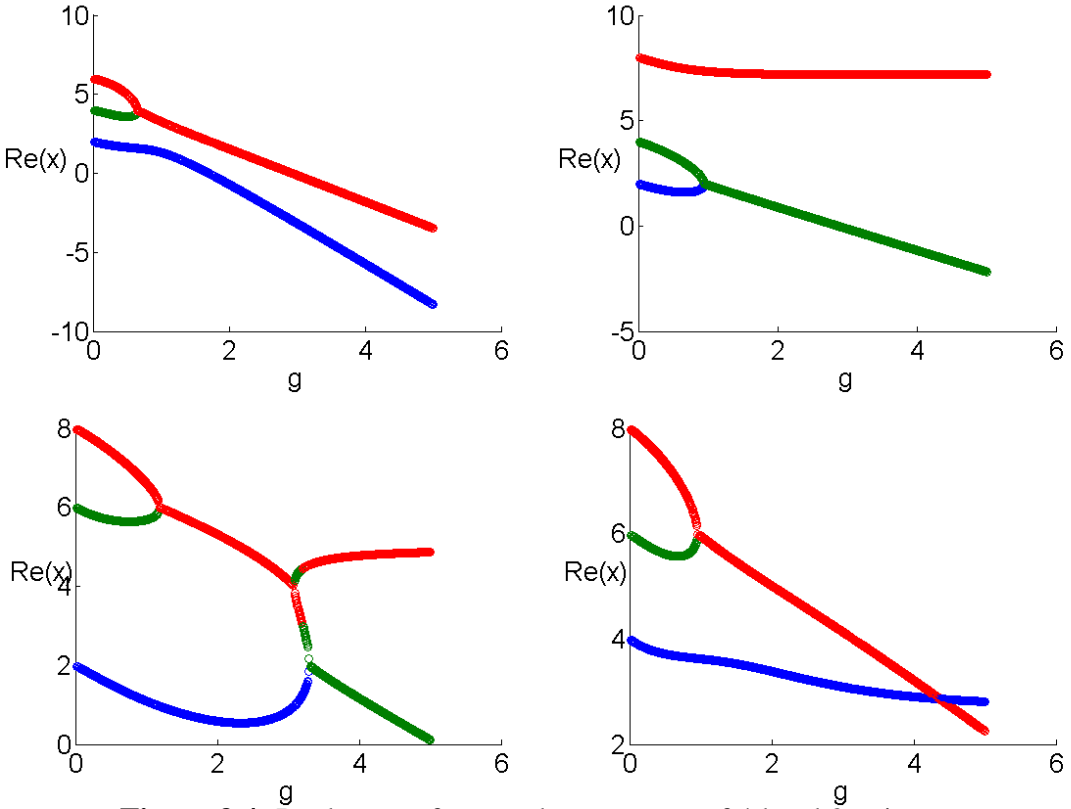


Figure 3.4: Real parts of spectral parameters of 4 level 3 pairs vs. g

For $N_p = 3$ problem, we can see again that the lowest collapsed pairs tends to go to $-\infty$ as g goes to ∞. And also, it can be seen that collapsed pairs can split up and then collapse with another pair. at the vicinities of singularities, effects of extrapolation can be observerd. Extrapolation has a small routine. First it checks whether the equation is solved right. If the solution is wrong, the routine steps back for a small amount and try again. For a defined value of attempts, if equation is still not solved, then the routine takes a larger step and continue from there.

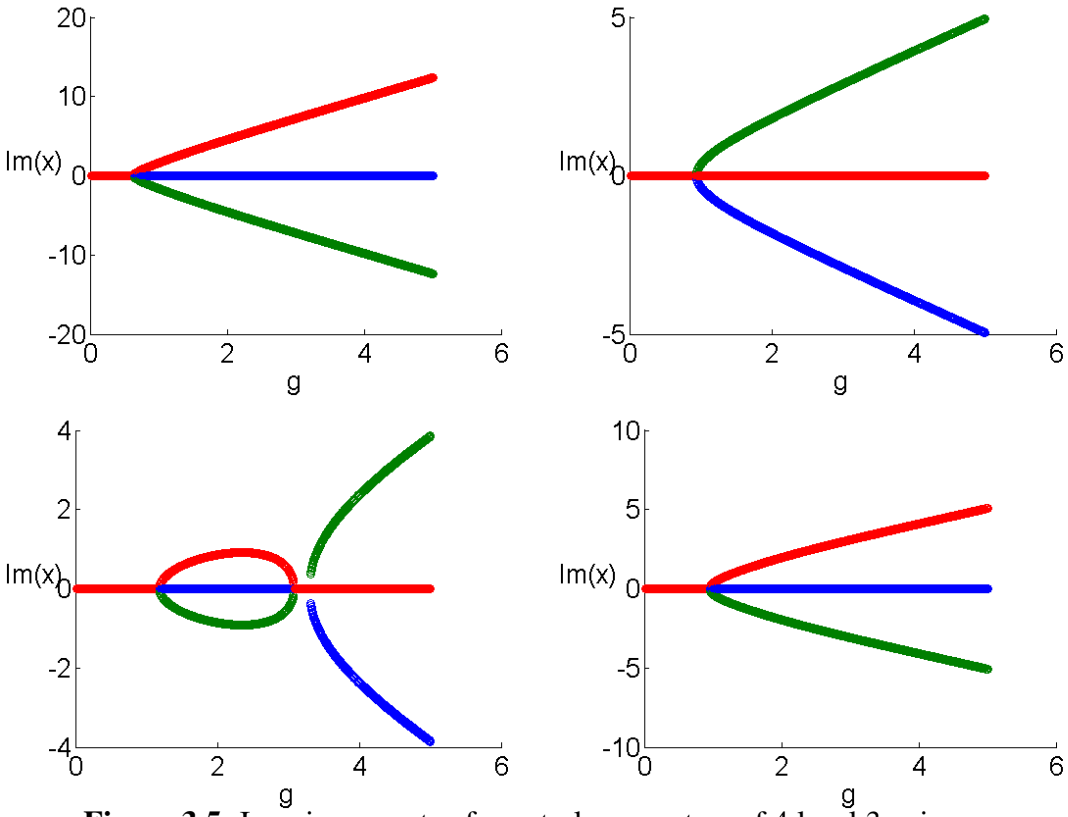


Figure 3.5: Imaginary parts of spectral parameters of 4 level 3 pairs vs. g

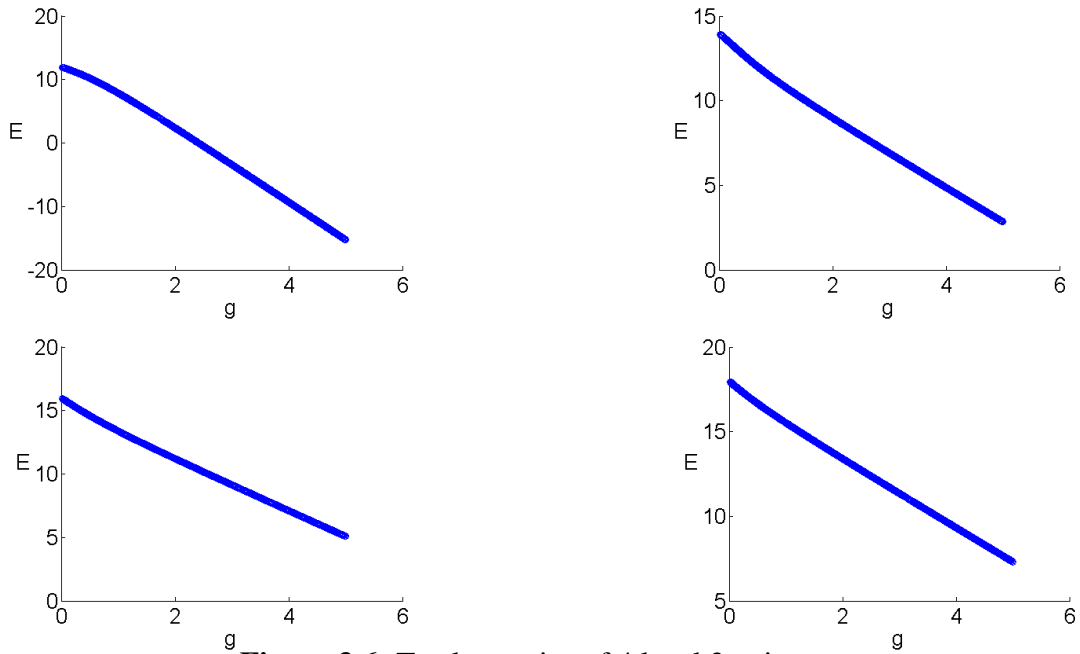


Figure 3.6: Total energies of 4 level 3 pairs vs. g

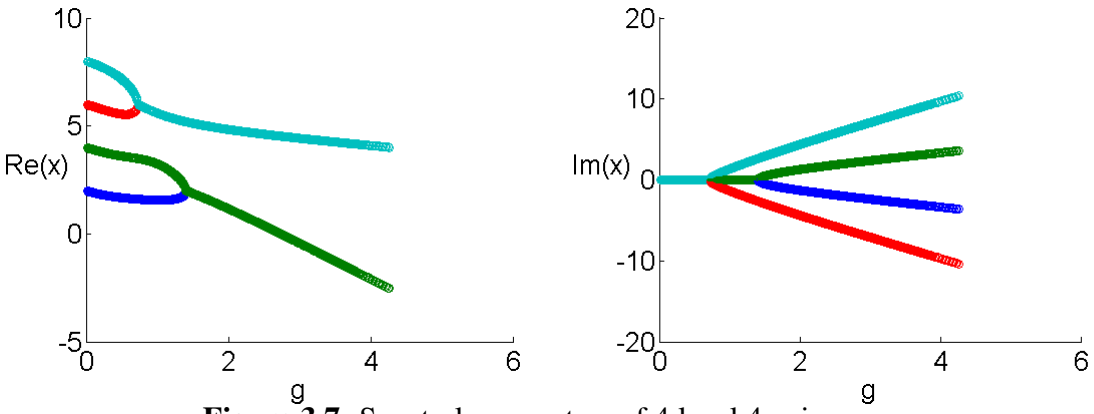


Figure 3.7: Spectral parameters of 4 level 4 pairs vs. g

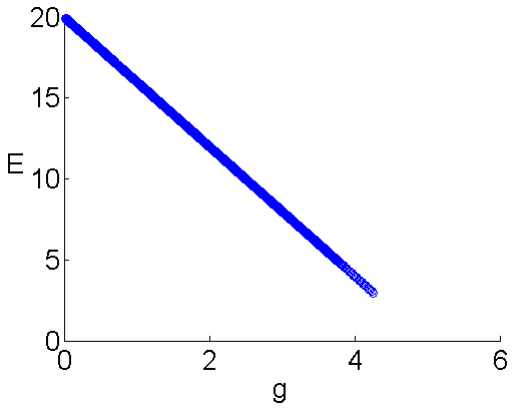


Figure 3.8: Total energies of 4 level 4 pairs vs. g

4. ELECTRON PAIRING MODEL WITH SPIN ORBIT INTERACTIONS

4.1 Spin-Orbit Coupled BCS Hamiltonian

Particles with spin-orbit interactions can be described by the following Hamiltonian

$$H = H_0 + H_{int} \tag{4.1}$$

where

$$H_0 = \sum_{\boldsymbol{k}} (c^{\dagger}_{\boldsymbol{k},\uparrow} c^{\dagger}_{\boldsymbol{k},\downarrow})(\varepsilon_{\boldsymbol{k}} + \alpha \boldsymbol{k} \cdot \boldsymbol{\sigma})(c_{\boldsymbol{k},\uparrow} c_{\boldsymbol{k},\downarrow})^T \tag{4.2}$$

is the single particle part including single particle energy $\varepsilon_{\boldsymbol{k}} = \varepsilon_{-\boldsymbol{k}}$, a Rashba type momentum-spin coupling $\boldsymbol{k} \cdot \boldsymbol{\sigma}$ with coupling constant α and an interaction term

$$H_{int} = -\sum_{\boldsymbol{k},\boldsymbol{k'}} V_0(\boldsymbol{k},\boldsymbol{k'}) c^{\dagger}_{\boldsymbol{k},\uparrow} c^{\dagger}_{-\boldsymbol{k},\downarrow} c_{-\boldsymbol{k'},\downarrow} c_{\boldsymbol{k'},\uparrow}$$

where $V_0(\boldsymbol{k},\boldsymbol{k'}) = V_0 > 0$ is chosen as a constant.

We consider a two dimensional system so that momenta $\boldsymbol{k} = (k_x, k_y)$ and spin operators $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$ have two components.

4.1.1 Single Particle Problem

The non-interacting part of the problem can be written in matrix form

$$H_0 = \sum_{\boldsymbol{k}} \begin{pmatrix} c^{\dagger}_{\boldsymbol{k},\uparrow} & c^{\dagger}_{\boldsymbol{k},\downarrow} \end{pmatrix} \mathbf{M} \begin{pmatrix} c_{\boldsymbol{k},\uparrow} \\ c_{\boldsymbol{k},\downarrow} \end{pmatrix} \tag{4.3}$$

where

$$\mathbf{M} \equiv \begin{pmatrix} \frac{\hbar^2 k^2}{2m} & \alpha(k_x - i k_y) \\ \alpha(k_x + i k_y) & \frac{\hbar^2 k^2}{2m} \end{pmatrix} \tag{4.4}$$

The solution of the single particle Fermion problem involves the diagonalization of the $\mathbf{M}$ matrix. The resulting unitary transformation gives the so-called helicity basis. The Hamiltonian becomes diagonal in this basis.

$$\mathbf{U}^{\dagger}\mathbf{M}\mathbf{U} = \begin{pmatrix} \varepsilon_{k,\Uparrow} & 0 \\ 0 & \varepsilon_{k,\Downarrow} \end{pmatrix} \tag{4.5}$$

or

$$H_0 = \sum_{\boldsymbol{k}} \left[\varepsilon_{k,\Uparrow} a^\dagger_{\boldsymbol{k},\Uparrow} a_{\boldsymbol{k},\Uparrow} + \varepsilon_{k,\Downarrow} a^\dagger_{\boldsymbol{k},\Downarrow} a_{\boldsymbol{k},\Downarrow} \right] \tag{4.6}$$

Introducing $s = \pm$ for the positive and negative helicities,

$$s = \begin{cases} +1 & \text{for} \quad \Uparrow \\ -1 & \text{for} \quad \Downarrow \end{cases} \tag{4.7}$$

the single particle part of the Hamiltonian can be written as:

$$H_0 = \sum_{\boldsymbol{k},s} \varepsilon_{k,s} a^\dagger_{k,s} a_{k,s} \tag{4.8}$$

where the eigenvalues are

$$\varepsilon_{\boldsymbol{k},s} = \frac{\hbar^2 k^2}{2m} + \alpha k s. \tag{4.9}$$

The basis transformation matrix, $\mathbf{U}$, can be obtained from eigenvectors of $\mathbf{M}$. (The general two-level system solution is given in App. A.2.)

$$\mathbf{U} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{-i\theta_{\boldsymbol{k}}} \\ e^{i\theta_{\boldsymbol{k}}} & -1 \end{pmatrix}$$

Also the operators transform similarly.

$$\begin{pmatrix} c_{\boldsymbol{k},\uparrow} \\ c_{\boldsymbol{k},\downarrow} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{-i\theta_{\boldsymbol{k}}} \\ e^{i\theta_{\boldsymbol{k}}} & -1 \end{pmatrix} \begin{pmatrix} a_{\boldsymbol{k},\Uparrow} \\ a_{\boldsymbol{k},\Downarrow} \end{pmatrix} \tag{4.10}$$

Above $\theta_{\boldsymbol{k}}$ denotes the polar angle of the $\boldsymbol{k}$ vector, i.e. $\theta_{\boldsymbol{k}} = \arctan(k_y/k_x)$. It has the following properties:

- $e^{i\theta_{\boldsymbol{k}}} = \frac{k_x + ik_y}{|k|}$
- $\theta_{-\boldsymbol{k}} = \theta_{\boldsymbol{k}} + \pi$ or $e^{i\theta_{\boldsymbol{k}}} = -e^{i\theta_{-\boldsymbol{k}}}$ $(\boldsymbol{k} \neq 0)$
- $e^{i\theta_{\boldsymbol{k}}} = 1$ for $\boldsymbol{k} = 0$

4.1.2 Interaction Terms In Helicity Basis

Interaction part of the Hamiltonian must be transformed as well.

$$\begin{aligned} H_{int} &= -\sum_{\boldsymbol{k},\boldsymbol{k'}} V_0 c^\dagger_{\boldsymbol{k},\uparrow} c^\dagger_{-\boldsymbol{k},\downarrow} c_{-\boldsymbol{k'},\downarrow} c_{\boldsymbol{k'},\uparrow} \\ &= -\frac{V_0}{4} \sum_{\boldsymbol{k},\boldsymbol{k'}} (a^\dagger_{\boldsymbol{k},\Uparrow} + e^{i\theta_{\boldsymbol{k}}} a^\dagger_{\boldsymbol{k},\Downarrow})(e^{-i\theta_{-\boldsymbol{k}}} a^\dagger_{-\boldsymbol{k},\Uparrow} - a^\dagger_{-\boldsymbol{k},\Downarrow}) \\ &\quad (e^{i\theta_{-k'}} a_{-\boldsymbol{k'},\Uparrow} - a_{-\boldsymbol{k'},\Downarrow})(a_{\boldsymbol{k'},\Uparrow} + e^{-i\theta_{\boldsymbol{k'}}} a_{\boldsymbol{k'},\Downarrow}) \end{aligned} \tag{4.11}$$

Since the interaction term is the product of two sums, we can examine each seperately. Writing

$$H_{int} = \sum_{\boldsymbol{k}} S^{\dagger}_{\boldsymbol{k}} \sum_{\boldsymbol{k}'} S_{\boldsymbol{k}'} \tag{4.12}$$

we have

$$\begin{aligned} S^{\dagger}_{\boldsymbol{k}} &= (a^{\dagger}_{\boldsymbol{k},\Uparrow} + e^{i\theta_{\boldsymbol{k}}} a^{\dagger}_{\boldsymbol{k},\Downarrow})(e^{-i\theta_{-k}} a^{\dagger}_{-\boldsymbol{k},\Uparrow} - a^{\dagger}_{-\boldsymbol{k},\Downarrow}) \\ &= -e^{-i\theta_{\boldsymbol{k}}} a^{\dagger}_{\boldsymbol{k},\Uparrow} a^{\dagger}_{-\boldsymbol{k},\Uparrow} - e^{i\theta_{\boldsymbol{k}}} a^{\dagger}_{\boldsymbol{k},\Downarrow} a^{\dagger}_{-\boldsymbol{k},\Downarrow} - a^{\dagger}_{\boldsymbol{k},\Uparrow} a^{\dagger}_{-\boldsymbol{k},\Downarrow} - a^{\dagger}_{\boldsymbol{k},\Downarrow} a^{\dagger}_{-\boldsymbol{k},\Uparrow} \end{aligned} \tag{4.13}$$

For $k \neq 0$:

$$\begin{aligned} S^{\dagger}_{\boldsymbol{k}} + S^{\dagger}_{-\boldsymbol{k}} &= -e^{-i\theta_{\boldsymbol{k}}} a^{\dagger}_{\boldsymbol{k},\Uparrow} a^{\dagger}_{-\boldsymbol{k},\Uparrow} - e^{i\theta_{\boldsymbol{k}}} a^{\dagger}_{\boldsymbol{k},\Downarrow} a^{\dagger}_{-\boldsymbol{k},\Downarrow} - a^{\dagger}_{\boldsymbol{k},\Uparrow} a^{\dagger}_{-\boldsymbol{k},\Downarrow} - a^{\dagger}_{\boldsymbol{k},\Downarrow} a^{\dagger}_{-\boldsymbol{k},\Uparrow} \\ &\quad -e^{-i\theta_{-\boldsymbol{k}}} a^{\dagger}_{-\boldsymbol{k},\Uparrow} a^{\dagger}_{\boldsymbol{k},\Uparrow} - e^{i\theta_{-\boldsymbol{k}}} a^{\dagger}_{-\boldsymbol{k},\Downarrow} a^{\dagger}_{\boldsymbol{k},\Downarrow} - a^{\dagger}_{-\boldsymbol{k},\Uparrow} a^{\dagger}_{\boldsymbol{k},\Downarrow} - a^{\dagger}_{-\boldsymbol{k},\Downarrow} a^{\dagger}_{\boldsymbol{k},\Uparrow} \end{aligned} \tag{4.14}$$

The terms with cross helicities cancel each other

$$-a^{\dagger}_{\boldsymbol{k},\Uparrow} a^{\dagger}_{-\boldsymbol{k},\Downarrow} - a^{\dagger}_{\boldsymbol{k},\Downarrow} a^{\dagger}_{-\boldsymbol{k},\Uparrow} - a^{\dagger}_{-\boldsymbol{k},\Uparrow} a^{\dagger}_{\boldsymbol{k},\Downarrow} - a^{\dagger}_{-\boldsymbol{k},\Downarrow} a^{\dagger}_{\boldsymbol{k},\Uparrow} = 0 \tag{4.15}$$

and we get

$$\begin{aligned} S^{\dagger}_{\boldsymbol{k}} + S^{\dagger}_{-\boldsymbol{k}} &= -e^{-i\theta_{\boldsymbol{k}}} a^{\dagger}_{\boldsymbol{k},\Uparrow} a^{\dagger}_{-\boldsymbol{k},\Uparrow} - e^{i\theta_{\boldsymbol{k}}} a^{\dagger}_{\boldsymbol{k},\Downarrow} a^{\dagger}_{-\boldsymbol{k},\Downarrow} - e^{-i\theta_{-\boldsymbol{k}}} a^{\dagger}_{-\boldsymbol{k},\Uparrow} a^{\dagger}_{\boldsymbol{k},\Uparrow} - e^{i\theta_{-\boldsymbol{k}}} a^{\dagger}_{-\boldsymbol{k},\Downarrow} a^{\dagger}_{\boldsymbol{k},\Downarrow} \\ &= e^{-i\theta_{\boldsymbol{k}}} (a^{\dagger}_{-\boldsymbol{k},\Uparrow} a^{\dagger}_{\boldsymbol{k},\Uparrow} - a^{\dagger}_{\boldsymbol{k},\Uparrow} a^{\dagger}_{-\boldsymbol{k},\Uparrow}) + e^{i\theta_{\boldsymbol{k}}} (a^{\dagger}_{-\boldsymbol{k},\Downarrow} a^{\dagger}_{\boldsymbol{k},\Downarrow} - a^{\dagger}_{\boldsymbol{k},\Downarrow} a^{\dagger}_{-\boldsymbol{k},\Downarrow}) \\ &= -2e^{-i\theta_{\boldsymbol{k}}} a^{\dagger}_{\boldsymbol{k},\Uparrow} a^{\dagger}_{-\boldsymbol{k},\Uparrow} - 2e^{i\theta_{\boldsymbol{k}}} a^{\dagger}_{\boldsymbol{k},\Downarrow} a^{\dagger}_{-\boldsymbol{k},\Downarrow} \end{aligned} \tag{4.16}$$

The $k = 0$ terms can be written with the use of properties of $e^{i\theta_{\boldsymbol{k}}}$ as

$$-a^{\dagger}_{\boldsymbol{0},\Uparrow} a^{\dagger}_{\boldsymbol{0},\Downarrow} + a^{\dagger}_{\boldsymbol{0},\Downarrow} a^{\dagger}_{\boldsymbol{0},\Uparrow} = -2a^{\dagger}_{\boldsymbol{0},\Uparrow} a^{\dagger}_{\boldsymbol{0},\Downarrow} \tag{4.17}$$

After these simplifications the interaction part of the Hamiltonian becomes

$$\begin{aligned} H_{int} &= -V_0 \sum_{\boldsymbol{k}} [e^{-i\theta_{\boldsymbol{k}}} a^{\dagger}_{\boldsymbol{k},\Uparrow} a^{\dagger}_{-\boldsymbol{k},\Uparrow} + e^{i\theta_{\boldsymbol{k}}} a^{\dagger}_{\boldsymbol{k},\Downarrow} a^{\dagger}_{-\boldsymbol{k},\Downarrow} + \delta_{k,0} (a^{\dagger}_{\boldsymbol{0},\Uparrow} a^{\dagger}_{\boldsymbol{0},\Downarrow})] \\ &\quad \times \sum_{\boldsymbol{k}'} [e^{i\theta_{\boldsymbol{k}'}} a_{-\boldsymbol{k}',\Uparrow} a_{\boldsymbol{k}',\Uparrow} + e^{-i\theta_{\boldsymbol{k}'}} a_{-\boldsymbol{k}',\Downarrow} a_{\boldsymbol{k}',\Downarrow} + \delta_{k',0} (a_{\boldsymbol{0},\Downarrow} a_{\boldsymbol{0},\Uparrow})] \end{aligned} \tag{4.18}$$

4.1.3 Mapping The Hamiltonian

In this sub-section we map the above Hamiltonian to the form given by Eq. 1.1.

The single particle levels $\boldsymbol{k}, s$ and $-\boldsymbol{k}, s$ are degenerate. These time-reversed states are involved in the pairing interaction. We switch to labeling j, m so that if $j, m \rightarrow \boldsymbol{k}, s$,

then $j,\bar{m} \rightarrow -\boldsymbol{k},s$ As a result of these transformations we find $\theta_{j,m} = -s\theta_{\boldsymbol{k}}$ with the property of

$$e^{i\theta_{0,+}} = e^{-i\theta_{0,-}} = 1 \tag{4.19}$$

Then we can define the pair operators as following:

$$\hat{n}_j \equiv \sum_m a^\dagger_{j,m} a_{j,m} \tag{4.20}$$

$$A^\dagger_j \equiv \sum_m \left(e^{i\theta_{j,m}} a^\dagger_{j,m} a^\dagger_{j,\bar{m}} + (-1)^m \delta_{j,0} a^\dagger_{0,m} a^\dagger_{0,\bar{m}} \right) \tag{4.21}$$

with the use of Eq. 4.19 we can rewrite Eq. 4.21 as

$$A^\dagger_j \equiv \sum_m e^{i\theta_{j,m}} a^\dagger_{j,m} a^\dagger_{j,\bar{m}} \tag{4.22}$$

Finally, the model Hamiltonian becomes

$$H = \sum_j \varepsilon_j \hat{n}_j + \frac{V_0}{4} \sum_{j,j'} A^\dagger_j A_{j'} \tag{4.23}$$

Now Eq. 4.23 is in the form of Eq. 1.1. Below we check the commutation relations of the pair operators to fully establish the mapping and apply the Richardson solution.

$$\begin{aligned}
\left[\hat{n}_j, A^\dagger_{j'}\right] &= \sum_m \left[a^\dagger_{j,m} a_{j,m}, \sum_n e^{i\theta_{j',n}} a^\dagger_{j',n} a^\dagger_{j',\bar{n}} \right] \\
&= \sum_{m,n} a^\dagger_{j,m} \left[a_{j,m}, e^{i\theta_{j',n}} a^\dagger_{j',n} a^\dagger_{j',\bar{n}} \right] \\
&= \sum_{m,n} e^{i\theta_{j',n}} a^\dagger_{j,m} \left(-a^\dagger_{j',n} \left\{ a_{j,m}, a^\dagger_{j',\bar{n}} \right\} + \left\{ a_{j,m}, a^\dagger_{j',n} \right\} a^\dagger_{j',\bar{n}} \right) \\
&= \sum_m e^{i\theta_{j,m}} \delta_{j,j'} \delta_{m,n} a^\dagger_{j,m} a^\dagger_{j,\bar{m}} \\
&= 2\delta_{j,j'} A^\dagger_j
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
\left[A_{j'}, A^\dagger_j\right] &= \sum_{m,n} e^{i(\theta_{j,m} - \theta_{j',n})} \left[a_{j',\bar{n}} a_{j',n}, a^\dagger_{j,m} a^\dagger_{j,\bar{m}} \right] \\
&= \sum_{m,n} e^{i(\theta_{j,m} - \theta_{j',n})} \left(a_{j',\bar{n}} \left[a_{j',n}, a^\dagger_{j,m} a^\dagger_{j,\bar{m}} \right] + \left[a_{j',\bar{n}}, a^\dagger_{j,m} a^\dagger_{j,\bar{m}} \right] a_{j',n} \right) \\
&= \sum_{m,n} e^{i(\theta_{j,m} - \theta_{j',n})} \left(\delta_{j,j'} \delta_{m,n} a_{j',\bar{n}} a^\dagger_{j,\bar{m}} - \delta_{j,j'} \delta_{m,n} a^\dagger_{j,m} a_{j',n} \right) \\
&= \sum_m \delta_{j,j'} \delta_{m,n} \left(1 - a^\dagger_{j,\bar{m}} a_{j,\bar{m}} - a^\dagger_{j,m} a_{j,m} \right) \\
&= 2\delta_{j,j'} \left(\Omega_j - 2\hat{n}_j \right)
\end{aligned} \tag{4.25}$$

Above Ω_j is the degeneracy of the level j. One can clearly see that these commutation relations are compatible with Eq. 2.15 and Eq. 2.16. As a result of this mapping, the SOC Fermi gas can be treated with the same approach as in Chapter 3.

4.2 Exact Solutions of Spin-Orbit Coupled Pairing Hamiltonian

Spin-orbit coupled systems may have more than two fold degeneracies which is an important factor when it comes to the numerical solution. However, it can be integrated to the solver easily. We showed that this system is exactly solvable but we encountered numerical diffuculties showing sensitivity to the initial starting values for different states. Additionally required step size for g also differs among configurations. Even a $\delta g = 0.001$ change in step size results to not converging solutions. As it can be seen from Fig. 4.1, in the ground state, the pairs immediately collapses and tends to go to $-\infty$ as g goes to ∞. Fig. 4.2 shows an example of an excited state. In general, the calculations involving excited states starting from degenerate levels are more susceptible to numerical precision.

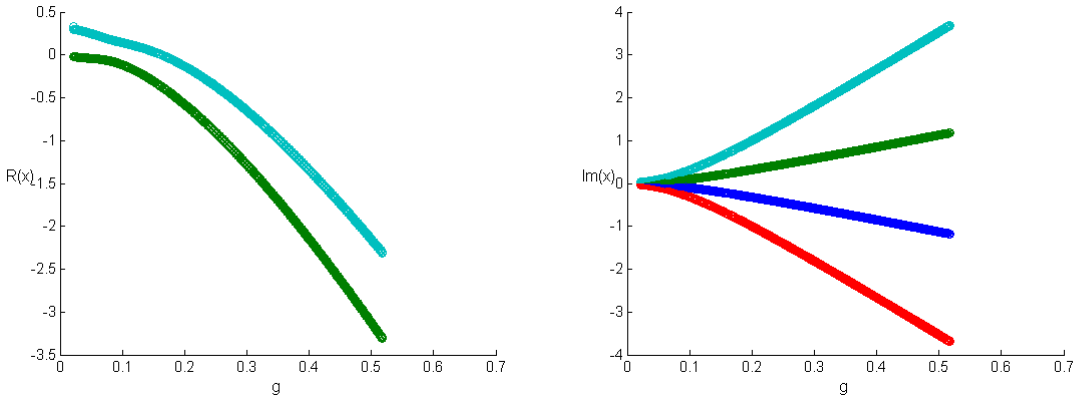


Figure 4.1: Spectral parameters of 4 level 4 pairs vs. g

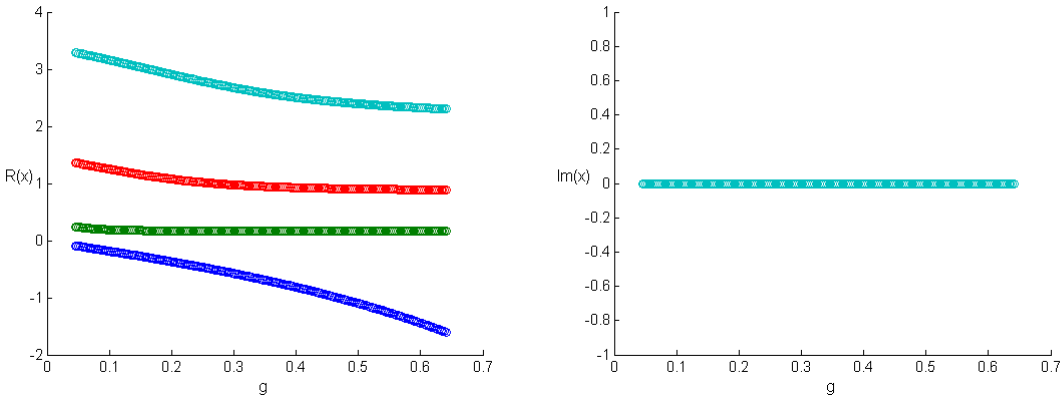


Figure 4.2: Spectral parameters of 4 level 4 pairs vs. g

5. CONCLUSIONS AND FUTURE ASPECTS

In this thesis, we followed Richardson's [10] and Dukelsky's [12] works in order to study an exactly solvable model which is a spin-orbit coupled Fermi gas with s-wave pairing interactions. Whereas Richardson's original calculation focuses on a specific Hamiltonian making particular transformations in order to avoid numerical singularities, the latter work provides a general framework for the method. A similar calculation has been published with a slightly different formalism. [13]

We have implemented a Rashba-type SOC on a pairing Hamiltonian and transformed it to helicity basis so that we mapped it to the class of the exactly solvable Hamiltonians. We solved non-linear algebraic equations for the spectral parameters of the model. These parameters are seperated into clusters which have a population limit related with system's level degeneracy. The numerical solution begins at the non-interacting limit and proceeds as the interaction parameter (g) changes slowly. For each value of g these non-linear equations are solved with a root-finding algorithm. The procedure continues until the desired value of the interaction parameter is reached. One of the important features of the solver is the extrapolation step which has to be done with utmost care.

We have come to the conclusion that SOC pairing Hamiltonian is indeed exactly solvable by mapping it to the general class of exactly solvable Richardson models. The model under consideration belongs to the class of topological superconductors and can host Majorana fermions. As future work, we are plannig to upgrade the solver so that it can solve larger systems. We are plan to investigate the system under a Zeeman field and check its topological phases in a fully interacting system with an exact solution.

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APPENDICES

APPENDIX A.1 : Combinatorics Identities
APPENDIX A.2 : Spin 1/2 Particle In an Arbitrary Magnetic Field
APPENDIX A.3 : The Mean Field Hamiltonian

APPENDIX A.1

Following relations can be derived from binomial theorem

$$\boxed{\sum_{n=0}^{p} a^n b^{p-n} \binom{p}{n} = (a+b)^n} \qquad \textbf{(A.1a)}$$

choosing $a = -1$ and $b = 1$.

$$\boxed{\begin{aligned} \sum_{n=0}^{p} (-1)^n (1)^{p-n} \binom{p}{n} &= 0 \qquad \textbf{(A.2a)} \\ 1 - \sum_{n=1}^{p} (-1)^n (1)^{p-n} \binom{p}{n} &= 0 \qquad \textbf{(A.2b)} \end{aligned}}$$

Another relation follows by shifting the summation index by one.

$$\boxed{\sum_{n=0}^{p-1} (-1)^n \binom{p}{n+1} = -\sum_{n=1}^{p} (-1)^n \binom{p}{n}} \qquad \textbf{(A.3a)}$$

APPENDIX A.2

The Hamiltonian of a spin 1/2 particle in a magnetic field can be written as

$$H = \gamma \boldsymbol{\sigma} \cdot \boldsymbol{B}. \tag{A.4}$$

where γ is a constant, $\boldsymbol{\sigma}$'s are Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.5}$$

and $\boldsymbol{B}$ denotes an arbitrary magnetic field.

In three dimensions,

$$\boldsymbol{B} = B(\sin\theta\cos\varphi\hat{i} + \sin\theta\sin\varphi\hat{j} + \cos\theta\hat{k}) \tag{A.6}$$

With these definitons Eq. **A.4** can be written explicitly as

$$\begin{aligned} H &= \gamma(\sigma_x B_x + \sigma_y B_y + \sigma_z B_z) \\ &= \gamma \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix} \\ &= \gamma B \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix} \end{aligned} \tag{A.7}$$

The energy values of this system are

$$E_\pm = \pm\gamma B. \tag{A.8}$$

The eigenvectors are given in terms of the polar and azimuthal angles of the magnetic field.

$$v_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\varphi} \end{pmatrix} \qquad v_- = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\varphi} \\ \cos\frac{\theta}{2} \end{pmatrix} \tag{A.9}$$

APPENDIX A.3

In this appendix we provide the mean-field Hamiltonian for the Richardson model. The suitable mean-field approach is supposed provide good results for large particle number. A comparison with the exact solution provides definite confirmation of the validity of the mean-field approximation.

The mean-field approximation reduces the many-body problem defined by the Hamiltonian

$$H = \sum_{j,m} \varepsilon_j a^\dagger_{jm} a_{jm} - g \sum_{j,j',m,m\prime} a^\dagger_{jm} a^\dagger_{j\bar{m}} a_{j'\bar{m}'} a_{j'm'} \tag{A.10}$$

effectively to a single-particle problem where interactions are included as an average external potential. This approximation can be carried out by defining the mean field terms as in the following.

- $\Delta^*_{j,m} = \langle a^\dagger_{jm} a^\dagger_{j\bar{m}} \rangle$
- $n^H_{j,m} = \langle a^\dagger_{jm} a_{jm} \rangle$
- $n^{ex^*}_{j,m} = \langle a^\dagger_{jm} a_{j\bar{m}} \rangle$

Neglecting fluctuations about the mean-field, the effective Hamiltonian becomes a quadratic Hamiltonian

$$\begin{aligned} H &= \sum_{j,m} \varepsilon_j a^\dagger_{jm} a_{jm} - g \sum_{j,j',m,m\prime} \Big[\Delta^*_{m,\bar{m}} a_{j'\bar{m}'} a_{j'm'} + \Delta_{m,\bar{m}} a^\dagger_{jm} a^\dagger_{j\bar{m}} + |\Delta|^2_{m,\bar{m}} \\ &\quad + n^H_{j',\bar{m}} a^\dagger_{j'\bar{m}} a_{j'\bar{m}'} + n^H_{j,m} a^\dagger_{jm} a_{jm} + n^H_{j',\bar{m}} n^H_{j,m} - n^{ex^*}_{j,j'} a^\dagger_{j\bar{m}} a_{j'm'} - n^{ex}_{j,j'} a^\dagger_{jm} a_{j'm'} - |n^{ex}|^2_{j,j'} \Big] \end{aligned}$$

which can be written in matrix form

$$H = \frac{1}{2} A^\dagger H_M A \tag{A.11}$$

with the operator vector

$$A^\dagger = \begin{pmatrix} a^\dagger_{\mathbf{1},\Uparrow} & \dots & a^\dagger_{\boldsymbol{N},\Uparrow} & a^\dagger_{\mathbf{1},\Downarrow} & \dots & a^\dagger_{\boldsymbol{N},\Downarrow} & a_{\mathbf{1},\Uparrow} & \dots & a_{\boldsymbol{N},\Uparrow} & a_{\mathbf{1},\Downarrow} & \dots & a_{\boldsymbol{N},\Downarrow} \end{pmatrix} \tag{A.12}$$

and the Hamiltonian matrix

$$H_M = \begin{pmatrix} T & \Delta \\ \Delta^\dagger & -T^T \end{pmatrix} \tag{A.13}$$

where Δ matrix is pair-density and T matrix covers the kinetic, Hartree and exchange terms. The mean-field calculation involves the self-consistent solution for the mean-field values.

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